Evaluation of \mathfrak{sl}_N -foams

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Workshop on Quantum Topology - Lille



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Definition (R.–Wagner, '17)

$$\langle F \rangle_{N} = \sum_{c} \frac{(-1)^{\sum_{1 \le i < j \le N} \theta_{ij}^{+}(F,c)} \prod_{f} P_{f}(c(f))}{(-1)^{\sum_{i=1}^{N} i\chi(F_{i}(c))/2} \prod_{1 \le i < j \le N} (X_{i} - X_{j})^{\frac{\chi(F_{ij}(c))}{2}}}$$

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Definition (Kauffman Bracket, Jones polynomial)

$$\langle \emptyset \rangle_{\mathrm{K}} = 1 \qquad \left\langle \bigcirc \sqcup L \right\rangle_{\mathrm{K}} = [2]_{q} \langle L \rangle$$
$$\left\langle \swarrow \right\rangle_{\mathrm{K}} = \left\langle \smile \right\rangle_{\mathrm{K}} - q \left\langle \right\rangle \quad \left(\right\rangle_{\mathrm{K}}$$

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \left\langle D \right\rangle_{\mathrm{K}}$$

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Khovanov homology



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Khovanov homology



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Khovanov homology



Shift the homological degree by $-n_-$, the *q*-degree by $n_+ - 2n_-$. Take the homology.

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Proposition (Bar-Natan, '02)

Khovanov homology is strictly stronger than the Jones polynomial.





(source www.colab.sfu.ca/KnotPlot/KnotServer/)

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Theorem (Kronheimer–Mrowka, '10)

Khovanov homology detects the unknot.

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Theorem (Kronheimer–Mrowka, '10)

Khovanov homology detects the unknot.

Milnor conjecture (Kronheimer–Mrowka, '93, Rasmussen '04) The slice genus of the (p, q)-torus knot is equal to $\frac{(p-1)(q-1)}{2}$.



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- A recipe to deal with crossings
- An ad-hoc TQFT



A recipe to deal with crossings ~~ Rickard complexes

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An ad-hoc TQFT ~> evaluation of foams

The \mathfrak{sl}_N -link invariant

Proposition (Drinfel'd)

One can deform $U(\mathfrak{sl}_N)$ into $H := U_q(\mathfrak{sl}_N)$ such that it becomes a quasi-triangular Hopf $\mathbb{C}(q)$ -algebra with non-trivial braiding.

k	$\mathrm{id}_{\bigwedge_q^k V}, \ \mathrm{id}_{(\bigwedge_q^\ell V)^*}$							
D_1 D_2	$f_1 \circ f_2$							
D_1 D_2	$f_1\otimes f_2$							
kn le kn le	braiding							

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k ↑ ℓ	$\operatorname{id}_{\wedge_{a}^{k}V}, \operatorname{id}_{(\wedge_{a}^{\ell}V)^{*}}$	\overline{k}	evaluation
		k · k	coevaluation
D ₂	$t_1 \circ t_2$	$k + \ell$	$\wedge^k_q V \otimes \wedge^\ell_q V$
D_1 D_2	$f_1\otimes f_2$	ℓ+↓k	$\longrightarrow riangle_q^{k+\ell} V$
kn læ kn læ	braiding	l t k	$\wedge^{k+\ell}_q V \longrightarrow$
\land	Sidiang	$k \stackrel{\bullet}{+} \ell$	$\wedge^k_q V \otimes \wedge^\ell_q V$

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MOY calculus (Murakami–Ohtsuki–Yamada)

Lusztig ('94):





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 $\begin{array}{cccc} \mathcal{F}: & \mathsf{Feam}_N & \longrightarrow & \mathbb{Z}[X_1, \dots, X_N] - \mathsf{mod}_{\mathrm{gr}} \\ \text{Wish:} & & \mathrm{MOY}\text{-}\mathrm{graph} & \longmapsto & \mathrm{graded\ module} \\ & & & & \mathrm{feam} & \longmapsto & \mathrm{graded\ module\ map} \end{array}$

