

Categorification of the symmetric $U_q(\mathfrak{sl}_N)$ link invariant

Louis-Hadrien Robert

Emmanuel Wagner



UNIVERSITÉ
DE GENÈVE



Winter Braids 8 – Marseille

<http://www.unige.ch/math/folks/robert/wb8.pdf>

The \mathfrak{sl}_N -link invariant

$$\left\langle \begin{array}{c} \nearrow m \\ \searrow n \end{array} \right\rangle = \sum_{k=\max(0, m-n)}^m (-1)^{m-k} q^{k-m} \left\langle \begin{array}{ccc} & \nearrow n+k-m & \nearrow n \\ \nearrow n+k & \rightarrow & \nearrow m-k \\ \nwarrow n & \leftarrow k & \nwarrow m \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} \nearrow m \\ \searrow n \end{array} \right\rangle = \sum_{k=\max(0, m-n)}^m (-1)^{m-k} q^{m-k} \left\langle \begin{array}{ccc} & \nearrow n+k-m & \nearrow n \\ \nearrow n+k & \rightarrow & \nearrow m-k \\ \nwarrow n & \leftarrow k & \nwarrow m \end{array} \right\rangle$$

$$\langle \text{circle with arrow } k \rangle = \begin{bmatrix} N \\ k \end{bmatrix}$$

$$\langle \text{loop with } m+n \text{ and } m \text{ labels} \rangle = \begin{bmatrix} N-m \\ n \end{bmatrix} \langle \text{vertical arrow } m \rangle$$

$$\langle \text{Y-junction with } i, j, k \text{ and } i+j+k \rangle = \langle \text{Y-junction with } i, j, k \text{ and } i+j+k \rangle$$

$$\langle \text{loop with } m+n \text{ and } m \text{ labels} \rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \langle \text{vertical arrow } m+n \rangle$$

$$\langle \text{loop with } m+1 \text{ and } m \text{ labels} \rangle = \langle \text{vertical arrow } 1 \rangle + [N-m-1] \langle \text{Y-junction with } m-1 \text{ labels} \rangle$$

$$\langle \text{rectangle with } m, n+l, n+k, m+l-k \text{ labels} \rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \langle \text{rectangle with } m-j, n+l+j, n, m+l \text{ labels} \rangle$$

From Λ^\bullet to Sym^\bullet : N goes to $-N$.

$$\left\langle \begin{array}{c} \text{circle with arrow } k \end{array} \right\rangle = \begin{bmatrix} N \\ k \end{bmatrix}$$

$$\left\langle \begin{array}{c} m+n \uparrow \\ \text{loop } n \\ m \downarrow \end{array} \right\rangle = \begin{bmatrix} N-m \\ n \end{bmatrix} \left\langle \begin{array}{c} \uparrow \\ m \\ \downarrow \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ \text{node} \\ \uparrow \\ i+j+k \end{array} \right\rangle = \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ \text{node} \\ \uparrow \\ i+j+k \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} m+n \uparrow \\ \text{loop } n \\ m+n \downarrow \end{array} \right\rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \left\langle \begin{array}{c} \uparrow \\ m+n \\ \downarrow \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} 1 \quad m \\ \uparrow \quad \downarrow \\ m+1 \quad 1 \\ \downarrow \quad \uparrow \\ m \quad m \\ \uparrow \quad \downarrow \\ 1 \quad m \end{array} \right\rangle = \left\langle \begin{array}{c} \uparrow \\ 1 \\ \downarrow \end{array} \right\rangle + [N-m-1] \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \nearrow \\ \text{node} \\ \downarrow \\ m-1 \\ \uparrow \quad \downarrow \\ 1 \quad m \end{array} \right\rangle$$

$$\left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ n+k \quad m+l-k \\ \leftarrow \quad \rightarrow \\ k \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ m-j \quad n+l+j \\ \leftarrow \quad \rightarrow \\ n+j-m \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle$$

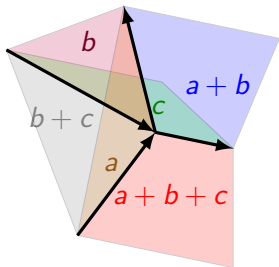
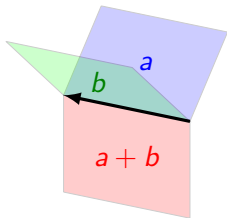
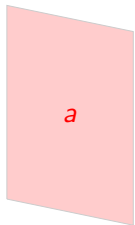
From Λ^\bullet to Sym^\bullet : N goes to $-N$.

