

Foams and categorification

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Workshop Categorification in Mathematical Physics

$$\left\langle \left(\begin{array}{c} \text{circle with arrow } k \end{array} \right) \right\rangle = \left[\begin{array}{c} N \\ k \end{array} \right]_q$$

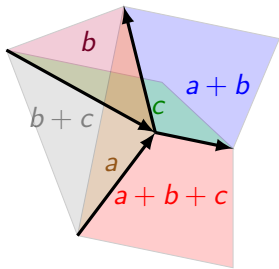
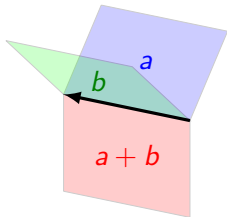
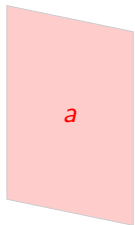
$$\left\langle \begin{array}{c} m+n \text{ (top)} \\ \text{loop } m \text{ (left), } n \text{ (right)} \\ m+n \text{ (bottom)} \end{array} \right\rangle = \left[\begin{array}{c} N-m \\ n \end{array} \right]_q \left\langle \begin{array}{c} \uparrow \\ m \end{array} \right\rangle$$

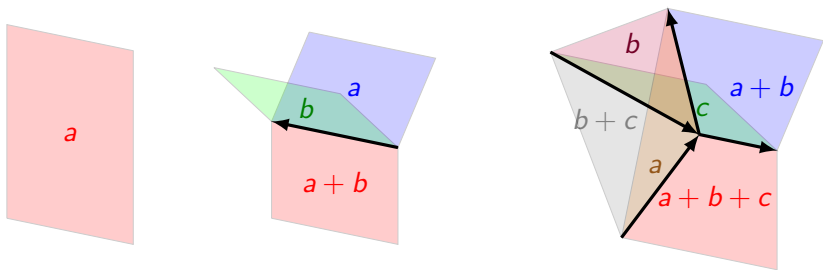
$$\left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \quad \nearrow \\ \uparrow \\ i+j+k \end{array} \right\rangle = \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \searrow \quad \nearrow \\ \uparrow \\ i+j+k \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} m+n \text{ (top)} \\ \text{loop } m \text{ (left), } n \text{ (right)} \\ m+n \text{ (bottom)} \end{array} \right\rangle = \left[\begin{array}{c} m+n \\ m \end{array} \right]_q \left\langle \begin{array}{c} \uparrow \\ m+n \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} 1 \quad m \\ \uparrow m+1 \quad \downarrow m+1 \\ \leftarrow \quad \rightarrow \\ \uparrow m+1 \quad \downarrow m+1 \\ 1 \quad m \end{array} \right\rangle = \left\langle \begin{array}{c} \uparrow \\ 1 \end{array} \right\rangle \left\langle \begin{array}{c} \downarrow \\ m \end{array} \right\rangle + [N-m-1]_q \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \searrow \\ \uparrow \\ m-1 \\ \nearrow \quad \nwarrow \\ 1 \quad m \end{array} \right\rangle$$

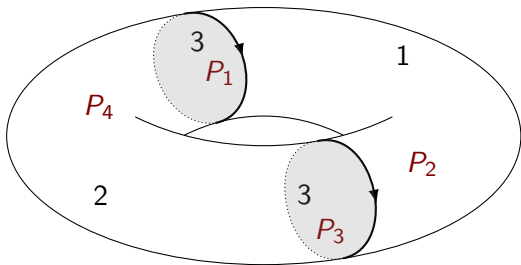
$$\left\langle \begin{array}{c} m \quad n+l \\ \uparrow n+k \quad \uparrow m+l-k \\ \leftarrow n+k-m \quad \rightarrow k \\ \uparrow n \quad \uparrow m+l \end{array} \right\rangle = \sum_{j=\max(0, m-n)}^m \left[\begin{array}{c} l \\ k-j \end{array} \right]_q \left\langle \begin{array}{c} m \quad n+l \\ \uparrow m-j \quad \uparrow n+l+j \\ \leftarrow j \quad \rightarrow n+j-m \\ \uparrow n \quad \uparrow m+l \end{array} \right\rangle$$





