Foams and categorification

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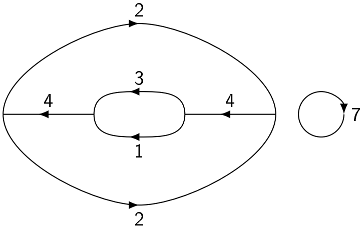
Узлы и теория представлений

Aims

- Introduce foams (as a category),
- Explain a formula,
- Relate to link homologies.

Definition (Murkami–Ohtsuki–Yamada)

A *MOY graph* is a trivalent, oriented, labeled, plane graph which respects the flow.



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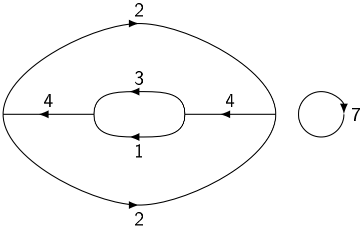
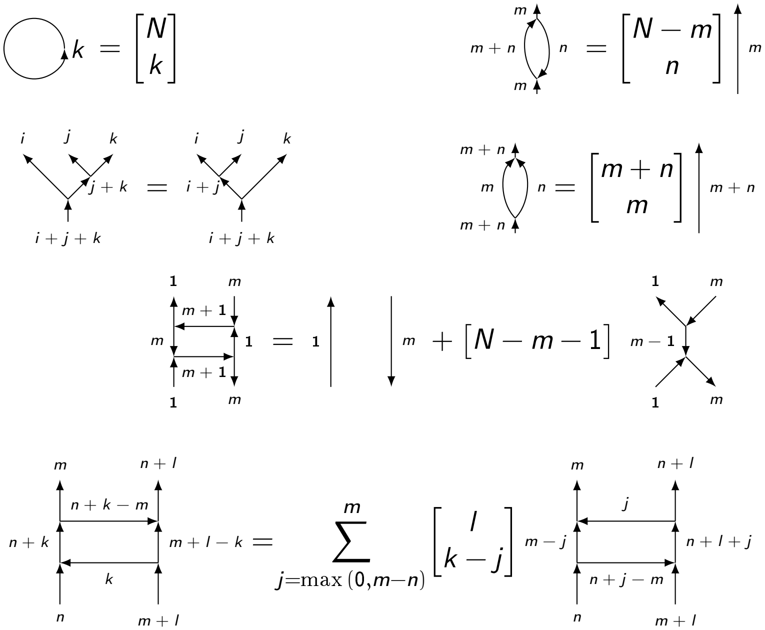


$$\mathcal{M}_N = \langle \text{MOY graphs} \rangle_{\mathbb{Z}[q,q^{-1}]} \Big/ \text{relations} \qquad N \geq 1$$



k = [N k]
m
m + n
n
= [N − m n]
m
i j k
j + k
=
i + j
i + j + k
m + n
m
n = [m + n m]
m + n
1 m
m + 1
m
1
= 1
m
+ [N − m − 1]
m − 1
m
n + l
n + k − m
n + k
m + l − k
=
k
n
m + l
j=max (0,m−n)
[l k − j]
m − j
j
n + l + j
n + j − m

Theorem (Murakami–Othuski–Yamada, Kauffman–Vogel, Wu)

The module $\mathcal{M}_N$ *has rank* 1 *and is generated by the empty graph.* $\Gamma = \langle \Gamma \rangle_N \emptyset$*. The polynomial* $\langle \Gamma \rangle_N$ *is in* $\mathbb{N}[q, q^{-1}]$*.*

