Evaluation of $\mathfrak{sl}_N$-foams

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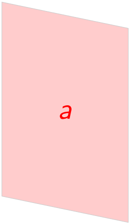
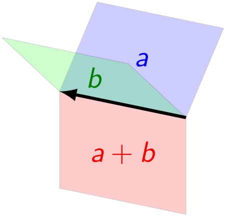
DER FORSCHUNG | DER LEHRE | DER BILDUNG

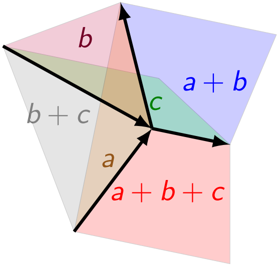
UNIVERSITÉ DE BOURGOGNE

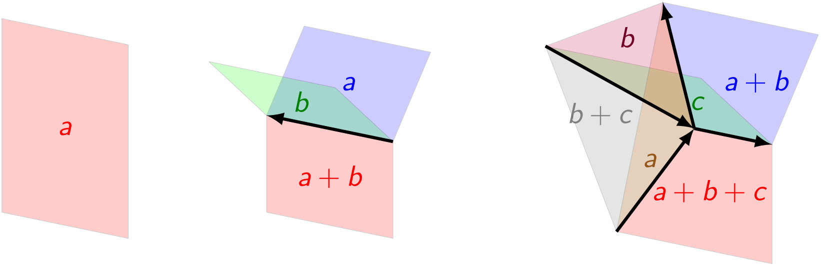
Winter Braids 7 – Caen

http://www.math.uni-hamburg.de/home/robert/wb7talk.pdf


a


a
b
$a+b$


b
$a+b$
c
$b+c$
a
$a+b+c$



Definition (R.–Wagner, '17)

$$\langle F\rangle_N = \sum_c \frac{(-1)^{\sum_{i=1}^N i\chi(F_i(c))/2+\sum_{1\le i<j\le N}\theta^+_{ij}(F,c)}\prod_f P_f(c(f))}{\prod_{1\le i<j\le N}(X_i-X_j)^{\frac{\chi(F_{ij}(c))}{2}}}$$

Definition (Kauffman Bracket, Jones polynomial)

$$\langle \emptyset \rangle_{\mathrm{K}} = 1 \qquad \left\langle \bigcirc \sqcup L \right\rangle_{\mathrm{K}} = [2]_q \langle L \rangle$$

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle_{\mathrm{K}}$$

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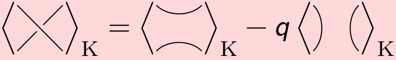

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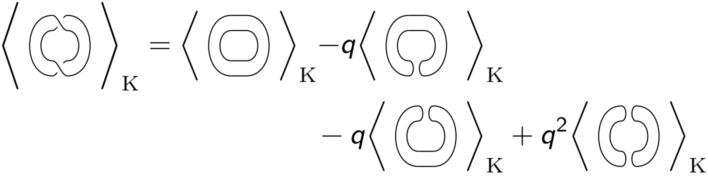

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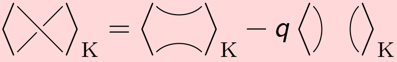

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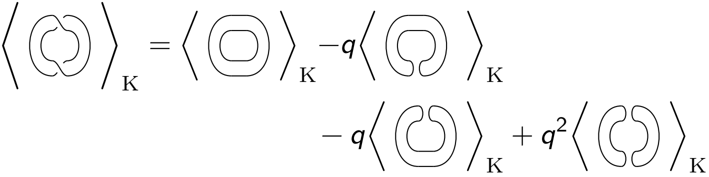
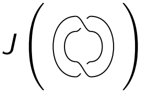
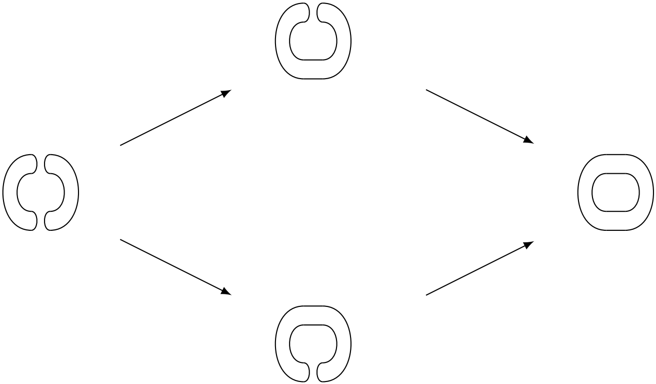


