

Foams and categorification

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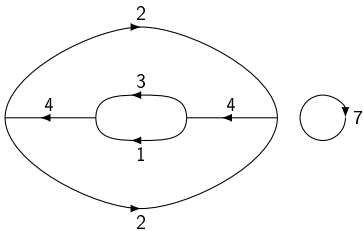
Узлы и теория представлений

Aims

- ▶ Introduce foams (as a category),
- ▶ Explain a formula,
- ▶ Relate to link homologies.

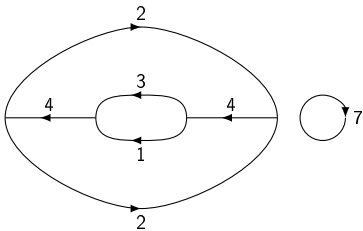
Definition (Murkami–Ohtsuki–Yamada)

A *MOY graph* is a trivalent, oriented, labeled, plane graph which respects the flow.



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$$\mathcal{M}_N = \langle \text{MOY graphs} \rangle_{\mathbb{Z}[q, q^{-1}]} / \text{relations} \quad N \geq 1$$

$$\text{circle with arrow} \xrightarrow{k} = \begin{bmatrix} N \\ k \end{bmatrix}$$

$$\begin{matrix} m \uparrow \\ m+n \uparrow \end{matrix} \begin{matrix} \leftarrow \\ \rightarrow \end{matrix} \begin{matrix} n \\ m \end{matrix} = \begin{bmatrix} N-m \\ n \end{bmatrix} \uparrow^m$$

$$\begin{matrix} i & j & k \\ \swarrow & \searrow & \nearrow \\ & j+k & \\ \uparrow & & \\ i+j+k & & \end{matrix} = \begin{matrix} i & j & k \\ \swarrow & \nearrow & \\ & i+j & \\ \uparrow & & \\ i+j+k & & \end{matrix}$$

$$\begin{matrix} m+n \uparrow \\ m \uparrow \\ m+n \uparrow \end{matrix} \begin{matrix} \leftarrow \\ \rightarrow \end{matrix} \begin{matrix} n \\ m \end{matrix} = \begin{bmatrix} m+n \\ m \end{bmatrix} \uparrow^{m+n}$$

$$\begin{matrix} 1 & m \\ \uparrow & \downarrow \\ m+1 & \\ \leftarrow & \rightarrow \\ m & 1 \\ \downarrow & \uparrow \\ m+1 & \\ \uparrow & \downarrow \\ 1 & m \end{matrix} = \uparrow^1 \quad \downarrow^m + [N-m-1] \quad \begin{matrix} 1 & m \\ \swarrow & \nearrow \\ & m-1 \\ \nearrow & \swarrow \\ 1 & m \end{matrix}$$

$$\begin{matrix} m & n+l \\ \uparrow & \uparrow \\ n+k & \leftarrow n+k-m \\ \leftarrow & \rightarrow \\ k & \\ \uparrow & \uparrow \\ n & m+l \end{matrix} = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \begin{matrix} m & n+l \\ \uparrow & \uparrow \\ m-j & \leftarrow j \\ \leftarrow & \rightarrow \\ n+j-m & \\ \uparrow & \uparrow \\ n & m+l \end{matrix}$$

Theorem (Murakami–Ohtuski–Yamada, Kauffman–Vogel, Wu)

The module \mathcal{M}_N has rank 1 and is generated by the empty graph. $\Gamma = \langle \Gamma \rangle_N \emptyset$. The polynomial $\langle \Gamma \rangle_N$ is in $\mathbb{N}[q, q^{-1}]$.