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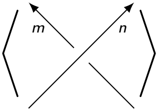
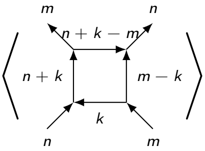
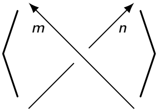


$$= \sum_{k=\max(0,m-n)}^{m} (-1)^{m-k} q^{k-m}$$





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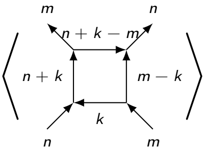
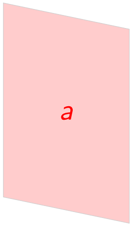
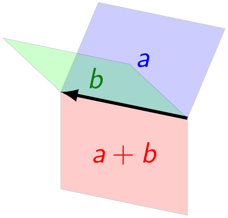
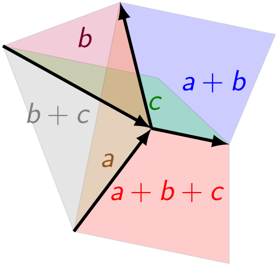


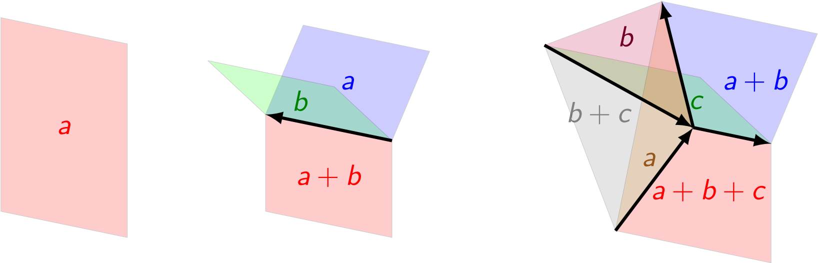
Theorem (Reshetikhin–Turaev, Murakami–Ohtsuki–Yamada)

This produces a polynomial link invariant called the $\mathfrak{sl}_N$ link invariant.


a

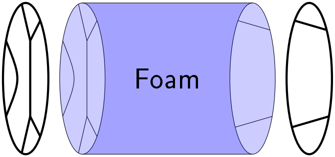

a
b
a + b

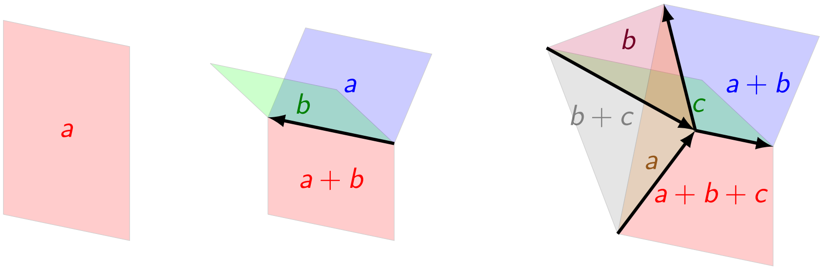

b
a + b
c
b + c
a
a + b + c



Definition

Category Foam:
Objects: MOY graphs,
Morphisms: foamy cobordisms.





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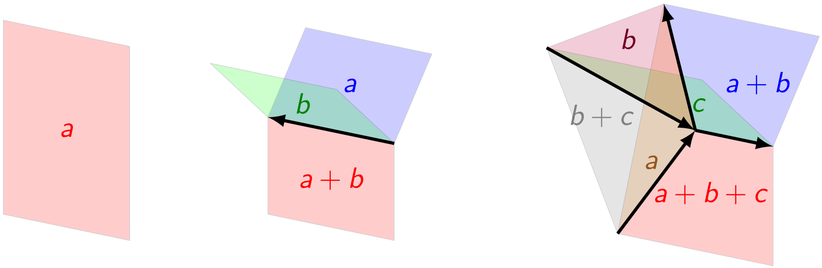
Wish

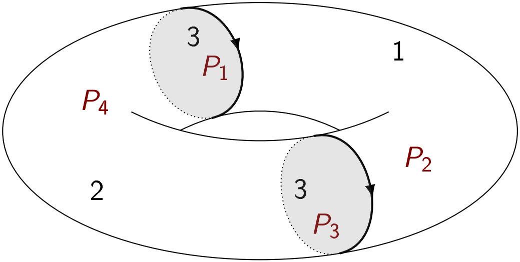
TQFT-like functor $\mathcal{G}_N$.
$\mathcal{G}_N : \text{Foam} \to R\text{-mod}_{\mathrm{gr}}$
$\mathrm{rk}_q \mathcal{G}_N(\Gamma) = \langle \Gamma \rangle_N \in \mathbb{N}[q, q^{-1}]$



Definition (R.–Wagner, ’17)

Let F be a closed foam.

