Analysis 2 (first part)

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These notes are mostly based on Jean-Marc Schlenker's lecture notes.

CHAPTER 1

Riemann integration

In this chapter, $a < b$ are two real numbers and I denote the interval $[a; b]$.

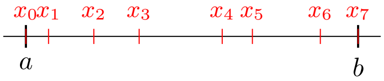
1. Integrations of step functions

1.1. Step functions.

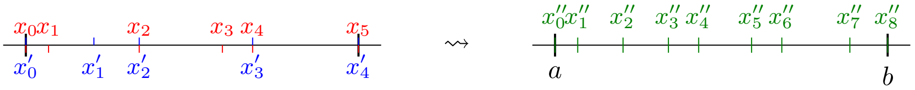
Definition 1.1.
(1) A *partition* $\sigma = (x_0, x_1, \dots, x_n)$ of I is a finite sequence $a = x_0 < x_1 < \dots < x_n = b$. The positive number $\delta(\sigma) = \max_{1\leq i\leq n-1}(x_{i+1} - x_i)$ is called the *mesh* or the *norm* of the partition σ.
(2) Let $\sigma = (x_0, x_1, \dots x_n)$ and $\sigma = (x'_0, x'_1, \dots x'_{n'})$ two partitions of $[a; b]$. The partition σ is *finer* than σ' if:

$$\{x'_i | i = 0, 1, \dots, n'\} \subseteq \{x_j | j = 0, 1, \dots, n\}.$$

Example 1.2. An example of a partition $\sigma = (x_0, x_1, \dots, x_n)$ with $n = 7$.



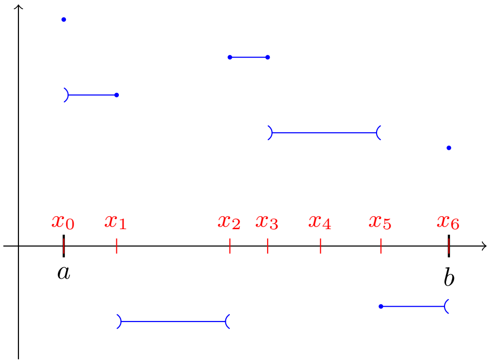
Remark 1.3.
(1) If σ is finer than σ', $\delta(\sigma) \leq \delta(\sigma')$. The converse does not necessarily holds.
(2) If $\sigma = (x_0, x_1, \dots, x_n)$ and $\sigma' = (x'_0, x'_1, \dots, x'_{n'})$ are two partitions of $[a; b]]$ and there exists another partition $\sigma'' = (x''_0, x''_1, \dots, x''_{n''})$ which is finer than both σ and σ'. This is illustrated below.



(3) "Being finer" is an order relation on the set of partitions of a given interval.

Definition 1.4. A function $f\colon [a; b] \to \mathbb{R}$ is a *step function* (or *staircase function*) if there exists a partition $\sigma = (x_0, \dots, x_n)$ of $[a; b]$ such that f is constant on $]x_i; x_{i+1}[$ for all $1 \leq i \leq n - 1$. In that case, one says that σ is *adapted* to f.

Example 1.5. A graphical example of a step function and an adapted partition is given below. Note that the partition does not need to be "optimal" to be adapted.



Remark 1.6.
(1) If a partition σ is adapted to f and σ' is finer than σ, then σ' is also adapted to f.
(2) The set of step functions from $[a;b]$ to $\mathbb{R}$ is a sub $\mathbb{R}$-vector space of the vector space $\mathbb{R}^{[a;b]}$ of functions from $[a;b]]$ to $\mathbb{R}$. In particular, if $f,g\colon [a;b] \to \mathbb{R}$ are two step functions and λ,μ are two reals, then $\lambda f + \mu g$ is a step function.

1.2. Integrations of step functions.

Notation 1.7. Let $f\colon [a;b] \to \mathbb{R}$ a step function et $\sigma = (x_0, x_1, \dots x_n)$ an adapted partition for f. One denotes $I(f,\sigma)$ the real defined by the following formula:

$$I(f,\sigma) = \sum_{i=0}^{n-1} v_i(x_{i+1} - x_i),$$

where v_i is the value of f on $]x_i; x_{i+1}[$.

