Evaluation of $\mathfrak{sl}_N$-foams

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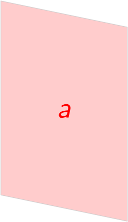
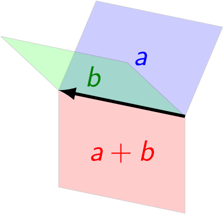
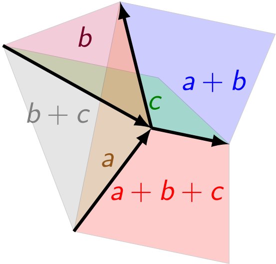
UHH Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

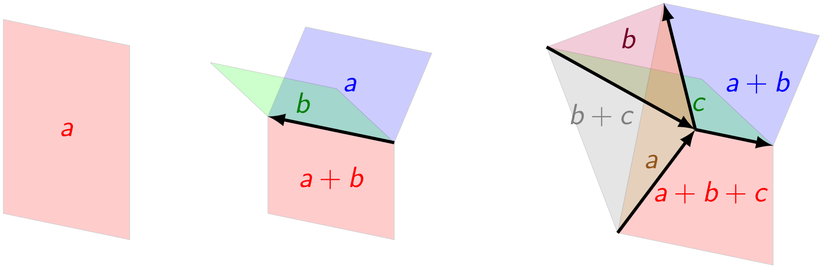
uB UNIVERSITÉ DE BOURGOGNE

Workshop on Quantum Topology – Lille


a


a
b
$a + b$


b
$a + b$
c
$b + c$
a
$a + b + c$



Definition (R.–Wagner, '17)

$$\langle F\rangle_N = \sum_c \frac{(-1)^{\sum_{1\le i<j\le N}\theta^+_{ij}(F,c)}\prod_f P_f(c(f))}{(-1)^{\sum_{i=1}^N i\chi(F_i(c))/2}\prod_{1\le i<j\le N}(X_i-X_j)^{\frac{\chi(F_{ij}(c))}{2}}}$$

Definition (Kauffman Bracket, Jones polynomial)

$$\langle \emptyset \rangle_{\mathrm{K}} = 1 \qquad \left\langle \bigcirc \sqcup L \right\rangle_{\mathrm{K}} = [2]_q \langle L \rangle$$

$$\left\langle \times \right\rangle_{\mathrm{K}} = \left\langle \asymp \right\rangle_{\mathrm{K}} - q \left\langle)\;(\right\rangle_{\mathrm{K}}$$

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle_{\mathrm{K}}$$

Definition (Kauffman Bracket, Jones polynomial)

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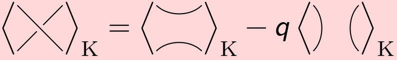
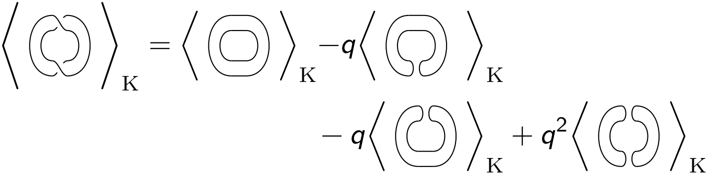


$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle_{\mathrm{K}}$$



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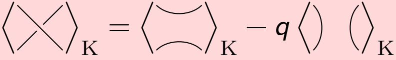
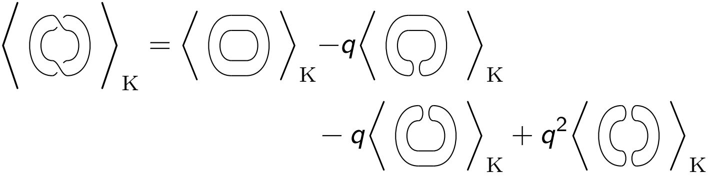
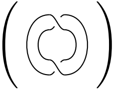
$$\langle \emptyset \rangle_{\mathrm{K}} = 1 \qquad \left\langle \bigcirc \sqcup L \right\rangle_{\mathrm{K}} = [2]_q \langle L \rangle$$