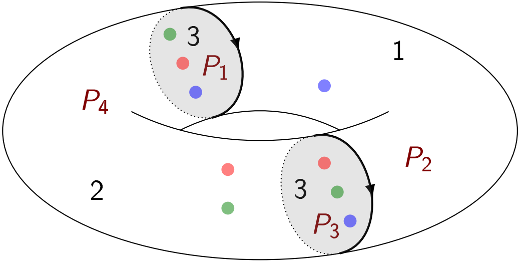
$$\tau_N(F) = \sum_c \frac{(-1)^{\sum_{i=1}^N i\chi_i(F(c))/2 + \sum_{1\le i<j\le N} \theta_{ij}^+(F(c))} \prod_f P_f(c(f))}{\prod_{1\le i<j\le N}(X_i - X_j)^{\frac{\chi_{ij}(F(c))}{2}}}$$



$P_1 = t_1 + t_2 + t_3$ $\qquad P_2 = t_1^2$

$P_3 = 1$ $\qquad P_4 = t_1 t_2 (t_1 + t_2)$

$$N = 4$$
$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



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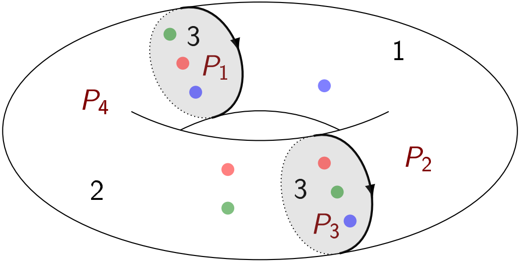

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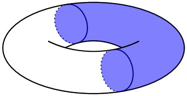
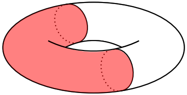
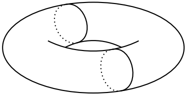
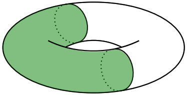
$\mathbb{P}$	Monochrome	$\chi_\bullet$
X_1		2
X_2		2
X_3		0
X_4		2

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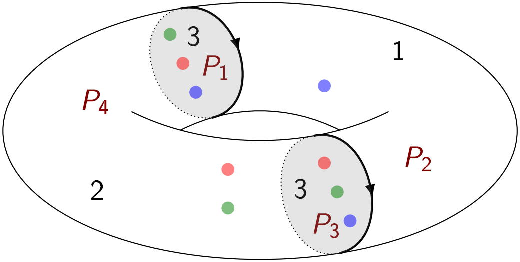

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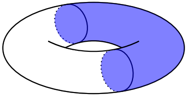
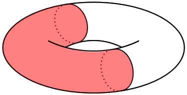
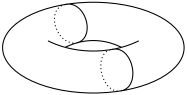
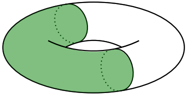
X_1		2
X_2		2
X_3		0
X_4		2

	X_1X_2	X_1X_3	X_1X_4	X_2X_3	X_2X_4	X_3X_4
$\chi_{\bullet\bullet}$	0	2	0	2	0	2
$\theta^+_{\bullet\bullet}$	2	0	2	0	0	0

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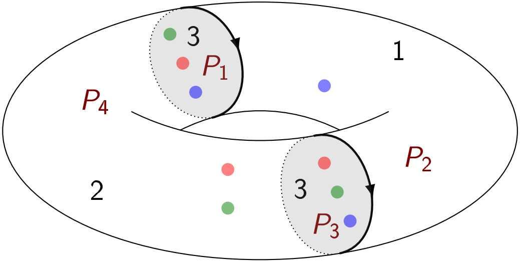

