

# Foams and categorification

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$$\left\langle \left( \begin{array}{c} \text{circle with arrow } k \end{array} \right) \right\rangle = \left[ \begin{array}{c} N \\ k \end{array} \right]_q$$

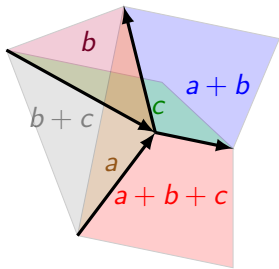
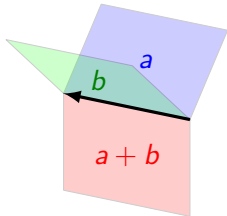
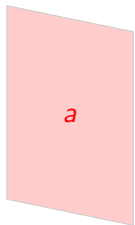
$$\left\langle \left( \begin{array}{c} m+n \xrightarrow{m} \\ \xrightarrow{m} n \end{array} \right) \right\rangle = \left[ \begin{array}{c} N-m \\ n \end{array} \right]_q \left\langle \left( \begin{array}{c} \uparrow \\ m \end{array} \right) \right\rangle$$

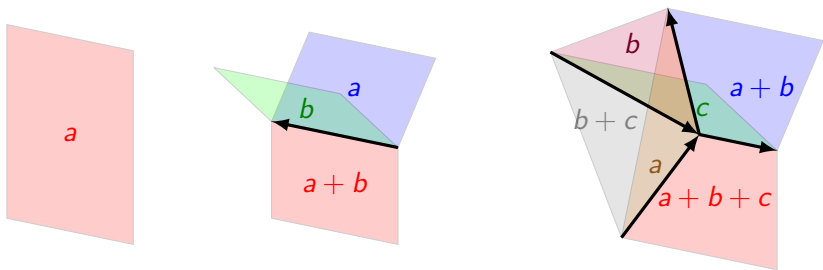
$$\left\langle \left( \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \quad \nearrow \\ \uparrow \\ i+j+k \end{array} \right) \right\rangle = \left\langle \left( \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \quad \nearrow \\ \uparrow \\ i+j+k \end{array} \right) \right\rangle$$

$$\left\langle \left( \begin{array}{c} m+n \xrightarrow{m} \\ \xrightarrow{m} n \end{array} \right) \right\rangle = \left[ \begin{array}{c} m+n \\ m \end{array} \right]_q \left\langle \left( \begin{array}{c} \uparrow \\ m+n \end{array} \right) \right\rangle$$

$$\left\langle \left( \begin{array}{c} 1 \quad m \\ \uparrow m+1 \quad \downarrow m+1 \\ \leftarrow \quad \rightarrow \\ \uparrow m+1 \quad \downarrow m+1 \\ 1 \quad m \end{array} \right) \right\rangle = \left\langle \left( \begin{array}{c} \uparrow \\ 1 \end{array} \right) \right\rangle \left\langle \left( \begin{array}{c} \downarrow \\ m \end{array} \right) \right\rangle + [N-m-1]_q \left\langle \left( \begin{array}{c} 1 \quad m \\ \swarrow \quad \searrow \\ \downarrow \\ m-1 \\ \nearrow \quad \nwarrow \\ 1 \quad m \end{array} \right) \right\rangle$$

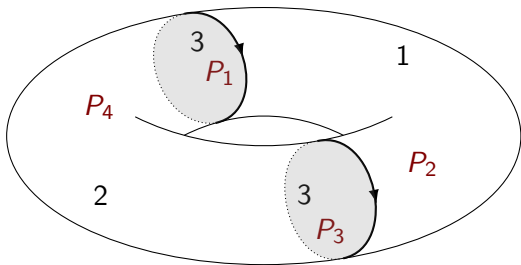
$$\left\langle \left( \begin{array}{c} m \quad n+l \\ \uparrow n+k \quad \uparrow m+l-k \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \uparrow n \quad \uparrow m+l \end{array} \right) \right\rangle = \sum_{j=\max(0, m-n)}^m \left[ \begin{array}{c} l \\ k-j \end{array} \right]_q \left\langle \left( \begin{array}{c} m \quad n+l \\ \uparrow n+j-m \quad \uparrow n+l+j \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \uparrow n \quad \uparrow m+l \end{array} \right) \right\rangle$$