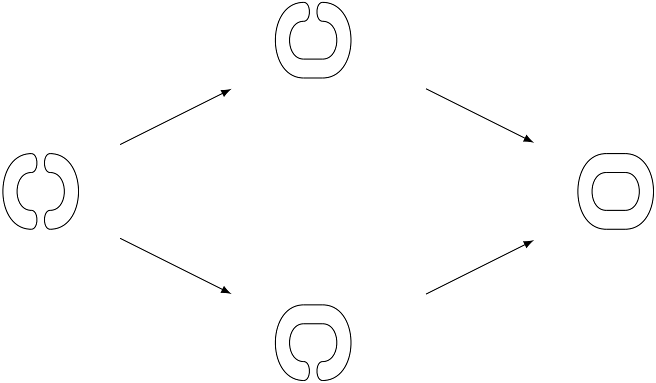


$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle_{\mathrm{K}}$$

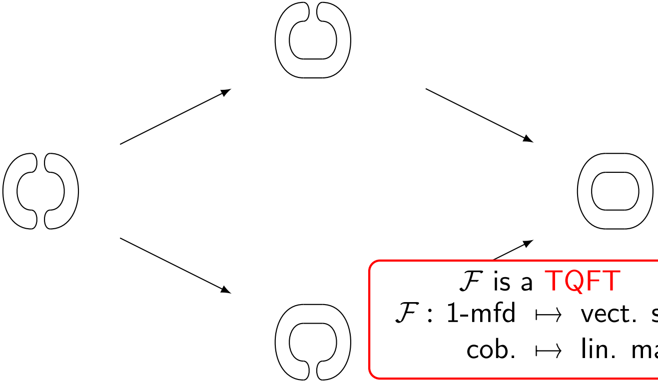


J  $= q^6 + q^4 + q^2 + 1$

Khovanov homology



Khovanov homology

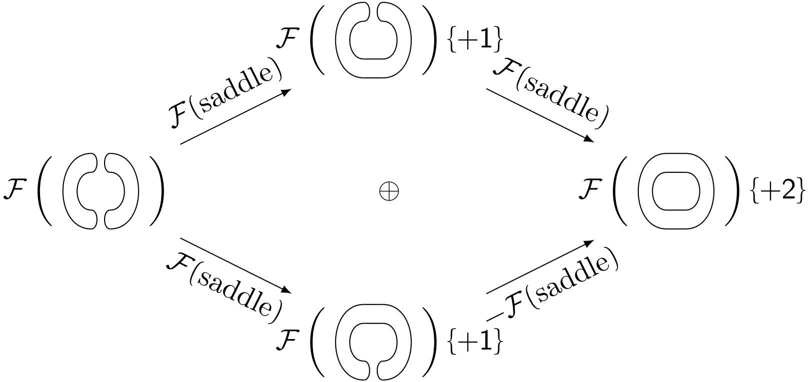


$\mathcal{F}$ is a TQFT

$\mathcal{F}$: 1-mfd $\mapsto$ vect. sp.

cob. $\mapsto$ lin. map

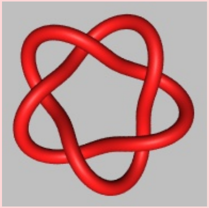
Khovanov homology



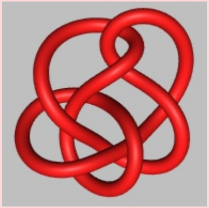
Shift the homological degree by $-n_-$, the q-degree by $n_+ - 2n_-$.
Take the homology.

Proposition (Bar-Natan, '02)

Khovanov homology is strictly stronger than the Jones polynomial.



5_1



10_{132}

(*source* `www.colab.sfu.ca/KnotPlot/KnotServer/`)

Proposition (Bar-Natan, '02)

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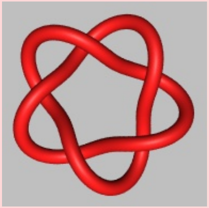
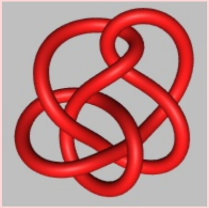


5_1



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(source `www.colab.sfu.ca/KnotPlot/KnotServer/`*)*

Theorem (Kronheimer–Mrowka, '10)

Khovanov homology detects the unknot.