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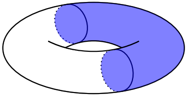
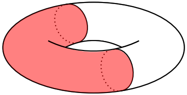
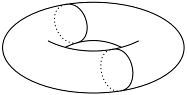
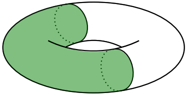
$$\frac{(-1)^{1+2+4}(X_1 + X_2 + X_4)X_1{}^2(X_2X_4(X_2 + X_4))}{(X_1 - X_3)(X_2 - X_3)(X_3 - X_4)}$$

$\mathbb{P}$	Monochrome	$\chi_\bullet$
X_1		2
X_2		2
X_3		0
X_4		2

	X_1X_2	X_1X_3	X_1X_4	X_2X_3	X_2X_4	X_3X_4
$\chi_{\bullet\bullet}$	0	2	0	2	0	2
$\theta^+_{\bullet\bullet}$	2	0	2	0	0	0

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Theorem (R.–Wagner (2017))

For any closed foam $\tau_N(F) \in R = \mathbb{Z}[X_1, \ldots, X_N]^{S_N}$.

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Universal Construction

An evaluation of closed cobordisms $\rightsquigarrow$ (Maybe) a TQFT

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Universal Construction

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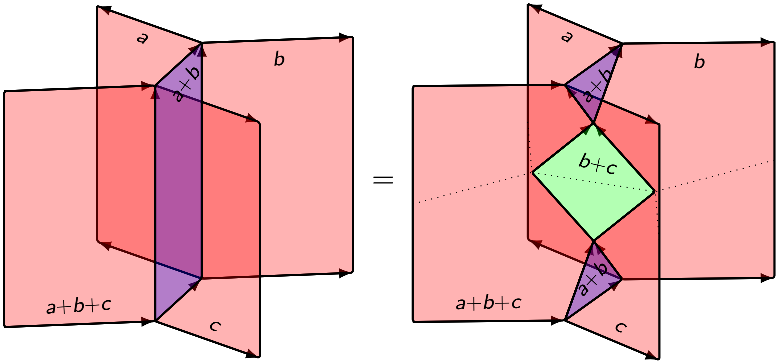
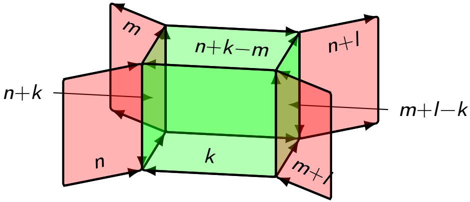
Theorem (R.–Wagner, '17)

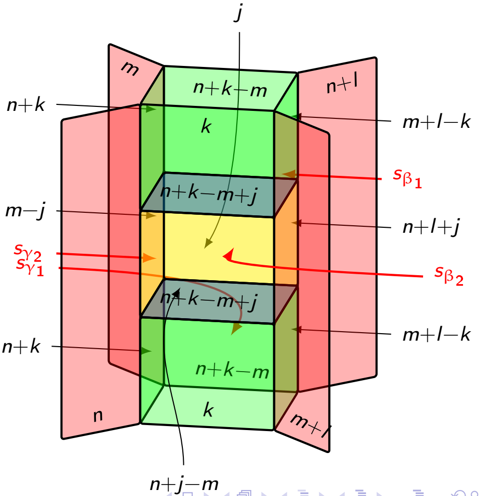
The evaluation τ_N *together with the* Universal Construction *yields an ad-hoc TQFT* $\mathcal{G}_N$.