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 $\begin{array}{cccc} \mathcal{F}: & \mathsf{Foam}_N & \longrightarrow & \mathbb{Z}[X_1, \dots, X_N] - \mathsf{mod}_{\mathrm{gr}} \\ \text{Wish:} & & \mathrm{MOY}\text{-}\mathrm{graph} & \longmapsto & \mathrm{graded\ module} \\ & & & \mathrm{foam} & \longmapsto & \mathrm{graded\ module\ map} \end{array}$

Universal Construction An evaluation \rightsquigarrow (Maybe) a TQFT

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 $\begin{array}{cccc} \mathcal{F}: & \mathsf{Foam}_N & \longrightarrow & \mathbb{Z}[X_1, \dots, X_N] - \mathsf{mod}_{\mathrm{gr}} \\ \text{Wish:} & & \mathrm{MOY}\text{-}\mathrm{graph} & \longmapsto & \mathrm{graded\ module} \\ & & & \mathrm{foam} & \longmapsto & \mathrm{graded\ module\ map} \end{array}$

Universal Construction An evaluation ↔ (Maybe) a TQFT

Theorem (R.–Wagner, '17)

The evaluation defined on the first slide together with the Universal Construction yields an ad-hoc TQFT.

Universal Construction (Blanchet, Habbeger, Masbaum, Vogel)

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Given: {closed cobordisms} $\longrightarrow R$

Universal Construction (Blanchet, Habbeger, Masbaum, Vogel)

Given: {closed cobordisms} $\longrightarrow R$

$$\Gamma \longmapsto \mathcal{F}(\Gamma) := \bigoplus_{\emptyset} R_F / \frac{\sum_i \lambda_i F_i = 0 \text{ if}}{\sum_i \lambda_i \tau(F_i G) = 0 \text{ for all }_{\Gamma} G_{\emptyset}}$$
$$F_{\Gamma_1} G_{\Gamma_2} \longmapsto \mathcal{F}(G) : \begin{pmatrix} \mathcal{F}(\Gamma_1) \to \mathcal{F}(\Gamma_2) \\ [_{\emptyset} F_{\Gamma_1}] \mapsto [_{\emptyset} F G_{\Gamma_2}] \end{pmatrix}$$