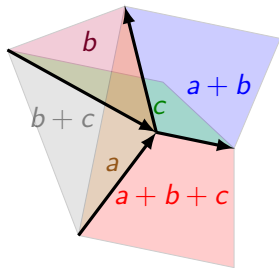
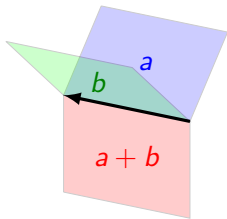
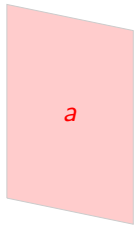
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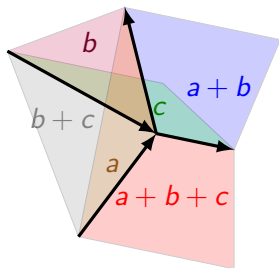
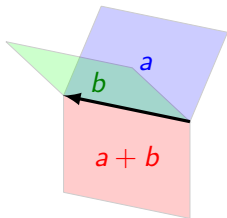
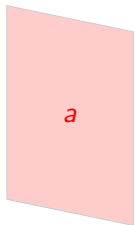
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$$\begin{aligned}
 \left\langle \begin{array}{c} \nearrow^m \searrow^n \\ \searrow^m \nearrow^n \end{array} \right\rangle &= \sum_{k=\max(0, m-n)}^m (-1)^{m-k} q^{k-m} \left\langle \begin{array}{c} \nearrow^{n+k-m} \searrow^{m-k} \\ \searrow^k \nearrow^k \end{array} \right\rangle \\
 \left\langle \begin{array}{c} \nearrow^m \searrow^n \\ \nearrow^n \searrow^m \end{array} \right\rangle &= \sum_{k=\max(0, m-n)}^m (-1)^{m-k} q^{m-k} \left\langle \begin{array}{c} \nearrow^{n+k-m} \searrow^{m-k} \\ \searrow^k \nearrow^k \end{array} \right\rangle
 \end{aligned}$$

Theorem (Reshetikhin–Turaev, Murakami–Ohtsuki–Yamada)

This produces a polynomial link invariant called the \mathfrak{sl}_N link invariant.



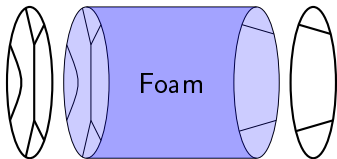


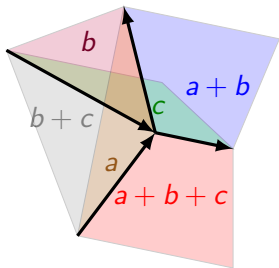
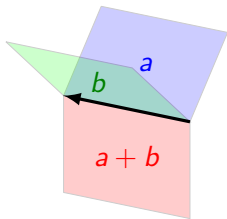
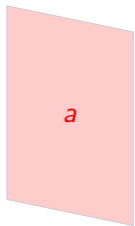
Definition

Category Foam:

Objects: MOY graphs,

Morphisms: foamy cobordisms.





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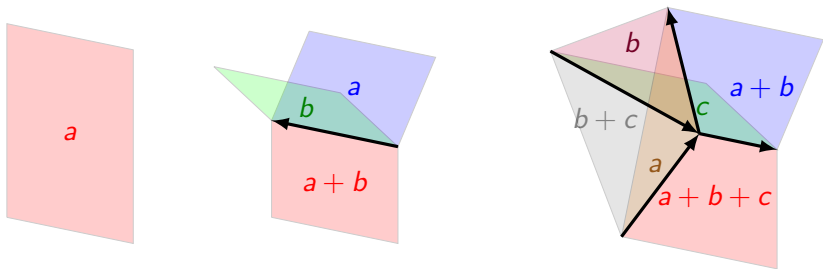
Morphisms: foamy cobordisms.

Wish

TQFT-like functor \mathcal{G}_N .

$\mathcal{G}_N : \text{Foam} \rightarrow R\text{-mod}_{\text{gr}}$

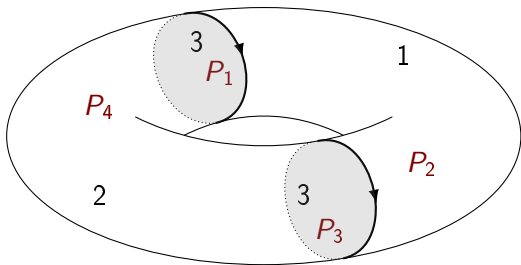
$\text{rk}_q \mathcal{G}_N(\Gamma) = \langle \Gamma \rangle_N \in \mathbb{N}[q, q^{-1}]$



Definition (R.-Wagner, '17)

Let F be a closed foam.