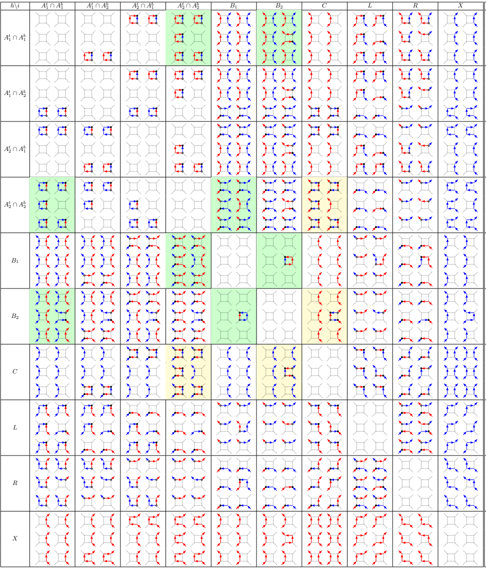

a
b
a+b
a+b+c
c
=
a
b
a+b
b+c
a+b
a+b+c
c



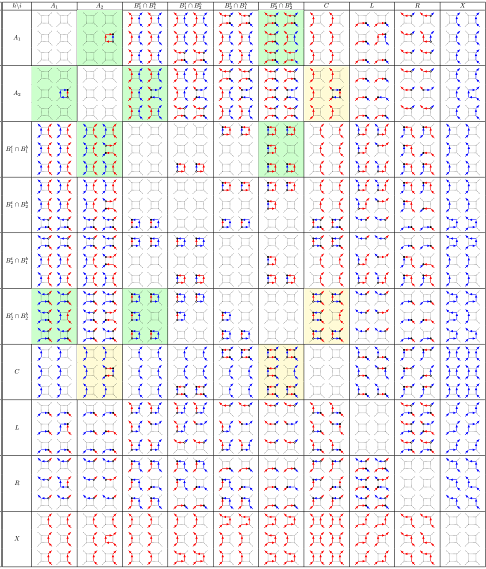
$$= \sum_{j=\max(0,m-n)}^{m} \; \sum_{\alpha\in T(k-j,l-k+j)}$$

$$(-1)^{|\alpha|+(l-k+j)(m-j)} \sum_{\substack{\beta_1,\beta_2\\ \gamma_1,\gamma_2}} c^{\alpha}_{\beta_1\beta_2}\, c^{\widehat{\alpha}}_{\gamma_1\gamma_2}$$





h\i
$A_1^t \cap A_1^b$
$A_1^t \cap A_2^b$
$A_2^t \cap A_1^b$
$A_2^t \cap A_2^b$
B_1
B_2
C
L
R
X



h\i
A_1
A_2
$B_1^t \cap B_1^b$
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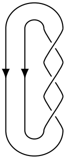
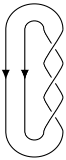
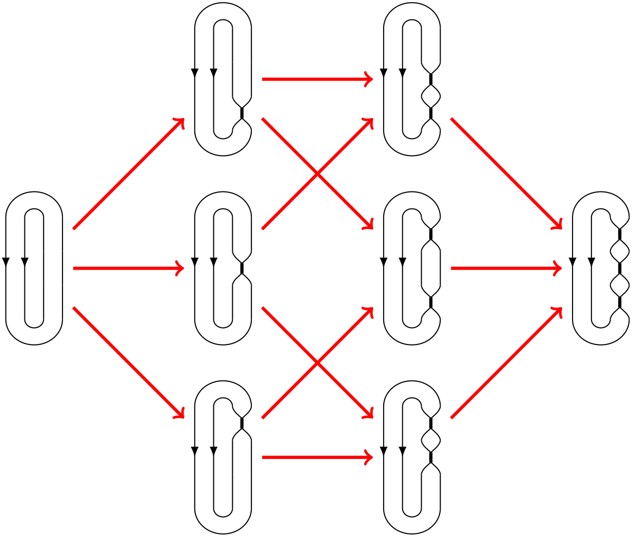
C
L
R
X

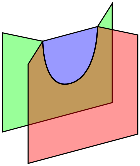
Chain complex



Chain complex







Theorem (R.–Wagner (2017))

This construction yields an homological link invariant: the colored equivariant $\mathfrak{sl}_N$ homology.
It categorifies the $\mathfrak{sl}_N$ link polynomial.