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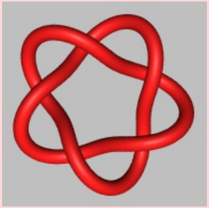

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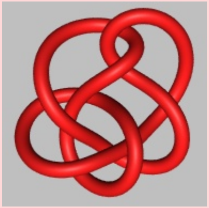

10_{132}

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Theorem (Kronheimer–Mrowka, '10)

Khovanov homology detects the unknot.

Milnor conjecture (Kronheimer–Mrowka, '93, Rasmussen '04)

The slice genus of the (p, q)-torus knot is equal to $\frac{(p-1)(q-1)}{2}$.

$$\begin{tikzcd}
\text{Links} \arrow[r, "\text{Khovanov homology}"] \arrow[dr, swap, "\text{Jones polynomial } (\mathfrak{sl}_2\text{-invariant})"] & \text{Seq. of graded } \mathbb{Z}\text{-mod} \arrow[d, "\chi_q"] \arrow[d, phantom, shift right=8ex, "\circlearrowright"] \\
 & \text{Laurent pol.}
\end{tikzcd}$$

- A recipe to deal with crossings
- An ad-hoc TQFT

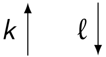
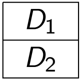
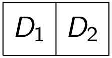
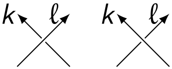
$$\begin{tikzcd}
\text{Links} \arrow[r, "\mathfrak{sl}_N\text{-homology}"] \arrow[dr, "\mathfrak{sl}_N\text{-invariant}"'] \arrow[dr, phantom, shift left=6, "\circlearrowright"] & \text{Seq. of graded } \mathbb{Z}\text{-mod} \arrow[d, "\chi_q"] \\
 & \text{Laurent pol.}
\end{tikzcd}$$

- A recipe to deal with crossings ⇝ Rickard complexes
- An ad-hoc TQFT ⇝ evaluation of foams

The $\mathfrak{sl}_N$-link invariant

Proposition (Drinfel'd)

One can deform $U(\mathfrak{sl}_N)$ into $H := U_q(\mathfrak{sl}_N)$ such that it becomes a quasi-triangular Hopf $\mathbb{C}(q)$-algebra with non-trivial braiding.

	$\mathrm{id}_{\wedge_q^k V}$, $\mathrm{id}_{(\wedge_q^\ell V)^*}$
	$f_1 \circ f_2$
	$f_1 \otimes f_2$
	braiding

The $\mathfrak{sl}_N$-link invariant

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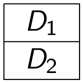
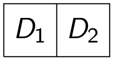
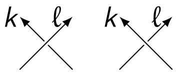
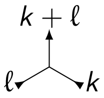
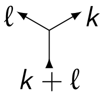
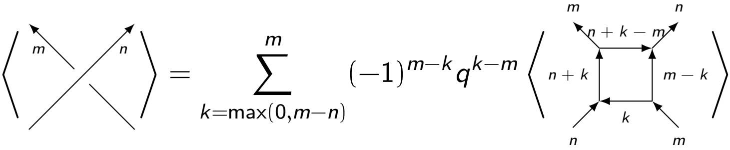
	$\mathrm{id}_{\bigwedge_q^k V}$, $\mathrm{id}_{(\bigwedge_q^\ell V)^*}$
	$f_1 \circ f_2$
	$f_1 \otimes f_2$
	braiding

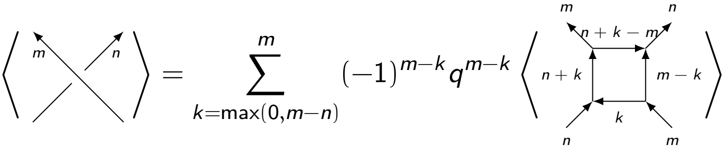
	evaluation
	coevaluation
	$\bigwedge_q^k V \otimes \bigwedge_q^\ell V \longrightarrow \bigwedge_q^{k+\ell} V$
	$\bigwedge_q^{k+\ell} V \longrightarrow \bigwedge_q^k V \otimes \bigwedge_q^\ell V$

