

# Evaluation of $\mathfrak{sl}_N$ -foams

Louis-Hadrien Robert

Emmanuel Wagner

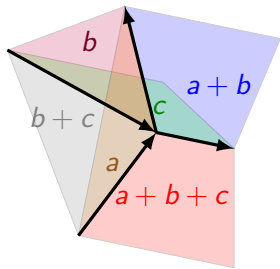
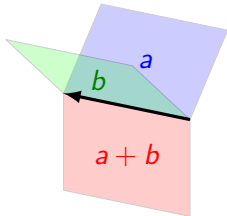
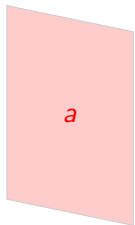


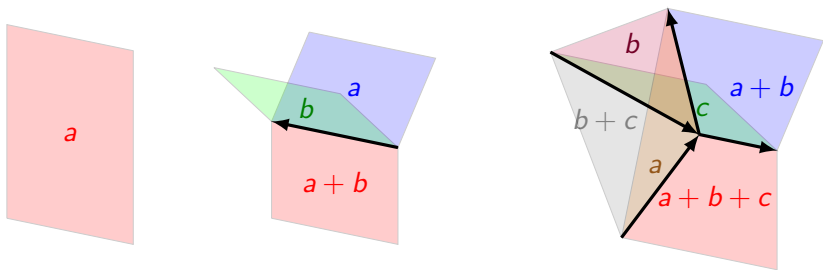
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Winter Braids 7 – Caen

<http://www.math.uni-hamburg.de/home/robert/wb7talk.pdf>





## Definition (R.-Wagner, '17)

$$\langle F \rangle_N = \sum_c \frac{(-1)^{\sum_{i=1}^N i\chi(F_i(c))/2 + \sum_{1 \leq i < j \leq N} \theta_{ij}^+(F,c)} \prod_f P_f(c(f))}{\prod_{1 \leq i < j \leq N} (X_i - X_j)^{\frac{\chi(F_{ij}(c))}{2}}}$$

## Definition (Kauffman Bracket, Jones polynomial)

$$\langle \emptyset \rangle_{\mathbb{K}} = 1 \quad \langle \bigcirc \sqcup L \rangle_{\mathbb{K}} = [2]_q \langle L \rangle$$

$$\langle \text{crossing} \rangle_{\mathbb{K}} = \langle \text{cup} \rangle_{\mathbb{K}} - q \langle \text{cap} \rangle_{\mathbb{K}}$$

$$J(L) = (-1)^{n-} q^{n+ - 2n-} \langle D \rangle_{\mathbb{K}}$$

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$$\langle \text{crossing} \rangle_K = \langle \text{smoothing} \rangle_K - q \langle \text{other smoothing} \rangle_K$$

$$J(L) = (-1)^{n-} q^{n+-2n-} \langle D \rangle_K$$

$$\begin{aligned} \langle \text{link} \rangle_K &= \langle \text{link} \rangle_K - q \langle \text{link} \rangle_K \\ &\quad - q \langle \text{link} \rangle_K + q^2 \langle \text{link} \rangle_K \end{aligned}$$

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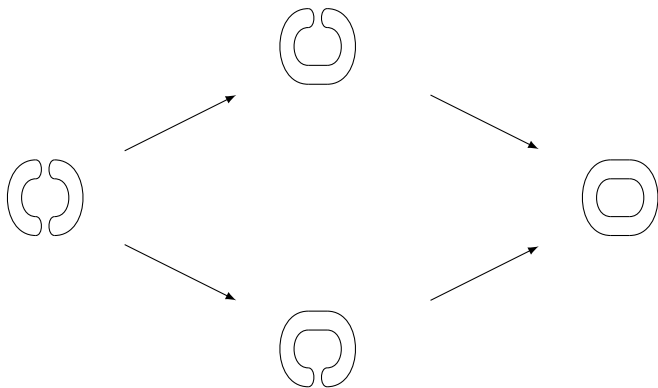
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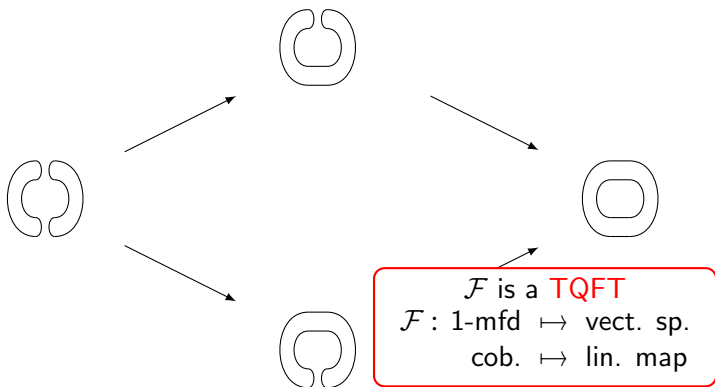
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$$J(\text{link}) = q^6 + q^4 + q^2 + 1$$