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$h \setminus i$	$A_1^i \cap A_1^i$	$A_1^* \cap A_2^*$	$A_2^* \cap A_1^*$	$A_2 \cap A_2$	B ₁	B ₂	C	L	R	X	$h \setminus i$	A1	A1	$B_1^* \cap B_1^*$	$B_1^* \cap B_2^*$	$B_2^* \cap B_1^*$	$B'_1 \cap B'_2$	C	L	R	X
	耳 耳	回口	耳耳	耳耳	五五	戸戸	耳耳	月月	耳耳	「耳耳」		戸戸	日日	耳耳	耳耳	AA	ДД	五五	ДД	戸戸	耳耳
$A_1^t \cap A_1^b$	ЦЦ	ЦЦ	ДД	ДĬ	ЫÄН	ΗЩ	ΈĔ	ЩЩ	ДД	μμ	.41	ЦЦ	ЦЩ	HH	HH	HH	μH	ΈĔ.	ЦЩ	ЩЩ	μĦ
	ЦΠ	ΠΠ	ΠH	αa	HH	ΉH	HH	Яд	ЪЪ	ΠH		ΗH	HH	HH	ΗH	HH	ΉH	ΈH	ЦΠ	HH	ЦЦ
$A_1^i\cap A_2^b$	ЦЦ	ЦЦ	ЩЩ	ЩЩ	HH	ЦH	ΗH	ДД	ЦЦ	ЦЦ		ДД	ДД	HH	HH	μд	ЦД	ΈĔ	ЦЦ	ΗН	ЦЦ
	ЦΠ	ΉН	ΠН	Π	HHH	HH	HH	HΗ	ਖ਼ਿਸ਼	HH.	A_2	ΉĦ	HH.	HH	HH	HHH	ΗH	Я́Ы	ΗЩ	HH	ЦЩ
	μa	ΠH	μ'n	ΠH	HHH	ШH	ΉH	HΗ	'nн	चिच		ΉH	HH.	ĤĤ	ШH	ਸਿੰਸ	ΗH	EE	ЦЦ	HH	E E
	щщ	ΠΠ	ΠH	ΠΠ	HHH	HH	ΉĦ	ΗH	ΠĦ	विच		HH	HH	ΠĦ	ЦЦ	μ'n	ШĦ	ΠÌ	ਸ਼ਿਸ਼	ΠH	HH
$A_{2}^{i}\cap A_{1}^{i}$	YY.	RA.	HH H	ΠΩΥΥ	ਬਿੱ ਬਿ	ਸਿੰਬ	HH	물물	'ਜ ਜ	물물	$B_1^i \cap B_2^i$	ਸਿੰਸ	물문	HH.	YY.	R.	¥Υ	물물	'ਸ ਦ	물물	답답
	YY.	G G	188	Πa	ਜਿ ਦ	답답	답답	답답	ਿੱਖ	G G		답답	답답	요요		88	BB	금급	'ਚ 'ਚ	답답	답답
	ΠΠ	ΠT	ĤĤ	Ϋ́́	ਜਿ ਸ	ਦਿੰਦੇ	ΉН	Ϋ́Υ	ਜਿੰਦੀ	ਰ ਹ		ਜਿੱਥੇ	मिम	ĤĤ	Ϋ́́	60	D D	ĤĤ	ਬਿੱਬ	H H	답답
$A_2^{i} \cap A_2^{i}$	Ϋ́Ξ	2 2	1 u u	1 H H	물물	요요	물문	물물	'nн	E E	$B_1^t \cap B_2^b$	ਸਿੰਸ	물문	Η̈́́́́	HH H	88	H H	물물	ਜ ਸ	물문	물물
	aa.	RA.	la a	Ϋ́́Η	임엽	민요요	ЪЪ	엄엽	ਜਿ ਜਿ	[중요]		답답	단단	БЪ	YY.	E E	HH.	[요요]	ਜਿ ਜਿ	업업	담답
	HH	HH	ਸਿੰਸ	ਸੁਸ਼	Ϋ́́	ЧH	HH	ਜਿਸ	ΫĦ	HH		ਸ਼ਿਸ਼	ਸਿੰਦ	ΠΠ	пп	Ϋ́́	Η̈́́́́́	ĦН	ਸਿੱਸ	법법	ਬਿਬ
B_1	답답	답답	답답	요요	199	Ϋ́	199	ਿੱਖ	답답	답답	$B_2^i \cap B_1^b$	답답	답답	임임	YY.	198	ΠG	엽엽	'ਜੋ ਦੇ	답답	단단
	HH	말꾼	HH	요요	199	'HH	HHH	E T	ΗЩ	물물		ਸਿੰਸ	HI H	Η̈́́́́	E E	HH H	BB	물급	ਰ ਦੇ	HH	물물
	ੱਖ ਖ	쉽쉽	물문	ਜਿੰਦੇ ਦੇ	Ϋ́Ϋ́Υ	YY	ਰਿਰ	ਉਉ	업업	영업		'ਜ ਜ	ਦਿੰਦੇ	ΠD	ΠΠ	88	ŶŶ	'ਜ ਜ	ਜਿੱਚ	꿈꿈	চিচি
B_2	유요	답답	199	188	188	88	답답	l C C	Ϋ́Υ	담읍	$B_2^i \cap B_2^b$	କ୍ଳକ	188	88	188	말음	88	요엽	କଳ	88	음음
	유요	말요	답답	요요	188	199	H T	199	ΗЩ	답답		요요	l 🛛 🖓	E E	88	E E	199	요요	ਦਿੰਦੇ	답답	ାଟଟ
	Ϋ́́Υ	유운	ΗH	HH	fr	ਸਿੱਸ	Y Y	ਜਿੰਜ	ЯH	ਜਿੰਦੇ ਜ		T T	유율	ĤĤ	fff	ਜਿਜ	ΠП	99	ਸਿੱਸ	유유	ਜਿੱਜ