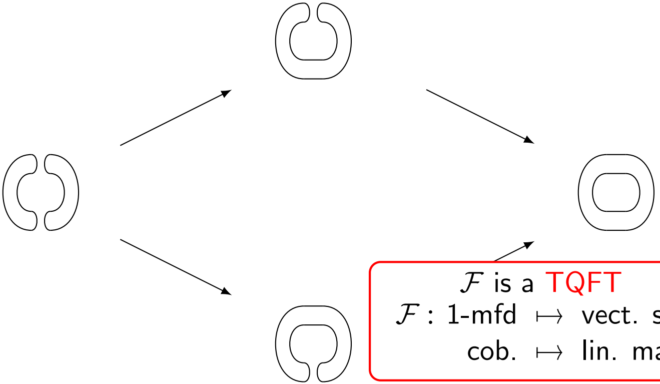


$$= q^6 + q^4 + q^2 + 1$$

Khovanov homology



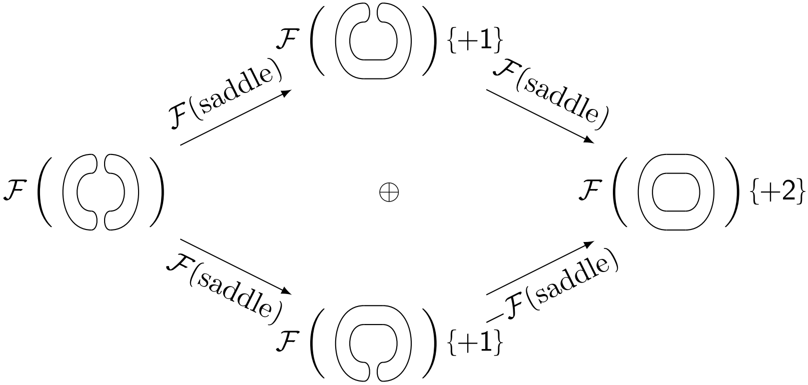
Khovanov homology



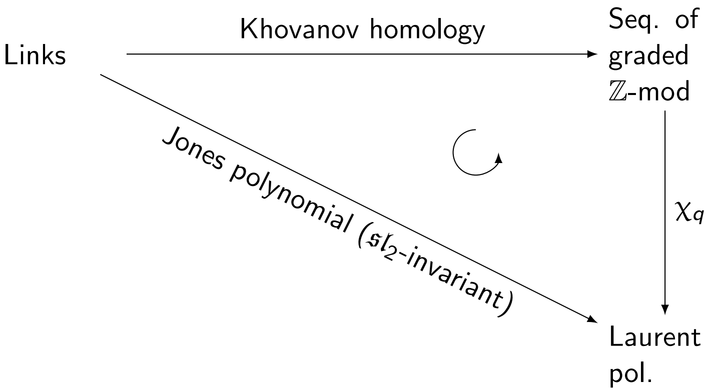
$\mathcal{F}$ is a TQFT

$$\begin{aligned}\mathcal{F}: \text{1-mfd} &\mapsto \text{vect. sp.}\\ \text{cob.} &\mapsto \text{lin. map}\end{aligned}$$

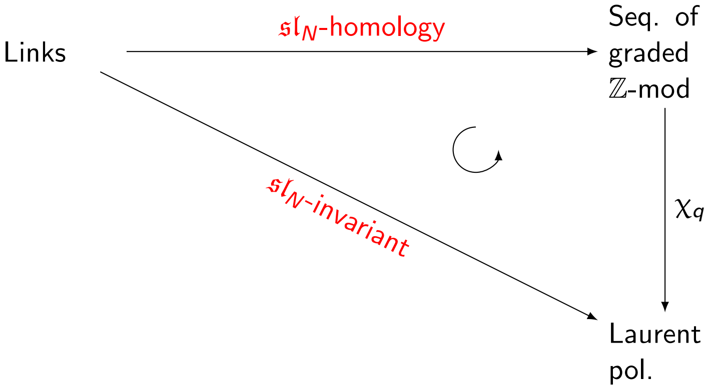
Khovanov homology



Shift the homological degree by $-n_{-}$. Take the homology.

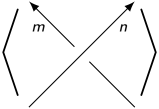


- A recipe to deal with crossings
- An ad-hoc TQFT

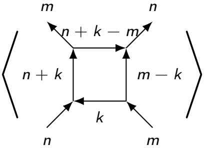


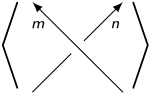
- A recipe to deal with crossings $\rightsquigarrow$ Rickard complexes
- An ad-hoc TQFT $\rightsquigarrow$ evaluation of foams