Proposition 1.8. *Let σ and σ' be two partitions adapted to a step function $f\colon [a;b] \to \mathbb{R}$, then $I(f,\sigma) = I(f,\sigma')$. In other word the quantity $I(f,\sigma)$ depends only on f and not on the (adapted) partition chosen to compute it.*

Notation 1.9. If $f\colon [a;b] \to \mathbb{R}$ is a step function, denote $\int_a^b f = \int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t := I(f,\sigma)$ for any partition σ adapted to f. We extend these notations and declare that $\int_a^a = 0$ and that $\int_b^a = -\int_a^b f$.

Proof of Proposition 1.8. It is enough to consider the case where σ' is finer than σ and we can even further reduce to the case where σ' is obtained from σ by adding one point in the partition σ. Denote $\sigma = (x_0, x_1, \dots, x_n)$,

$$\sigma' = (x_0, \dots, x_j, y, x_{j+1}, \dots, x_n) = (x'_0, x'_1, \dots, x'_{n+1})$$

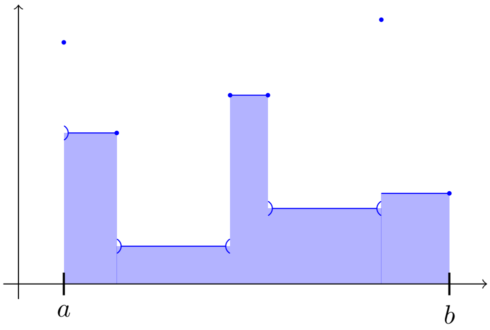
and v_i and v'_i the values of f on $]x_i; x_{i+1}[$ and on $]x'_i; x'_{i+1}[$.

One has:

$$\begin{aligned}
I(f,\sigma) &= \sum_{i=1}^{n-1} v_i(x_{i+1} - x_i) \\
&= \sum_{i=1}^{j-1} v_i(x_{i+1} - x_i) + v_j(x_{j+1} - x_j) + \sum_{i=j+1}^{n-1} v_i(x_{i+1} - x_i) \\
&= \sum_{i=1}^{j-1} v'_i(x'_{i+1} - x'_i) + v_j(y - x_j) + v_j(x_{j+1} - y) + \sum_{i=j+2}^{n} v'_i(x'_{i+1} - x'_i) \\
&= \sum_{i=1}^{j-1} v'_i(x'_{i+1} - x'_i) + v'_j(x'_j - x'_{j+1}) + v'_j(x_{j+2} - x_{j+1}) \sum_{i=j+2}^{n} v'_i(x'_{i+1} - x'_i) \\
&= \sum_{i=0}^{n} v'_i(x'_{i+1} - x'_i) \\
&= I(f,\sigma').
\end{aligned}$$

□

Remark 1.10. If $f\colon [a;b] \to \mathbb{R}_{\geq 0}$ is a step function, then $\int_a^b f$ measures the area under the curve of f: each terms in the sum corresponds to the area of a rectangle.



1.3. Property of the integral of step functions.

Proposition 1.11. *Let $f, g\colon [a;b] \to \mathbb{R}$ be two step functions.*

(1)
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(2) If λ is a real number, then:
$$\int_a^b (\lambda f) = \lambda \int_a^b f.$$

(3) If $f \leq g$, then:
$$\int_a^b f \leq \int_a^b g.$$

(4) For all $c \in [a;b]$,
$$\int_a^b f = \int_a^c f + \int_c^b f$$

(5) If f and g are equal everywhere but on a finite number of points, then
$$\int_a^b f = \int_a^b g.$$

Remark 1.12. (1) The points (1) and (2) say that $f \mapsto \int_a^b f$ is a linear form on the space of step functions.
(2) The point (3) says that $f \mapsto \int_a^b f$ is increasing.
(3) In point 4, we slightly abuse notations: $\int_a^c f$ should read $\int_a^c f_{|[a;c]}$ and $\int_c^b f$ should read $\int_a^c f_{|[c;b]}$. This relation is called the Chasles relation.