## Definition (R.-Wagner, '17)

$$\tau_N(F) = \sum_c \frac{(-1)^{\sum_{i=1}^N iX_i(F(c))/2 + \sum_{1 \leq i < j \leq N} \theta_{ij}^+(F(c))} \prod_f P_f(c(f))}{\prod_{1 \leq i < j \leq N} (X_i - X_j)^{\frac{x_{ij}(F(c))}{2}}}$$



$$P_1 = t_1 + t_2 + t_3$$

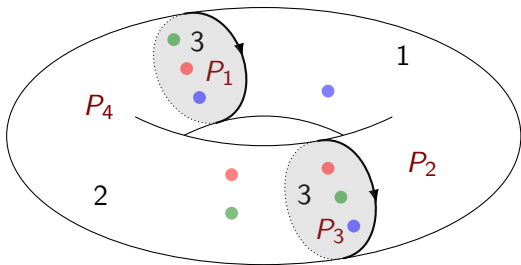
$$P_2 = t_1^2$$

$$P_3 = 1$$

$$P_4 = t_1 t_2 (t_1 + t_2)$$

$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



$$P_1 = t_1 + t_2 + t_3$$

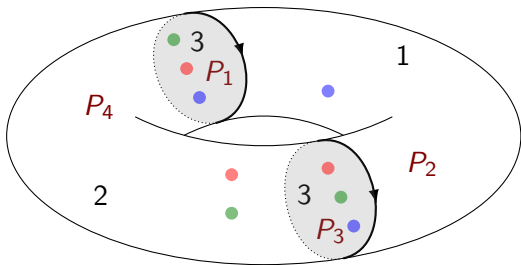
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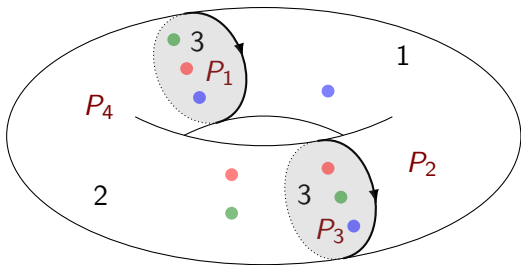
$$P_3 = 1$$

$$P_4 = t_1 t_2 (t_1 + t_2)$$

$\mathbb{P}$	Monochrome	$\chi_\bullet$
$X_1$		2
$X_2$		2
$X_3$		0
$X_4$		2

$$N = 4$$

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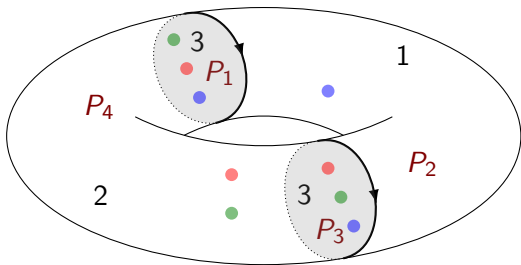
$\mathbb{P}$	Monochrome	$\chi_\bullet$
$X_1$		2
$X_2$		2
$X_3$		0
$X_4$		2

	$X_1 X_2$	$X_1 X_3$	$X_1 X_4$	$X_2 X_3$	$X_2 X_4$	$X_3 X_4$
$\chi_{\bullet\bullet}$	0	2	0	2	0	2
$\theta_{\bullet\bullet}^+$	2	0	2	0	0	0

$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$





$$P_1 = t_1 + t_2 + t_3 \quad P_2 = t_1^2$$

$$P_3 = 1 \quad P_4 = t_1 t_2 (t_1 + t_2)$$

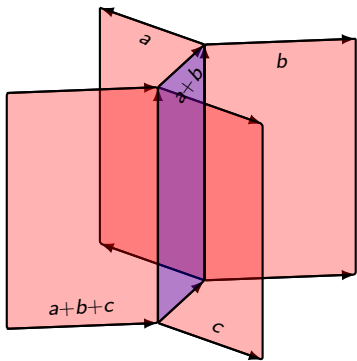
$$\frac{(-1)^{1+2+4} (X_1 + X_2 + X_4) X_1^2 (X_2 X_4 (X_2 + X_4))}{(X_1 - X_3)(X_2 - X_3)(X_3 - X_4)}$$