C	188	ାଟ୍ରକ୍ର	188	'କ୍ରକ୍ର	188	:ଟଟ	188	ାଟଟ	ାଳକ	।ଟଟ	c	ାଳକ	ାଟ୍ରକ୍ର	66	188	199	'କ୍ରକ୍ର	유요	'ଟଟ'	음읍	ାଟ୍ଟଟ
	음음	요요	음음	요요	188	음음	188	ାଟଟ	요요	답답		음음	요요	음음	[요요	요요	요요	유요	요요	음음	답답
	च च	ਜਿੰਜ	ЪЪ	T T	ਜਿੰਦੇ	ਜਿੰਦੀ	ΗH	Ϋ́Υ	ਜਿ ਸ	ਰਿੰਡ		Ϋ́́Υ	Ϋ́́Υ	ਸਿੱਧ	ਸ਼ਿਸ਼	ਚਿੱਚੋ	66	HH	TT T	'ਜ ਜ	ਤਿੰਦੇ
L		음음	음음	188	188	\square	166	188	। କ୍ରକ୍ର	[요요]	L	음음	188	$\mathbf{G}\mathbf{G}$	ାଟ୍ଟିକ	<u>ାଟ୍</u> ଟକ	$\Theta \Theta$	ାଟ୍ଟକ୍ରା	88	요요	188
	ାଳକ	188	188	188	କୁକ	। କଳ	ାକ୍ଳକ୍ର	188	ାଳକ	ାଳକା		요요	188	' ਜੇ ਜਿ	ାଳକ	ାଟ୍ରିକ	$\Theta \Theta$	[윤윤]	88	ାଟ୍ରକ୍ର	ାଟ୍ରକ୍ର
R	유유	 <u>-</u>	\ominus	99	88	88	99	88	99	유유		-	00	$\widehat{\mathbf{H}}\widehat{\mathbf{H}}$	RR	88	88	우유	777	99	66
	ရှင္ရ	ရှင်္ခ	ရှုမှု	ရြရှ	ାଳକ	00	ାକୁକୁ	ျငင္ရ		ାଳକା	R	ାକକ	ရြှငူ	<u> </u>	IQQ.	ရူဇူဇူ		ရှင်ရှ	90	99	ାକଳ
	I Q Q.	ရင္ရင္ရ	1 2 2	ရြရှ	100	00	199	စုဓ		ାକ୍ଳକ୍ର		ရြင္ရ	ရြှငူ	<u> </u>		ାକଳ		ုမှုမှု	କୁକୁ		ାକକ
X	음음	QQ	ਿੰਦੇ	중중	남음음	음음	惊유	승승	TT	- 2 2		- 2 2	199	유유	1 9 9	중중	ଟ୍ବଟ୍ର	H	- 22	66	199
	IQQ.	ရခူခူ	199	ုခူခူ	କଳ	ାଳକ	199	ାନନ	ାନକ		x	IQQ.	ାକୁକୁ	ନନ	କ୍ଳକ୍ଲ	ାକକ	କଳ	유유	କଳ	ାକକ	IQQ.
	IQQ.	ရခုခု	199	ုခုခု	କ୍ରକ	ျမှုမှု	IQQ.	ရခုခူ	କ୍ରକ୍ର			IQQ.	199	i A A	ရရှ	ାକ୍ଳକ	ရရှ	유유	QQ.	ရြရှ	
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Proposition

The module associated with a MOY-graph with a symmetry axis is a Frobenius algebra.

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Proposition

The module associated with a MOY-graph with a symmetry axis is a Frobenius algebra.

Proposition (R.–Wagner, '17)

The Frobenius algebra associated with



is isomorphic to the T-equivariant cohomology ring of

$$\mathfrak{Flag}(\mathbb{C}^{a_1} \subset \mathbb{C}^{a_1+a_2} \subset \cdots \subset \mathbb{C}^{a_1+\cdots+a_{k-1}} \subset \mathbb{C}^N).$$





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Corollary (R.–Wagner, '17)

The Littlewood–Richardson coefficients are given by:

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