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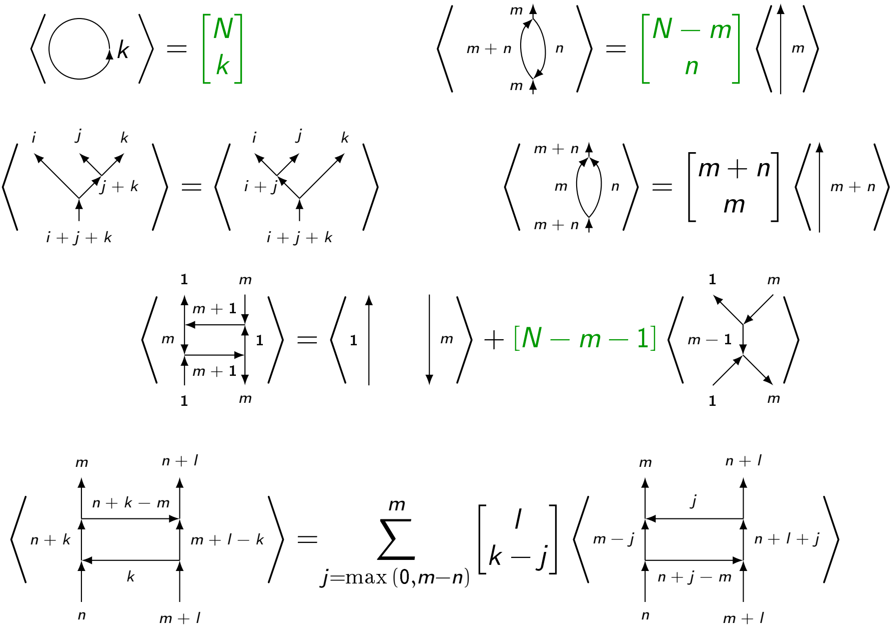
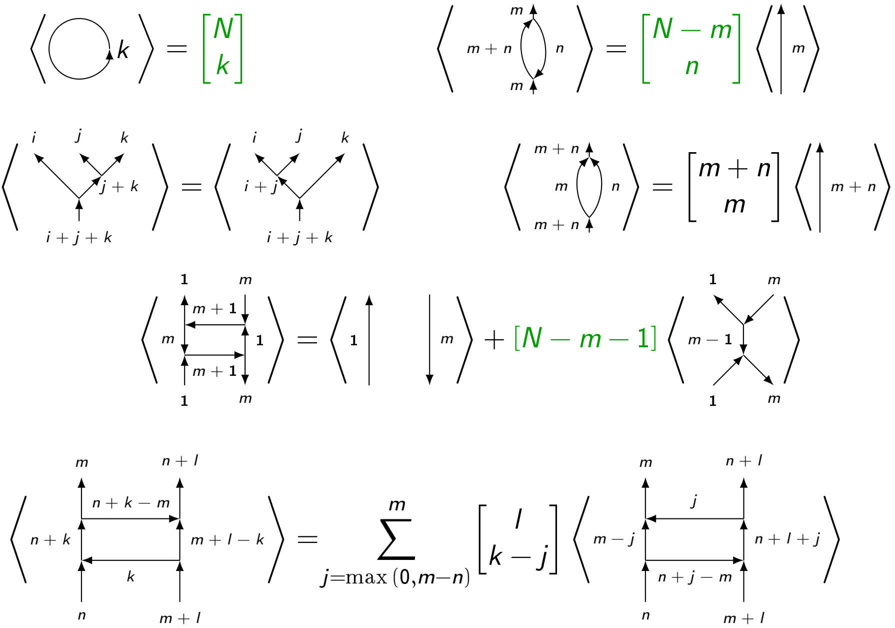
Remark