$\mathbb{P}$	Monochrome	$\chi_\bullet$
$X_1$		2
$X_2$		2
$X_3$		0
$X_4$		2

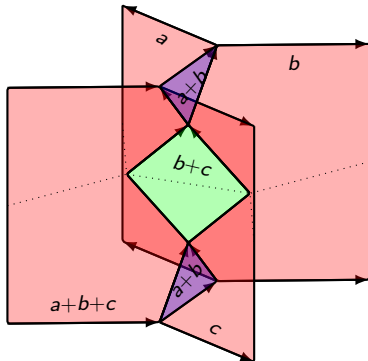
	$X_1 X_2$	$X_1 X_3$	$X_1 X_4$	$X_2 X_3$	$X_2 X_4$	$X_3 X_4$
$\chi_{\bullet\bullet}$	0	2	0	2	0	2
$\theta_{\bullet\bullet}^+$	2	0	2	0	0	0

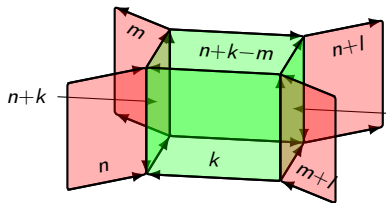
$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



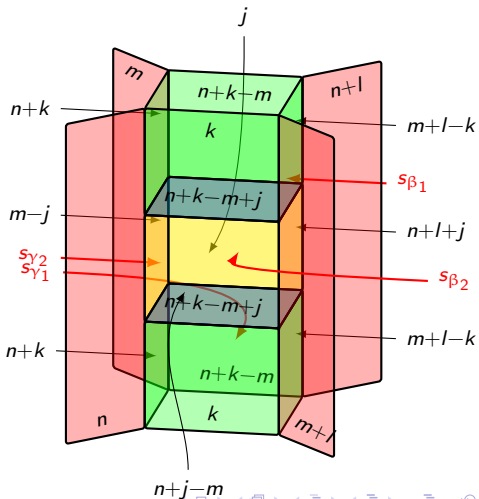
=





$$m+l-k = \sum_{j=\max(0, m-n)}^m \sum_{\alpha \in T(k-j, l-k+j)}$$

$$(-1)^{|\alpha|+(l-k+j)(m-j)} \sum_{\substack{\beta_1, \beta_2 \\ \gamma_1, \gamma_2}} c_{\beta_1 \beta_2}^{\alpha} c_{\gamma_1 \gamma_2}^{\hat{\alpha}}$$



$A_1$	$A_2$	$A_3$	$A_4$	$B_1$	$B_2$	$C$	$L$	$R$	$X$	$A_1$	$A_2$	$A_3$	$A_4$	$C$	$L$	$R$	$X$	
										$A_1$								
										$A_2$								
										$A_3$								
										$A_4$								
										$B_1$								
										$B_2$								
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										$L$								
										$R$								
										$X$								

$$\langle \text{circle with arrow } k \rangle = \begin{bmatrix} N \\ k \end{bmatrix}$$

$$\langle \text{loop with } m+n \text{ and } n \text{ labels} \rangle = \begin{bmatrix} N-m \\ n \end{bmatrix} \langle \text{vertical arrow } m \rangle$$

$$\langle \text{Y-junction with } i, j, k \text{ and } i+j+k \rangle = \langle \text{Y-junction with } i, j, k \text{ and } i+j+k \rangle$$

$$\langle \text{loop with } m+n \text{ and } n \text{ labels} \rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \langle \text{vertical arrow } m+n \rangle$$

$$\langle \text{loop with } m+1 \text{ and } 1 \text{ labels} \rangle = \langle \text{vertical arrow } 1 \rangle + [N-m-1] \langle \text{Y-junction with } m-1 \text{ labels} \rangle$$

$$\langle \text{rectangle with } n+k, m, n+l, m+l-k \text{ labels} \rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \langle \text{rectangle with } m-j, n+l+j \text{ labels} \rangle$$