$$\tau_N(F) = \sum_c \frac{(-1)^{\sum_{i=1}^N iX_i(F(c))/2 + \sum_{1 \leq i < j \leq N} \theta_{ij}^+(F(c))} \prod_f P_f(c(f))}{\prod_{1 \leq i < j \leq N} (X_i - X_j)^{\frac{X_{ij}(F(c))}{2}}}$$



$$P_1 = t_1 + t_2 + t_3$$

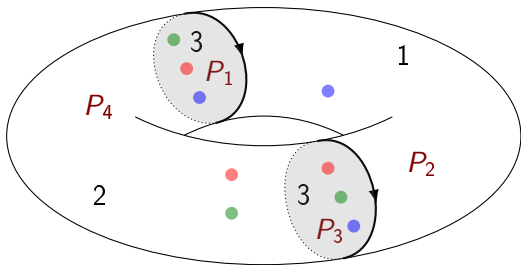
$$P_2 = t_1^2$$

$$P_3 = 1$$

$$P_4 = t_1 t_2 (t_1 + t_2)$$

$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



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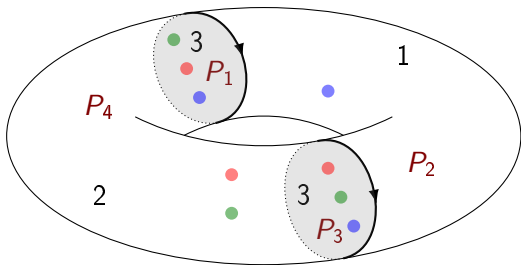
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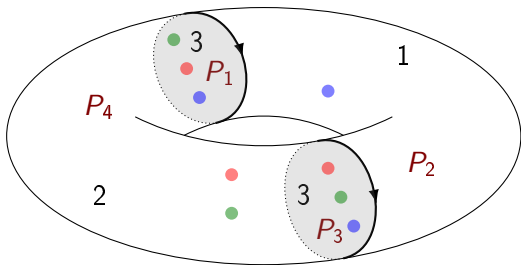
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\mathbb{P}	Monochrome	χ_\bullet
X_1		2
X_2		2
X_3		0
X_4		2

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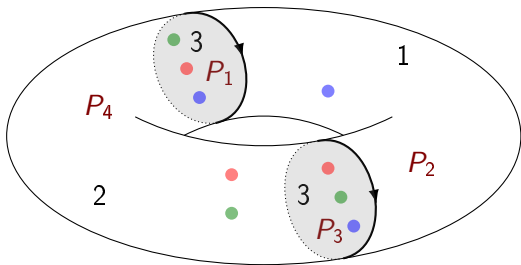
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X_4		2

	$X_1 X_2$	$X_1 X_3$	$X_1 X_4$	$X_2 X_3$	$X_2 X_4$	$X_3 X_4$
$\chi_{\bullet\bullet}$	0	2	0	2	0	2
$\theta_{\bullet\bullet}^+$	2	0	2	0	0	0

$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



$$P_1 = t_1 + t_2 + t_3$$

$$P_2 = t_1^2$$

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$$P_4 = t_1 t_2 (t_1 + t_2)$$

$$\frac{(-1)^{1+2+4} (X_1 + X_2 + X_4) X_1^2 (X_2 X_4 (X_2 + X_4))}{(X_1 - X_3)(X_2 - X_3)(X_3 - X_4)}$$

\mathbb{P}	Monochrome	χ_\bullet
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X_2		2
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	$X_1 X_2$	$X_1 X_3$	$X_1 X_4$	$X_2 X_3$	$X_2 X_4$	$X_3 X_4$
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Theorem (R.-Wagner (2017))

For any closed foam $\tau_N(F) \in R = \mathbb{Z}[X_1, \dots, X_N]^{S_N}$.

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Universal Construction

An evaluation of closed cobordisms \rightsquigarrow (Maybe) a TQFT

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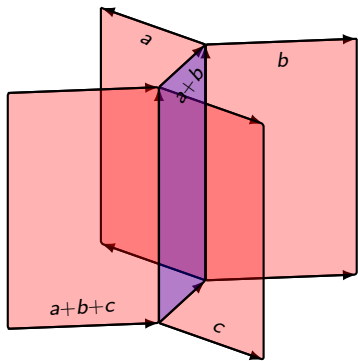
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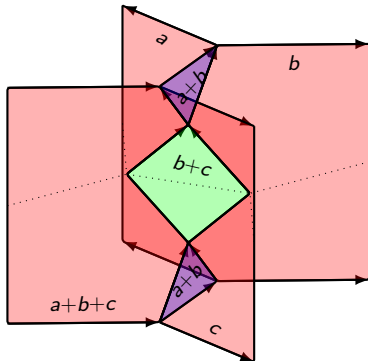
An evaluation of closed cobordisms \rightsquigarrow (Maybe) a TQFT

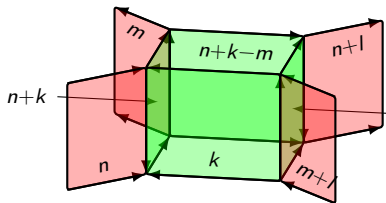
Theorem (R.–Wagner, '17)

The evaluation τ_N together with the **Universal Construction** yields an *ad-hoc* TQFT \mathcal{G}_N .