MOY calculus (Murakami–Ohtsuki–Yamada)

Lusztig ('94):



$$= \sum_{k=\max(0,m-n)}^{m} (-1)^{m-k} q^{k-m}$$



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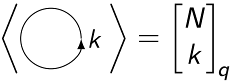
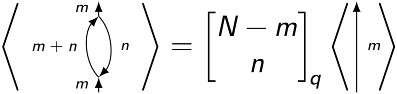
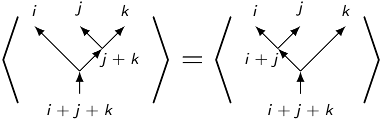
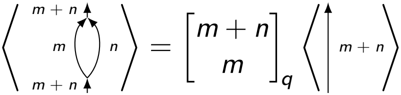


$$\begin{bmatrix}N\\k\end{bmatrix}_q$$

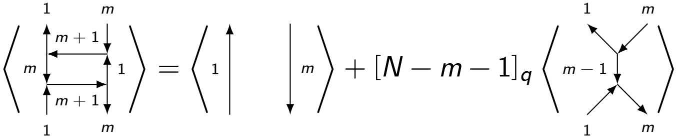


$$\begin{bmatrix}N-m\\n\end{bmatrix}_q$$

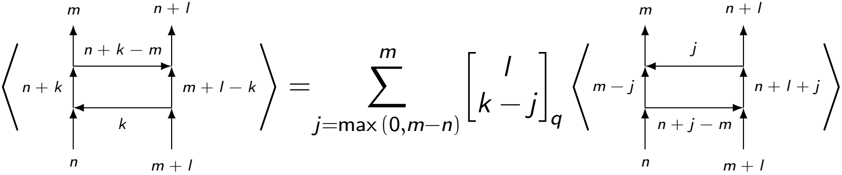




$$\begin{bmatrix}m+n\\m\end{bmatrix}_q$$



$$[N-m-1]_q$$



$$\sum_{j=\max(0,m-n)}^{m}\begin{bmatrix}l\\k-j\end{bmatrix}_q$$

$$\begin{array}{lrrcl} & \mathcal{F}: & \mathsf{Foam}_N & \longrightarrow & \mathbb{Z}[X_1,\ldots,X_N]-\mathsf{mod}_{\mathrm{gr}} \\ \text{Wish:} & & \text{MOY-graph} & \longmapsto & \text{graded module} \\ & & \text{foam} & \longmapsto & \text{graded module map} \end{array}$$

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Universal Construction

An evaluation $\rightsquigarrow$ (Maybe) a TQFT

Wish:

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Universal Construction

An evaluation $\rightsquigarrow$ (Maybe) a TQFT

Theorem (R.–Wagner, '17)

The evaluation defined on the first slide together with the Universal Construction yields an ad-hoc TQFT.

Universal Construction

(Blanchet, Habbeger, Masbaum, Vogel)

Given: $\{\text{closed cobordisms}\} \longrightarrow R$

$$\Gamma \quad \longmapsto \quad \mathcal{F}(\Gamma) := \bigoplus_{{}_{\emptyset}F_{\Gamma}} R_F$$

$$ {}_{\Gamma_1}G_{\Gamma_2} \quad \longmapsto \quad \mathcal{F}(G) \colon \left(\begin{array}{ccc} \mathcal{F}(\Gamma_1) & \rightarrow & \mathcal{F}(\Gamma_2) \\ {}_{\emptyset}F_{\Gamma_1} & \mapsto & {}_{\emptyset}FG_{\Gamma_2} \end{array} \right)$$