# Khovanov homology



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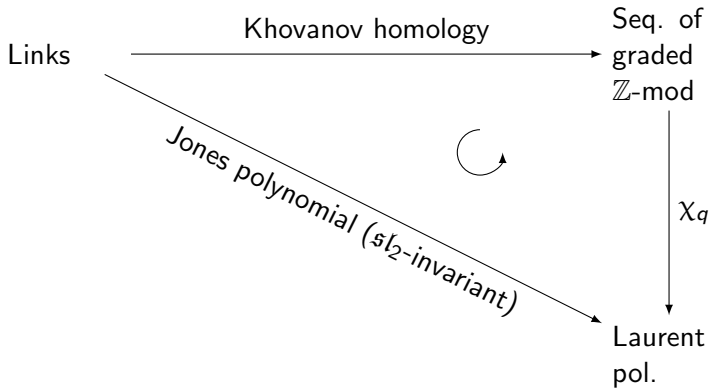




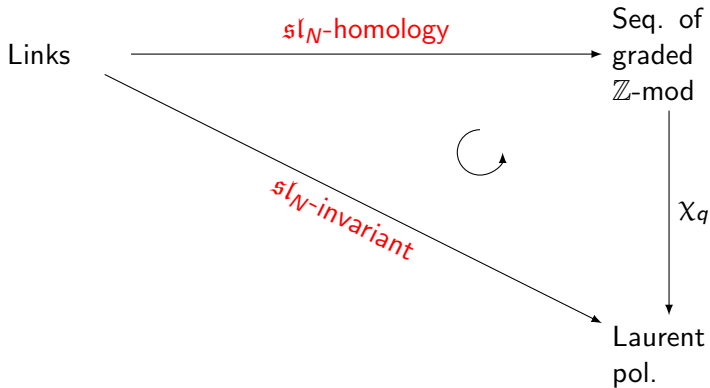
# Khovanov homology

$$\begin{array}{ccc} & \mathcal{F} \left( \text{link} \right) \{+1\} & \\ \mathcal{F}(\text{saddle}) \nearrow & & \searrow \mathcal{F}(\text{saddle}) \\ \mathcal{F} \left( \text{link} \right) & \oplus & \mathcal{F} \left( \text{link} \right) \{+2\} \\ \mathcal{F}(\text{saddle}) \searrow & & \nearrow -\mathcal{F}(\text{saddle}) \\ & \mathcal{F} \left( \text{link} \right) \{+1\} & \end{array}$$

Shift the homological degree by  $-n_-$ . Take the homology.



- ▶ A recipe to deal with crossings
- ▶ An ad-hoc TQFT



- ▶ A recipe to deal with crossings  $\rightsquigarrow$  Rickard complexes
- ▶ An ad-hoc TQFT  $\rightsquigarrow$  evaluation of foams

# The $\mathfrak{sl}_N$ -link invariant

$$\left\langle \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right\rangle = \sum_{k=\max(0, m-n)}^m (-1)^{m-k} q^{k-m} \left\langle \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right\rangle$$

The diagram on the left shows a crossing of two strands. The upper-left strand is labeled  $m$  and the upper-right strand is labeled  $n$ . The right side of the equation is a sum over  $k$  from  $\max(0, m-n)$  to  $m$ . Each term in the sum is  $(-1)^{m-k} q^{k-m}$  multiplied by a square diagram. The square diagram has four strands: the top-left is labeled  $m$ , the top-right is labeled  $n$ , the bottom-left is labeled  $n+k$ , and the bottom-right is labeled  $m-k$ . The strands are connected by horizontal and vertical lines forming a square.

$$\left\langle \begin{array}{c} \nwarrow \\ \nearrow \end{array} \right\rangle = \sum_{k=\max(0, m-n)}^m (-1)^{m-k} q^{m-k} \left\langle \begin{array}{c} \nwarrow \\ \nearrow \end{array} \right\rangle$$

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$\mathcal{F} :$

	Foam <sub>N</sub>	$\longrightarrow$	$\mathbb{Z}[X_1, \dots, X_N] - \text{mod}_{\text{gr}}$
Wish:	MOY-graph	$\longmapsto$	graded module
	foam	$\longmapsto$	graded module map