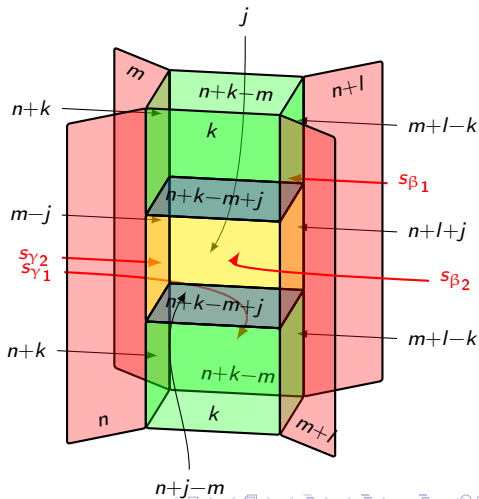
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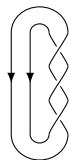
$$m+l-k = \sum_{j=\max(0, m-n)}^m \sum_{\alpha \in T(k-j, l-k+j)}$$

$$(-1)^{|\alpha|+(l-k+j)(m-j)} \sum_{\substack{\beta_1, \beta_2 \\ \gamma_1, \gamma_2}} c_{\beta_1 \beta_2}^{\alpha} c_{\gamma_1 \gamma_2}^{\hat{\alpha}}$$

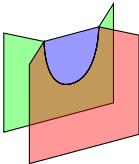
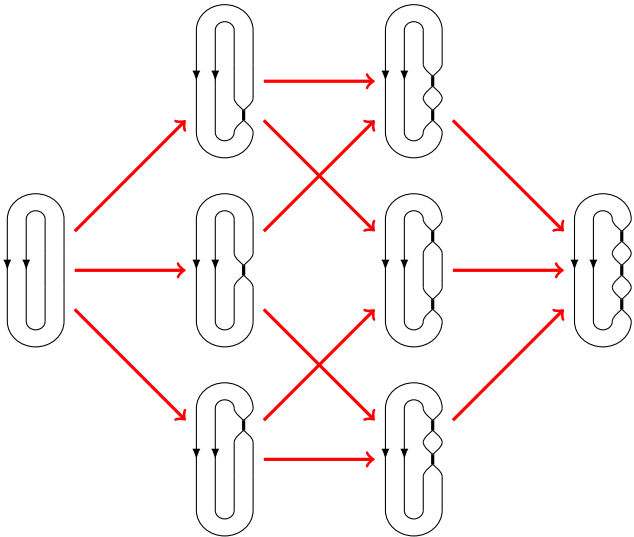
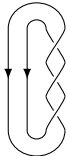


A_1	A_2	A_3	A_4	B_1	B_2	C	L	R	X	A_1	A_2	A_3	A_4	C	L	R	X	
										A_1								
										A_2								
										A_3								
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Chain complex



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Theorem (R.-Wagner (2017))

*This construction yields an homological link invariant: **the colored equivariant \mathfrak{sl}_N homology**.*
It categorifies the \mathfrak{sl}_N link polynomial.

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Remark

The colors refers to exterior powers of the standard representation of $U_q(\mathfrak{sl}_N)$.

$$\langle \langle \text{circle with arrow } k \rangle \rangle = \begin{bmatrix} N \\ k \end{bmatrix}$$

$$\langle \langle \text{loop with } m+n \text{ and } m \text{ labels} \rangle \rangle = \begin{bmatrix} N-m \\ n \end{bmatrix} \langle \langle \text{vertical line } m \rangle \rangle$$

$$\langle \langle \text{Y-junction with } i, j, k \text{ and } i+j+k \rangle \rangle = \langle \langle \text{Y-junction with } i, j, k \text{ and } i+j+k \rangle \rangle$$

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$$\langle \langle \text{loop with } m+1 \text{ and } m \text{ labels} \rangle \rangle = \langle \langle \text{vertical line } 1 \rangle \rangle + [N-m-1] \langle \langle \text{Y-junction with } m-1 \text{ labels} \rangle \rangle$$

$$\langle \langle \text{rectangle with } m, n+l, n+k, m+l-k \text{ labels} \rangle \rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \langle \langle \text{rectangle with } m-j, n+l+j \text{ labels} \rangle \rangle$$