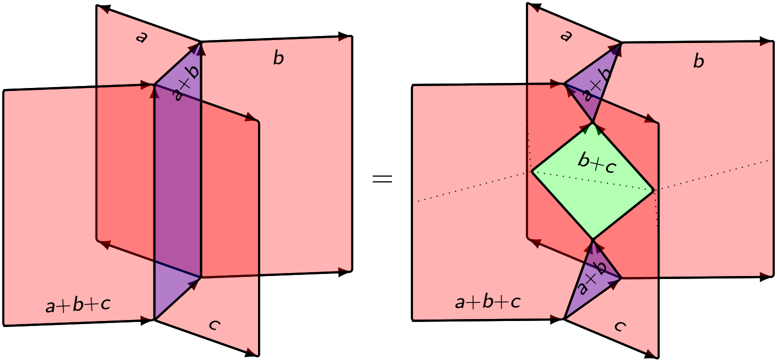
Universal Construction

(Blanchet, Habbeger, Masbaum, Vogel)

Given: $\{\text{closed cobordisms}\} \longrightarrow R$

$$\Gamma \quad \longmapsto \quad \mathcal{F}(\Gamma) := \bigoplus_{{}_\emptyset F_\Gamma} R_F \Bigg/ \begin{matrix} \sum_i \lambda_i F_i = 0 \text{ if} \\ \sum_i \lambda_i \tau(F_i G) = 0 \text{ for all } {}_\Gamma G_\emptyset \end{matrix}$$

$$[{}_{\Gamma_1}G_{\Gamma_2}] \quad \longmapsto \quad \mathcal{F}(G)\colon \begin{pmatrix} \mathcal{F}(\Gamma_1) & \to & \mathcal{F}(\Gamma_2) \\ [{}_\emptyset F_{\Gamma_1}] & \mapsto & [{}_\emptyset FG_{\Gamma_2}] \end{pmatrix}$$


a
b
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a+b+c
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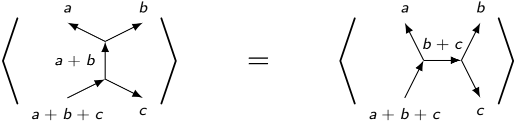
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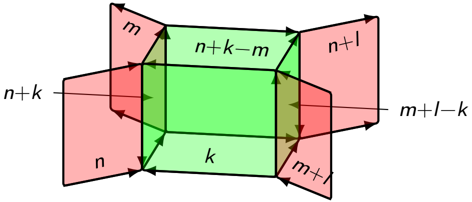
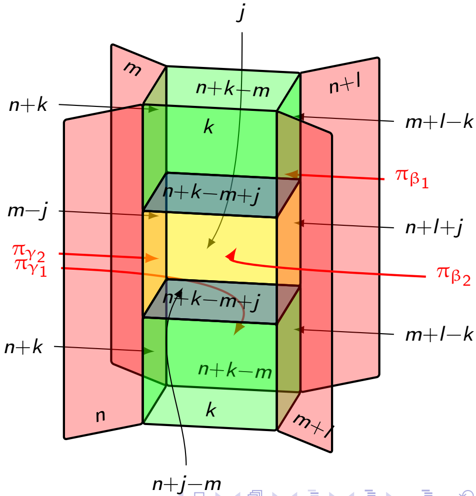
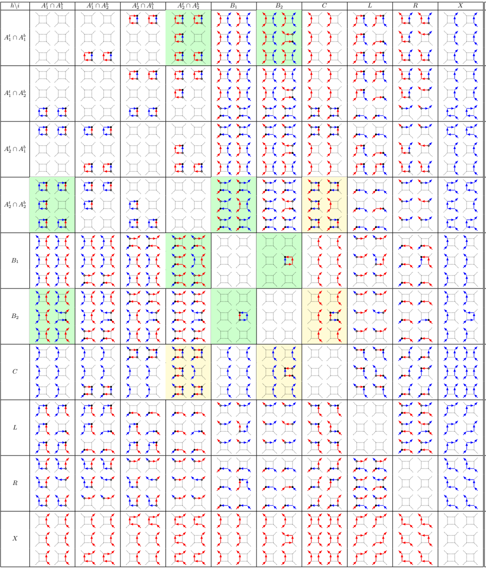
b + c
a + b + c
c



$$= \sum_{j=\max(0,m-n)}^{m} \; \sum_{\alpha \in T(k-j,l-k+j)}$$

$$(-1)^{|\alpha|+(l-k+j)(m-j)} \sum_{\substack{\beta_1,\beta_2\\ \gamma_1,\gamma_2}} c^{\alpha}_{\beta_1\beta_2} c^{\widehat{\alpha}}_{\gamma_1\gamma_2}$$