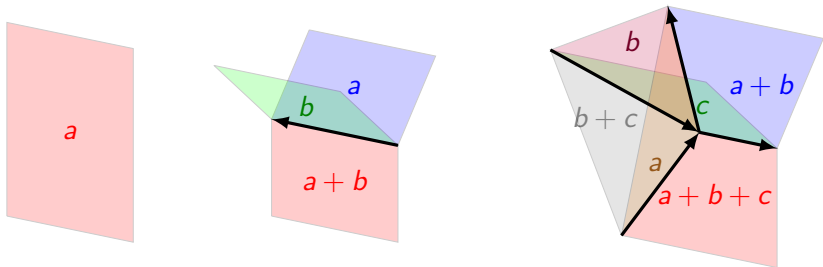
$$\left\langle \left\langle \begin{array}{c} \text{circle with arrow } k \end{array} \right\rangle \right\rangle = \begin{bmatrix} N + k - 1 \\ k \end{bmatrix} \quad \left\langle \left\langle \begin{array}{c} m+n \uparrow \\ \text{loop } n \\ m \downarrow \end{array} \right\rangle \right\rangle = \begin{bmatrix} N + m + n - 1 \\ n \end{bmatrix} \left\langle \left\langle \begin{array}{c} \uparrow \\ m \\ \downarrow \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ \text{node} \\ \uparrow \\ i+j+k \end{array} \right\rangle \right\rangle = \left\langle \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \\ \text{node} \\ \uparrow \\ i+j+k \end{array} \right\rangle \right\rangle \quad \left\langle \left\langle \begin{array}{c} m+n \uparrow \\ \text{loop } n \\ m \downarrow \\ m+n \uparrow \end{array} \right\rangle \right\rangle = \begin{bmatrix} m+n \\ m \end{bmatrix} \left\langle \left\langle \begin{array}{c} \uparrow \\ m+n \\ \downarrow \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} 1 \quad m \\ \uparrow \quad \downarrow \\ m+1 \quad 1 \\ \downarrow \quad \uparrow \\ m \quad m \end{array} \right\rangle \right\rangle = \left\langle \left\langle \begin{array}{c} \uparrow \\ 1 \\ \downarrow \end{array} \right\rangle \right\rangle + [N + m + 1] \left\langle \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \nearrow \\ \text{node} \\ \downarrow \\ m-1 \\ \swarrow \quad \nearrow \\ 1 \quad m \end{array} \right\rangle \right\rangle$$

$$\left\langle \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ n+k \quad m+l-k \\ \leftarrow \quad \rightarrow \\ k \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle \right\rangle = \sum_{j=\max(0, m-n)}^m \begin{bmatrix} l \\ k-j \end{bmatrix} \left\langle \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ m-j \quad n+l+j \\ \leftarrow \quad \rightarrow \\ n+j-m \\ \uparrow \quad \uparrow \\ n \quad m+l \end{array} \right\rangle \right\rangle$$





Theorem (Categorification of the Λ^\bullet -MOY calculus)

There exists a foamy TQFT $\mathcal{F}_N: \text{Foam} \rightarrow R_N\text{-mod}_{\text{gr}}$ such that

$$\dim_q \mathcal{F}_N(\Gamma) = \langle \Gamma \rangle.$$

It can be extended to an homological link invariant.

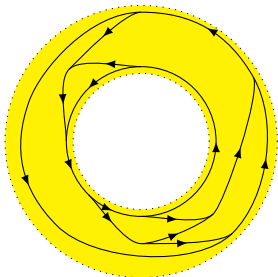
Bad news

It is not possible to categorify the Sym^\bullet -MOY calculus with such a TQFT.

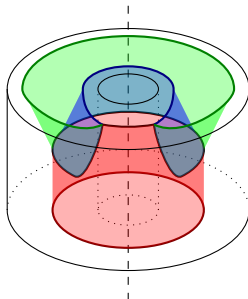
Bad news

It is not possible to categorify the Sym^\bullet -MOY calculus with such a TQFT.

We restrict the class of graphs and the class of foams:



Vinyl graphs



Tube-like foams

Theorem (R.-Wagner, '18)

There exists a foamy restricted TQFT $\mathcal{F}_N: \text{TLFoam} \rightarrow R_N\text{-mod}_{\text{gr}}$ such that

$$\dim_q \mathcal{F}_N(\Gamma) = \langle \Gamma \rangle .$$

It can be extended to an homological link invariant.

Thank you!

<http://www.unige.ch/math/folks/robert/wb8.pdf>