Definition (R.-Wagner, '17)

$$\tau_N(F) = \sum_c \frac{(-1)^{\sum_{i=1}^N i\chi_i(F(c))/2 + \sum_{1 \leq i < j \leq N} \theta_{ij}^+(F(c))} \prod_f P_f(c(f))}{\prod_{1 \leq i < j \leq N} (X_i - X_j)^{\frac{\chi_{ij}(F(c))}{2}}}$$



$$P_1 = t_1 + t_2 + t_3$$

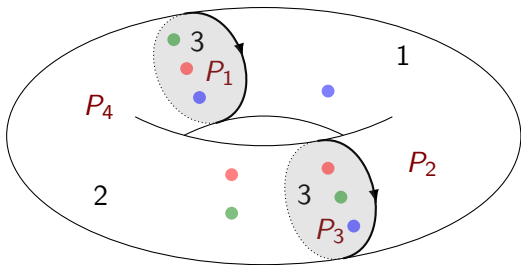
$$P_2 = t_1^2$$

$$P_3 = 1$$

$$P_4 = t_1 t_2 (t_1 + t_2)$$

$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



$$P_1 = t_1 + t_2 + t_3$$

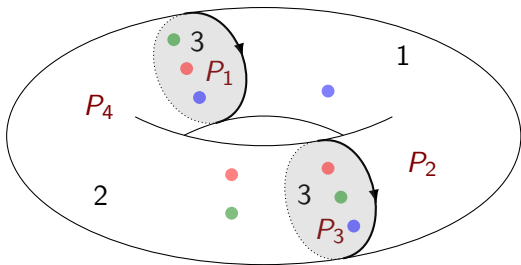
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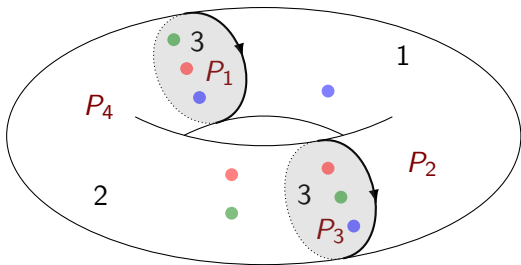
$$P_3 = 1$$

$$P_4 = t_1 t_2 (t_1 + t_2)$$

\mathbb{P}	Monochrome	χ_\bullet
X_1		2
X_2		2
X_3		0
X_4		2

$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



$$P_1 = t_1 + t_2 + t_3$$

$$P_2 = t_1^2$$

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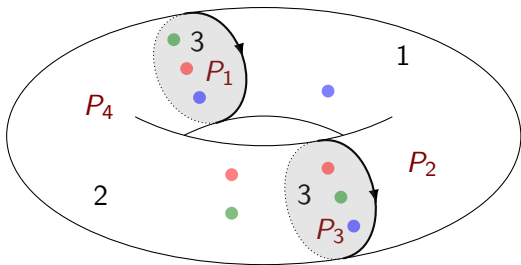
$$P_4 = t_1 t_2 (t_1 + t_2)$$

\mathbb{P}	Monochrome	χ_\bullet
X_1		2
X_2		2
X_3		0
X_4		2

	$X_1 X_2$	$X_1 X_3$	$X_1 X_4$	$X_2 X_3$	$X_2 X_4$	$X_3 X_4$
$\chi_{\bullet\bullet}$	0	2	0	2	0	2
$\theta_{\bullet\bullet}^+$	2	0	2	0	0	0

$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



$$P_1 = t_1 + t_2 + t_3 \quad P_2 = t_1^2$$

$$P_3 = 1 \quad P_4 = t_1 t_2 (t_1 + t_2)$$

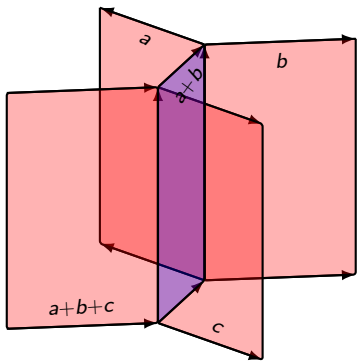
$$\frac{(-1)^{1+2+4} (X_1 + X_2 + X_4) X_1^2 (X_2 X_4 (X_2 + X_4))}{(X_1 - X_3)(X_2 - X_3)(X_3 - X_4)}$$

\mathbb{P}	Monochrome	χ_\bullet
X_1		2
X_2		2
X_3		0
X_4		2

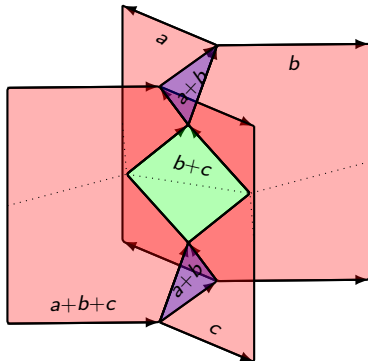
	$X_1 X_2$	$X_1 X_3$	$X_1 X_4$	$X_2 X_3$	$X_2 X_4$	$X_3 X_4$
$\chi_{\bullet\bullet}$	0	2	0	2	0	2
$\theta_{\bullet\bullet}^+$	2	0	2	0	0	0