The colors refers to exterior powers of the standard representation of $U_q(\mathfrak{sl}_N)$.



k
= [N k]
m
m + n
n
m
= [N − m n]
m
i
j
k
j + k
i + j + k
=
i
j
k
i + j
i + j + k
m + n
m
n
m + n
= [m + n m]
m + n
1
m
m + 1
m
1
m + 1
1
m
=
1
m
+ [N − m − 1]
1
m
m − 1
1
m
m
n + l
n + k − m
n + k
m + l − k
k
n
m + l
=
$\sum_{j=\max(0,m-n)}^{m} \begin{bmatrix} l \\ k-j \end{bmatrix}$
m
n + l
j
m − j
n + l + j
n + j − m
n
m + l

From $\Lambda^\bullet$ to $\mathrm{Sym}^\bullet$: N goes to $-N$.



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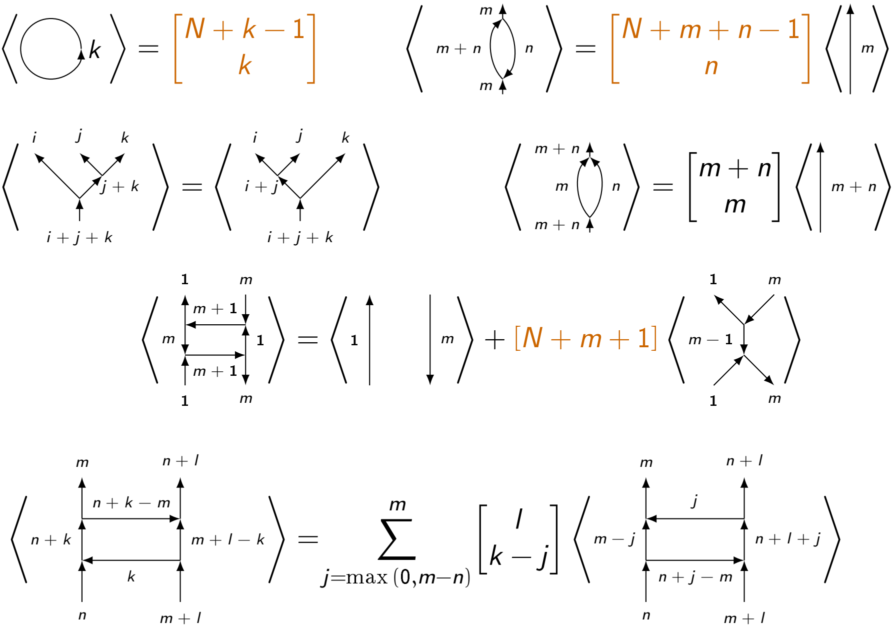

Bad news

It is not possible to categorify the $\mathrm{Sym}^\bullet$-MOY calculus with such a TQFT.

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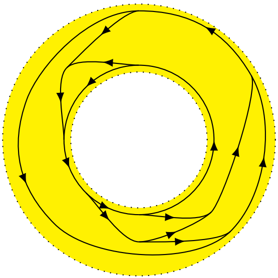
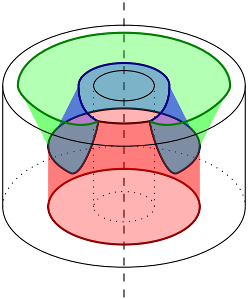
We restrict the class of graphs and the class of foams:



Vinyl graphs



Tube-like foams

Theorem (R.–Wagner, 2018)

There exists a foamy restricted TQFT

$$\mathcal{F}_N \colon \mathsf{TLFoam} \to \mathbb{Q}[X_1, \ldots, X_N]^{S_N}\text{-mod}_{\mathrm{gr}}$$

such that

$$\mathrm{rk}_q \mathcal{F}_N(\Gamma) = \langle \Gamma \rangle \, .$$

It can be extended to an homological link invariant called symmetric $\mathfrak{sl}_N$ homology.

Thank you !!