The $\mathfrak{sl}_N$-link invariant



$$= \sum_{k=\max(0,m-n)}^{m} (-1)^{m-k} q^{k-m}$$





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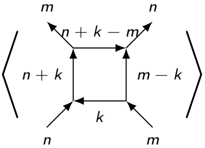
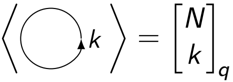
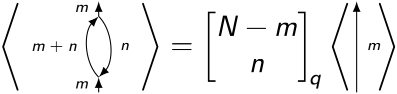
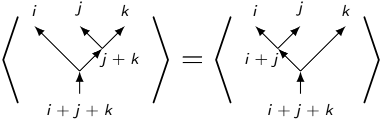
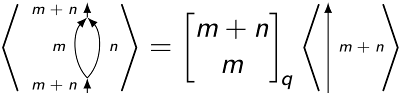
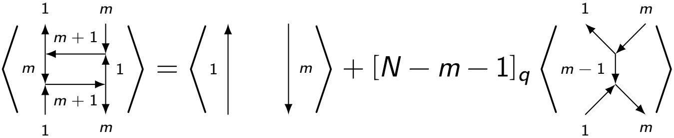


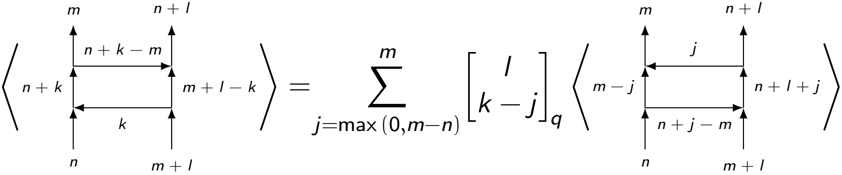

k
= [N k]_q


m
m + n
n
m
= [N − m n]_q
m


i
j
k
j + k
i + j + k
=
i
j
k
i + j
i + j + k


m + n
m
n
m + n
= [m + n m]_q
m + n


1
m
m + 1
m
1
m + 1
1
m
=
1
m
+ [N − m − 1]_q
1
m
m − 1
1
m


m
n + l
n + k − m
n + k
m + l − k
k
n
m + l
= Σ_{j=max (0,m−n)}^{m} [l k − j]_q
m
n + l
j
m − j
n + l + j
n + j − m
n
m + l

$$\begin{array}{rrcl} \mathcal{F}: & \mathsf{Foam_N} & \longrightarrow & \mathbb{Z}[X_1, \dots, X_N] - \mathsf{mod}_{\mathrm{gr}} \\ \text{Wish:} & \text{MOY-graph} & \longmapsto & \text{graded module} \\ & \text{foam} & \longmapsto & \text{graded module map} \end{array}$$

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Universal Construction

An evaluation $\rightsquigarrow$ (Maybe) a TQFT

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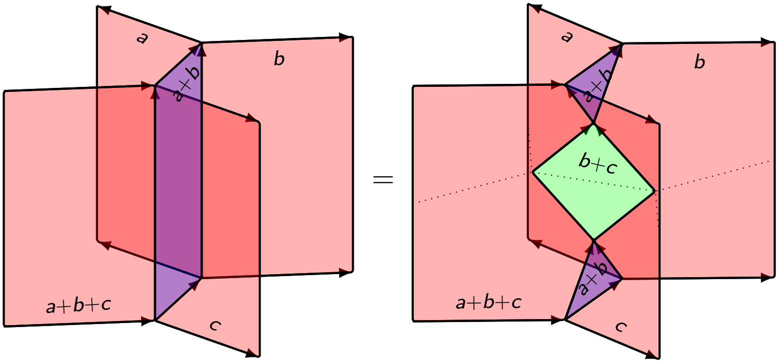
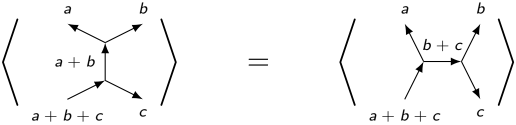
Universal Construction

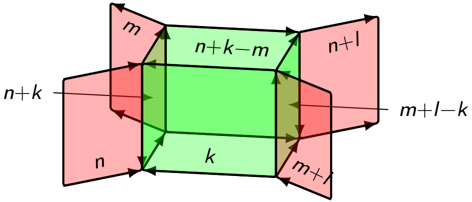
An evaluation $\rightsquigarrow$ (Maybe) a TQFT

Theorem (R.–Wagner, '17)

The evaluation defined on the first slide together with the ***Universal Construction****, yields an ad-hoc TQFT.*