$h\backslash i$
$A_1^t \cap A_1^b$
$A_1^t \cap A_2^b$
$A_2^t \cap A_1^b$
$A_2^t \cap A_2^b$
B_1
B_2
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$A_1^t \cap A_1^b$
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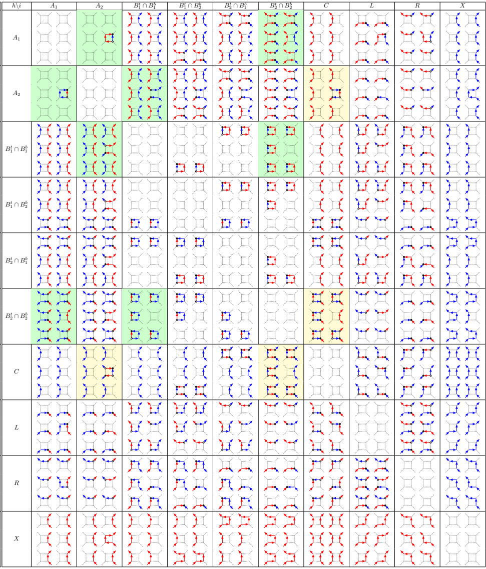
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C
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Proposition

The module associated with a MOY-graph with a symmetry axis is a Frobenius algebra.

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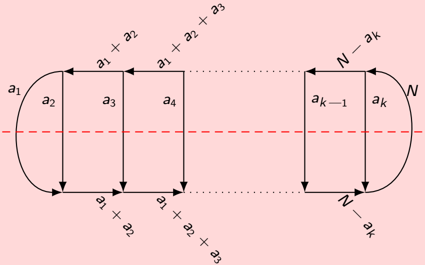
Proposition (R.–Wagner, ’17)

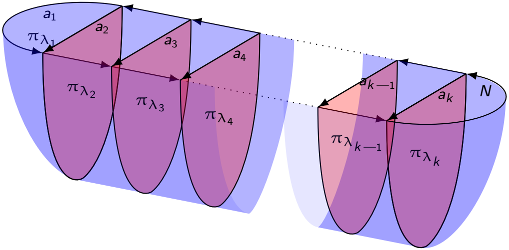
The Frobenius algebra associated with



is isomorphic to the T-equivariant cohomology ring of

$$\mathfrak{Flag}(\mathbb{C}^{a_1} \subset \mathbb{C}^{a_1+a_2} \subset \cdots \subset \mathbb{C}^{a_1+\cdots+a_{k-1}} \subset \mathbb{C}^{N}).$$

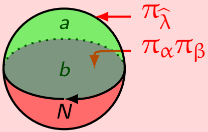
$$\prod_{i=1}^{k} \pi_{\lambda_i}(X_{a_i+1}, \ldots, X_{a_{i+1}}) \mapsto$$



Corollary (R.–Wagner, '17)

The Littlewood–Richardson coefficients are given by:

$$c^{\lambda}_{\alpha\beta} = (-1)^{|\widehat{\lambda}|+N(N+1)/2} \left\langle \qquad \right\rangle_N$$



$$= (-1)^{N(N+1)/2+|\widehat{\lambda}|} \sum_{\substack{A\sqcup B=\{X_1,\dots,X_N\} \\ |A|=a,|B|=b}} (-1)^{|B<A|} \frac{a_\alpha(A)\, a_\beta(A)\, a_{\widehat{\lambda}}(B)}{\Delta(X_1,\dots,X_N)}.$$

Thank you!