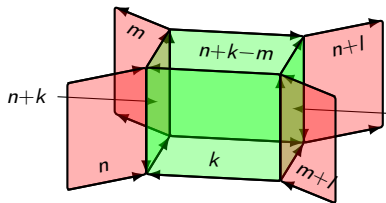
$$N = 4$$

$$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$$



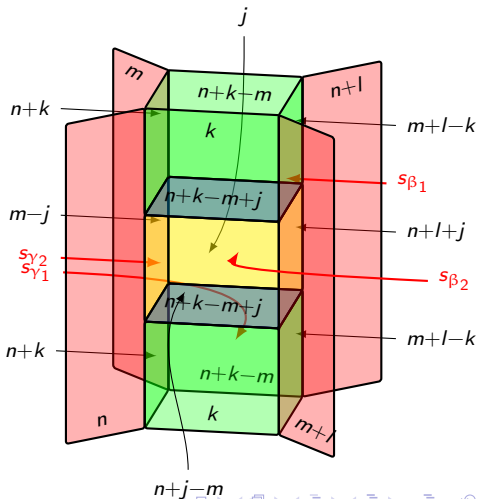
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$$m+l-k = \sum_{j=\max(0, m-n)}^m \sum_{\alpha \in T(k-j, l-k+j)}$$

$$(-1)^{|\alpha|+(l-k+j)(m-j)} \sum_{\substack{\beta_1, \beta_2 \\ \gamma_1, \gamma_2}} c_{\beta_1 \beta_2}^\alpha c_{\gamma_1 \gamma_2}^{\hat{\alpha}}$$



A_1	高口站	高口站	高口站	高口站	B_1	B_1	C	L	R	X	A_1	A_1	A_1	新口站	新口站	新口站	新口站	C	L	R	X	
新口站											A_1											
新口站											A_2											
新口站											新口站											
新口站											新口站											
B_1											新口站											
B_2											新口站											
C											C											
L											L											
R											R											
X											X											

$$\langle \text{circle with arrow } k \rangle = \begin{bmatrix} N \\ k \end{bmatrix}$$

$$\langle \text{loop with } m+n \text{ and } m \text{ labels} \rangle = \begin{bmatrix} N-m \\ n \end{bmatrix} \langle \text{vertical arrow } m \rangle$$

$$\langle \text{Y-junction with } i, j, k \text{ and } i+j+k \rangle = \langle \text{Y-junction with } i, j, k \text{ and } i+j+k \rangle$$

$$\langle \text{loop with } m+n \text{ and } m \text{ labels} \rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \langle \text{vertical arrow } m+n \rangle$$

$$\langle \text{loop with } m+1 \text{ and } m \text{ labels} \rangle = \langle \text{vertical arrow } 1 \rangle + [N-m-1] \langle \text{Y-junction with } m-1 \text{ labels} \rangle$$

$$\langle \text{rectangle with } m, n+l, n+k, m+l-k \text{ labels} \rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \langle \text{rectangle with } m-j, n+l+j, n, m+l \text{ labels} \rangle$$

From Λ^\bullet to Sym^\bullet : N goes to $-N$.

$$\left\langle \left\langle \begin{array}{c} \circlearrowright k \end{array} \right\rangle \right\rangle = \begin{bmatrix} N + k - 1 \\ k \end{bmatrix} \quad \left\langle \left\langle \begin{array}{c} m \uparrow \\ m+n \text{ } \curvearrowright \text{ } n \\ m \downarrow \end{array} \right\rangle \right\rangle = \begin{bmatrix} N + m + n - 1 \\ n \end{bmatrix} \left\langle \left\langle \begin{array}{c} \uparrow \\ m \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ \uparrow \\ i+j+k \end{array} \right\rangle \right\rangle = \left\langle \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ i+j \quad \uparrow \\ i+j+k \end{array} \right\rangle \right\rangle \quad \left\langle \left\langle \begin{array}{c} m+n \uparrow \\ m \text{ } \curvearrowright \text{ } n \\ m+n \downarrow \end{array} \right\rangle \right\rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \left\langle \left\langle \begin{array}{c} \uparrow \\ m+n \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} 1 \quad m \\ \uparrow \quad \downarrow \\ m+1 \quad \leftarrow \quad \rightarrow \quad 1 \\ \downarrow \quad \uparrow \\ 1 \quad m \end{array} \right\rangle \right\rangle = \left\langle \left\langle \begin{array}{c} \uparrow \\ 1 \end{array} \right\rangle \right\rangle \left\langle \left\langle \begin{array}{c} \downarrow \\ m \end{array} \right\rangle \right\rangle + [N + m + 1] \left\langle \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \nearrow \\ m-1 \quad \downarrow \\ \uparrow \quad \downarrow \\ 1 \quad m \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ n+k \quad \leftarrow \quad \rightarrow \quad m+l-k \\ \leftarrow \quad \rightarrow \quad k \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle \right\rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \left\langle \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ m-j \quad \leftarrow \quad \rightarrow \quad n+l+j \\ \leftarrow \quad \rightarrow \quad n+j-m \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle \right\rangle$$

Thank you !!