From  $\Lambda^\bullet$  to  $\text{Sym}^\bullet$ :  $N$  goes to  $-N$ .

$$\left\langle \begin{array}{c} \text{circle with arrow } k \end{array} \right\rangle = \begin{bmatrix} N \\ k \end{bmatrix}$$

$$\left\langle \begin{array}{c} m+n \uparrow \\ \text{loop } n \\ m \downarrow \end{array} \right\rangle = \begin{bmatrix} N-m \\ n \end{bmatrix} \left\langle \begin{array}{c} \uparrow \\ m \\ \downarrow \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ \text{node} \\ \uparrow \\ i+j+k \end{array} \right\rangle = \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ \text{node} \\ \uparrow \\ i+j+k \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} m+n \uparrow \\ \text{loop } n \\ m+n \downarrow \end{array} \right\rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \left\langle \begin{array}{c} \uparrow \\ m+n \\ \downarrow \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} 1 \quad m \\ \uparrow \quad \downarrow \\ m+1 \quad 1 \\ \downarrow \quad \uparrow \\ m \quad m \\ \uparrow \quad \downarrow \\ 1 \quad m \end{array} \right\rangle = \left\langle \begin{array}{c} \uparrow \\ 1 \\ \downarrow \end{array} \right\rangle + [N-m-1] \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \nearrow \\ \text{node} \\ \downarrow \\ 1 \quad m \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ n+k \quad m+l-k \\ \leftarrow \quad \rightarrow \\ k \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ m-j \quad n+l+j \\ \leftarrow \quad \rightarrow \\ n+j-m \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle$$

From  $\Lambda^\bullet$  to  $\text{Sym}^\bullet$ :  $N$  goes to  $-N$ .

$$\left\langle \left\langle \begin{array}{c} \circlearrowright k \end{array} \right\rangle \right\rangle = \begin{bmatrix} N + k - 1 \\ k \end{bmatrix} \quad \left\langle \left\langle \begin{array}{c} m \uparrow \\ m+n \quad \curvearrowright \quad n \\ m \downarrow \end{array} \right\rangle \right\rangle = \begin{bmatrix} N + m + n - 1 \\ n \end{bmatrix} \left\langle \left\langle \begin{array}{c} \uparrow \\ m \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ \uparrow \\ i+j+k \end{array} \right\rangle \right\rangle = \left\langle \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ i+j \quad \uparrow \\ i+j+k \end{array} \right\rangle \right\rangle \quad \left\langle \left\langle \begin{array}{c} m+n \uparrow \\ m \quad \curvearrowright \quad n \\ m+n \downarrow \end{array} \right\rangle \right\rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \left\langle \left\langle \begin{array}{c} \uparrow \\ m+n \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} 1 \quad m \\ \uparrow \quad \downarrow \\ m+1 \quad \leftarrow \quad \rightarrow \quad 1 \\ \downarrow \quad \uparrow \\ 1 \quad m \end{array} \right\rangle \right\rangle = \left\langle \left\langle \begin{array}{c} \uparrow \\ 1 \end{array} \right\rangle \right\rangle \left\langle \left\langle \begin{array}{c} \downarrow \\ m \end{array} \right\rangle \right\rangle + [N + m + 1] \left\langle \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \nearrow \\ m-1 \quad \downarrow \\ \uparrow \quad \downarrow \\ 1 \quad m \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ n+k \quad \leftarrow \quad \rightarrow \quad m+l-k \\ \leftarrow \quad \rightarrow \quad k \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle \right\rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \left\langle \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ m-j \quad \leftarrow \quad \rightarrow \quad n+l+j \\ \leftarrow \quad \rightarrow \quad n+j-m \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle \right\rangle$$

Thank you !!