Proof. Point (1): Let $\sigma = (x_0, x_1, \dots, x_n)$ be a partition of $[a;b]$ adapted to f and g and denote v_i (resp. w_i) the value of f (resp. g) on $]x_i; x_{i+1}[$ for all i in $\{0, \dots, n-1\}$. For all such i, $f+g$ is constant equal to $v_i + w_i$ on $]x_i; x_{i+1}[$, so that $f+g$ is a step function and that σ is adapted to $f+g$ and:

$$\begin{aligned}\int_a^b f = I(f+g,\sigma) &= \sum_{i=0}^{n-1} (v_i + w_i)(x_{i+1} - x_i) \\ &= \sum_{i=0}^{n-1} v_i(x_{i+1} - x_i) + \sum_{i=0}^{n-1} v_i(x_{i+1} - x_i) \\ &= I(f,\sigma) + I(g,\sigma) = \int_a^b f + \int_a^b g.\end{aligned}$$

Point (2): Let $\sigma = (x_0, \dots, x_n)$ be a partition of $[a;b]$ to f and denote v_i the value f on $]x_i;x_{i+1}[$ for all i in $\{0, \dots, n-1\}$. For all such i, λf is constant equal to λv_i on $]x_i;x_{i+1}[$ so that λf is a step function and that σ is adapted to λf. One has:

$$\begin{aligned}\int_a^b \lambda f = I(\lambda f, \sigma) &= \sum_{i=0}^{n-1} \lambda v_i (x_{i+1} - x_i) \\ &= \lambda \sum_{i=0}^{n-1} v_i (x_{i+1} - x_i) = \lambda I(f, \sigma) = \lambda \int_a^b f.\end{aligned}$$

Point (3): Because of the previous points, it is enough to prove that $\int_a^b f \geq 0$ whenever f is a nonnegative step function. Let $f \colon [a;b]$ be nonnegative step function and $\sigma = (x_0, \dots, x_n)$ be a partition of $[a;b]$ to f and denote $v_i \geq 0$ the value f on $]x_i;x_{i+1}[$ for all i in $\{0, \dots, n-1\}$. One has:

$$\begin{aligned}\int_a^b f = I(f, \sigma) &= \sum_{i=0}^{n-1} v_i (x_{i+1} - x_i) \\ &\geq \sum_{i=0}^{n-1} 0 (x_{i+1} - x_i) \geq 0.\end{aligned}$$

where the first inequality comes from the fact that the v_is are nonnegative and that $(x_{i+1} - x_i) \geq 0$.

Point (4): Let $\sigma = (x_0, \dots, x_n)$ be a partition adapted to f. We can suppose that $c = x_j$ for some j in $\{0, 1, \dots, n\}$. Denote by v_i the values of f on $]x_i;x_{i+1}[$. One has:

$$\begin{aligned}\int_a^b f = I(f, \sigma) &= \sum_{i=0}^{n-1} v_i (x_{i+1} - x_i) \\ &= \sum_{i=0}^{j-1} v_i (x_{i+1} - x_i) + \sum_{i=j}^{n-1} v_i (x_{i+1} - x_i) \\ &= I(f_{|[a;c]}, \sigma_1) + I(f_{|[c;b]}, \sigma_2) = \int_a^c f = \int_c^b f,\end{aligned}$$

where $\sigma_1 = (x_0, x_1, \dots, x_j)$ and $\sigma_2 = (x_j, x_{j+1}, \dots, x_n)$ are partitions of $[a;c]$ and $[c;b]$ respectively.

Point (5): Thanks to points (1) and (2) it is enough to prove that $\int_a^b f = 0$ whenever f is a function which equals 0 except for a finite number of points. Let f be such a function and $\sigma = (x_0, \dots, x_n)$ be a partition adapted to f. The values v_i of f on $]x_i;x_{i+1}[$ are all equal to 0 since $]x_i;x_{i+1}[$ contains infinitely many elements. Hence

$$\int_a^b f = I(f, \sigma) = \sum_{i=0}^{n-1} 0 (x_{i+1} - x_i) = 0.$$

□

2. Riemann integration

In this section, we let $f \colon [a;b] \to \mathbb{R}$ be a bounded function and choose m and M in $\mathbb{R}$ such that for all $x \in [a;b]$, $m \leq f(x) \leq M$.