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## Universal Construction

An evaluation  $\rightsquigarrow$  (Maybe) a TQFT

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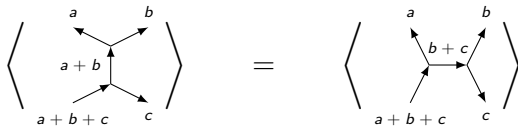
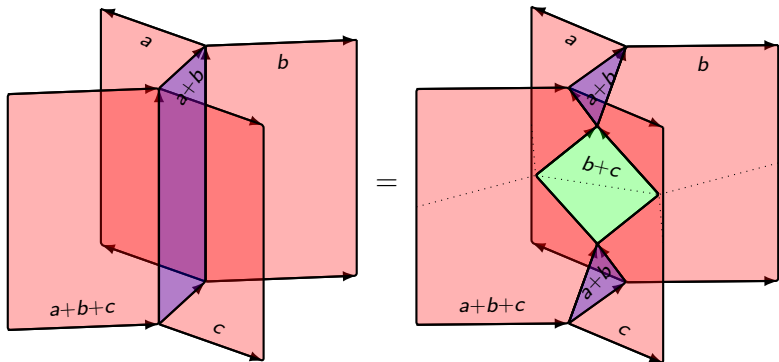
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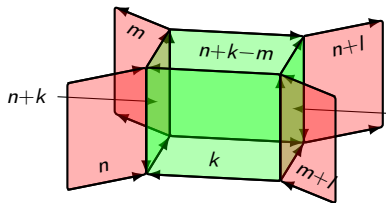
An evaluation  $\rightsquigarrow$  (Maybe) a TQFT

### Theorem (R.-Wagner, '17)

*The evaluation defined on the first slide together with the **Universal Construction**, yields an *ad-hoc* TQFT.*

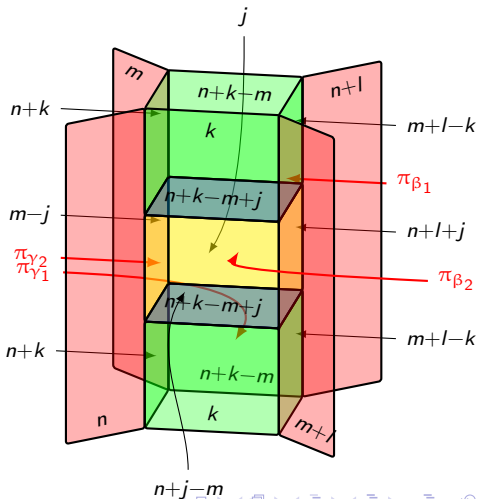






$$m+l-k = \sum_{j=\max(0, m-n)}^m \sum_{\alpha \in T(k-j, l-k+j)}$$

$$(-1)^{|\alpha|+(l-k+j)(m-j)} \sum_{\substack{\beta_1, \beta_2 \\ \gamma_1, \gamma_2}} c_{\beta_1 \beta_2}^{\alpha} c_{\gamma_1 \gamma_2}^{\hat{\alpha}}$$



$A_1$	$A_2$	$A_3$	$A_4$	$B_1$	$B_2$	$C$	$L$	$R$	$X$	$A_1$	$A_2$	$A_3$	$A_4$	$C$	$L$	$R$	$X$	
										$A_1$								
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## Proposition

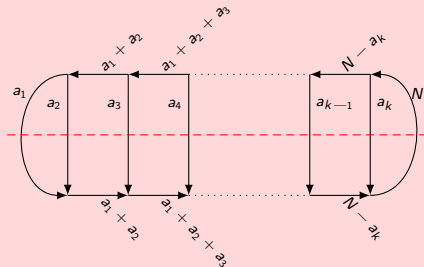
*The module associated with a MOY-graph with a symmetry axis is a Frobenius algebra.*

## Proposition

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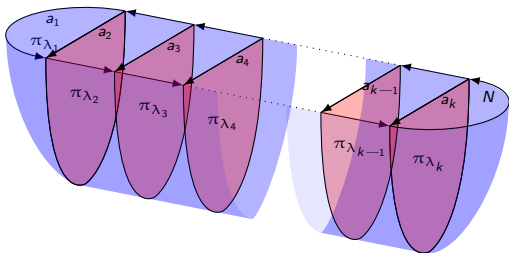
The Frobenius algebra associated with



is isomorphic to the cohomology ring of

$$\mathfrak{Flag}(\mathbb{C}^{a_1} \subset \mathbb{C}^{a_1+a_2} \subset \dots \subset \mathbb{C}^{a_1+\dots+a_{k-1}} \subset \mathbb{C}^N).$$

$$\prod_{i=1}^k \pi_{\lambda_i}(X_{a_i+1}, \dots, X_{a_{i+1}}) \mapsto$$



# Thank you!

<http://www.math.uni-hamburg.de/home/robert/wb7talk.pdf>