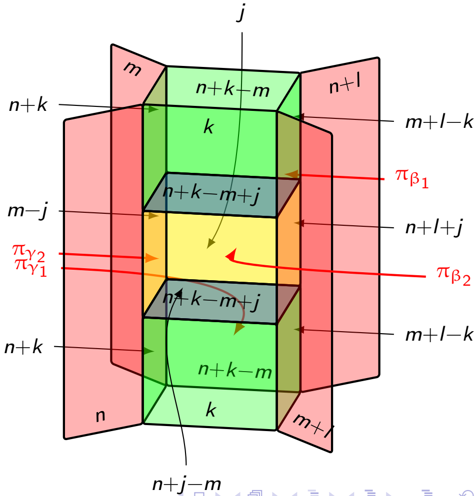

a
b
a+b
a+b+c
c
=
a
b
a+b
b+c
a+b
a+b+c
c

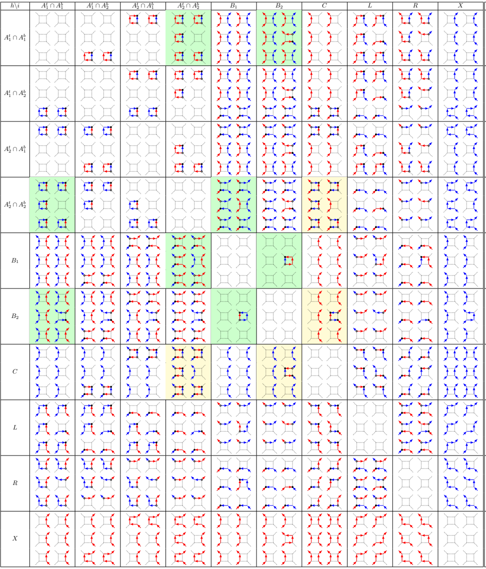

a
b
a + b
a + b + c
c
=
a
b
b + c
a + b + c
c



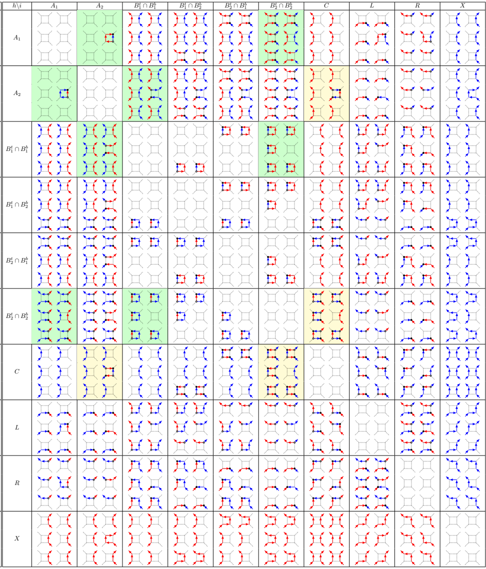
$$= \sum_{j=\max(0,m-n)}^{m} \; \sum_{\alpha \in T(k-j,\, l-k+j)}$$

$$(-1)^{|\alpha|+(l-k+j)(m-j)} \sum_{\substack{\beta_1,\beta_2 \\ \gamma_1,\gamma_2}} c^{\alpha}_{\beta_1\beta_2} c^{\widehat{\alpha}}_{\gamma_1\gamma_2}$$





$h\backslash i$
$A_1^t \cap A_1^b$
$A_1^t \cap A_2^b$
$A_2^t \cap A_1^b$
$A_2^t \cap A_2^b$
B_1
B_2
C
L
R
X



$h\backslash i$
A_1
A_2
$B_1^t \cap B_1^b$
$B_1^t \cap B_2^b$
$B_2^t \cap B_1^b$
$B_2^t \cap B_2^b$
C
L
R
X

Proposition

The module associated with a MOY-graph with a symmetry axis is a Frobenius algebra.

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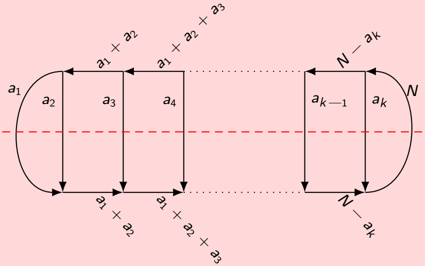
Proposition (R.–Wagner, '17)

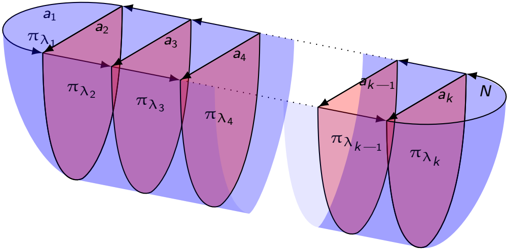
The Frobenius algebra associated with



is isomorphic to the cohomology ring of

$$\mathfrak{Flag}(\mathbb{C}^{a_1} \subset \mathbb{C}^{a_1+a_2} \subset \cdots \subset \mathbb{C}^{a_1+\cdots+a_{k-1}} \subset \mathbb{C}^N).$$

$$\prod_{i=1}^{k} \pi_{\lambda_i}(X_{a_i+1}, \ldots, X_{a_{i+1}}) \mapsto$$



Thank you!

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