From Λ^\bullet to Sym^\bullet : N goes to $-N$.

$$\left\langle \begin{array}{c} \circlearrowright k \\ \hline \end{array} \right\rangle = \begin{bmatrix} N \\ k \end{bmatrix}$$

$$\left\langle \begin{array}{c} m \uparrow \\ \circlearrowright n \\ m \downarrow \end{array} \right\rangle = \begin{bmatrix} N - m \\ n \end{bmatrix} \left\langle \begin{array}{c} \uparrow \\ \hline \end{array} \right\rangle_m$$

$$\left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \quad \nearrow \\ \uparrow \\ i+j+k \end{array} \right\rangle = \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \quad \nearrow \\ i+j \quad \uparrow \\ i+j+k \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} m+n \uparrow \\ \circlearrowright n \\ m+n \downarrow \end{array} \right\rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \left\langle \begin{array}{c} \uparrow \\ \hline \end{array} \right\rangle_{m+n}$$

$$\left\langle \begin{array}{c} 1 \quad m \\ \uparrow \quad \downarrow \\ m \quad \leftarrow \quad \rightarrow \quad 1 \\ \uparrow \quad \downarrow \\ 1 \quad m \end{array} \right\rangle = \left\langle \begin{array}{c} \uparrow \\ \hline \end{array} \right\rangle_1 \left\langle \begin{array}{c} \downarrow \\ \hline \end{array} \right\rangle_m + [N - m - 1] \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \searrow \quad \nearrow \\ m-1 \quad \downarrow \\ 1 \quad m \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ n+k \quad \leftarrow \quad \rightarrow \quad m+l-k \\ \leftarrow \quad \rightarrow \quad k \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ m-j \quad \leftarrow \quad \rightarrow \quad n+l+j \\ \leftarrow \quad \rightarrow \quad n+j-m \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle$$

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$$\left\langle \left\langle \begin{array}{c} \text{circle with arrow } k \end{array} \right\rangle \right\rangle = \begin{bmatrix} N + k - 1 \\ k \end{bmatrix} \quad \left\langle \left\langle \begin{array}{c} m+n \uparrow \\ \text{loop } n \\ m \downarrow \end{array} \right\rangle \right\rangle = \begin{bmatrix} N + m + n - 1 \\ n \end{bmatrix} \left\langle \left\langle \begin{array}{c} \uparrow \\ m \\ \downarrow \end{array} \right\rangle \right\rangle$$

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$$\left\langle \left\langle \begin{array}{c} 1 \quad m \\ \uparrow \quad \downarrow \\ m+1 \quad \uparrow \\ \leftarrow \quad \rightarrow \\ 1 \\ \downarrow \quad \uparrow \\ m+1 \quad \downarrow \\ 1 \quad m \end{array} \right\rangle \right\rangle = \left\langle \left\langle \begin{array}{c} \uparrow \\ 1 \\ \downarrow \end{array} \right\rangle \right\rangle + [N + m + 1] \left\langle \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \searrow \\ m-1 \quad \downarrow \\ \nearrow \quad \swarrow \\ 1 \quad m \end{array} \right\rangle \right\rangle$$

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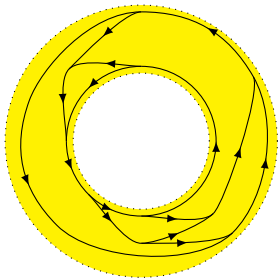
Bad news

It is not possible to categorify the Sym^\bullet -MOY calculus with such a TQFT.

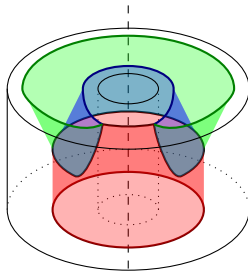
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It is not possible to categorify the Sym^\bullet -MOY calculus with such a TQFT.

We restrict the class of graphs and the class of foams:



Vinyl graphs



Tube-like foams

Theorem (R.-Wagner, 2018)

There exists a foamy restricted TQFT

$$\mathcal{F}_N: \text{TLFoam} \rightarrow \mathbb{Q}[X_1, \dots, X_N]^{S_N\text{-mod}_{\text{gr}}}$$

such that

$$\text{rk}_q \mathcal{F}_N(\Gamma) = \langle \Gamma \rangle.$$

It can be extended to an homological link invariant called symmetric \mathfrak{sl}_N homology.

Thank you !!