2.1. Definition.

Notation 2.1. Denote

$$\begin{aligned} e(f) &:= \{\underline{f}\colon\ [a;b] \to \mathbb{R} \text{ step function such that } \underline{f} \leq f\} \quad \text{and} \\ E(f) &:= \{\overline{f}\colon\ [a;b] \to \mathbb{R} \text{ step function such that } \overline{f} \geq f\}. \end{aligned}$$

These two sets are non-empty since $(t \mapsto m)$ is in $e(f)$ and $(t \mapsto M)$ is in $E(f)$. Moreover, for all $\underline{f}$ in $e(f)$, $\int_a^b \underline{f} \leq \int_a^b M = M(b-a)$ and for all $\overline{f}$ in $E(f)$, $\int_a^b \overline{f} \geq \int_a^b m = m(b-a)$. This implies that

$$\begin{aligned} I_-(f) &= \sup\left\{\int_a^b \underline{f} \ | \underline{f} \in e(f)\right\} \quad \text{and} \\ I_+(f) &= \inf\left\{\int_a^b \overline{f} \ | \overline{f} \in e(f)\right\}. \end{aligned}$$

are well-defined.

For all $\underline{f}$ in $e(f)$ and $\overline{f}$ in $E(f)$, one has $\underline{f} \leq f \leq \overline{f}$. This implies in particular, that $I_-(f) \leq I_+(f)$.

Definition 2.2. The function f is *Riemann-integrable* or simply *integrable* if $I_-(f) = I_+(f)$. In that case, one writes:

$$\int_a^b f := \int_a^b f(t)\mathrm{d}t := I_-(f).$$

Remark 2.3. Any step function f is integrable since one has: $I_-(f) = I_+(f) = I(f, \sigma)$ for any partition σ adapted to f.

Proposition 2.4. *A function $f\colon [a;b] \to \mathbb{R}$ is integrable if and only if, for all $\epsilon > 0$, there exists $\underline{f}$ in $(e(f))$ and $\overline{f}$ in $E(f)$ such that:*

$$\int_a^b (\overline{f} - \underline{f}) = \int_a^b \overline{f} - \int_a^b \underline{f} \leq \epsilon.$$

Proof. Let us first suppose that f is integrable and denote $J = \int_a^b f$. Let $\epsilon > 0$. Since

$$J = \sup\left\{\int_a^b \underline{f} \ | \underline{f} \in e(f)\right\}$$

there exists $\underline{f}$ in $e(f)$ such that $J - \int_a^b \underline{f} \leq \frac{\epsilon}{2}$. Similarly, since

$$J = \inf\left\{\int_a^b \overline{f} \ | \overline{f} \in E(f)\right\}$$

there exists $\overline{f}$ in $E(f)$ such that $\int_a^b \overline{f} - J \leq \frac{\epsilon}{2}$.

For such an $\underline{f}$ and such an $\overline{f}$, one has:

$$\int_a^b (\overline{f} - \underline{f}) \leq \epsilon.$$

Conversely, let $\epsilon > 0$. We can find $\underline{f}$ in $e(f)$ and $\underline{f}$ in $E(f)$ such that

$$\int_a^b \overline{f} - \int_a^b \underline{f} \leq \epsilon.$$

Since $I_+(f) \leq \int_a^b \overline{f}$ and $I_-(f) \geq \int_a^b \underline{f}$, this implies that

$$I_+(f) - I_-(f) \leq \epsilon.$$

Since this holds for any ϵ, one has $I_+(f) \leq I_-(f)$ and therefore $I_+(f) = I_-(f)$ and f is by definition integrable. $\square$

Example 2.5. Let us prove that the function

$$\begin{array}{rrcl} g = \chi_{\mathbb{Q}\cap[0;1]}\colon & [0;1] & \to & \mathbb{R} \\ & x & \mapsto & \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise,} \end{cases} \end{array}$$

is not integrable.

Let $\underline{g}$ be a step function in $e(g)$ and $\sigma = (x_0, x_1, \ldots, x_n)$ be a partition of $[0;1]$ adapted do $\underline{g}$. For any i in $\{0, 1, \ldots, n-1\}$, $]x_i, x_{i+1}[$ contains an element of $[0;1] \setminus \mathbb{Q}$, so that $\underline{g} \leq 0$ on $]x_i; x_i + 1[$ and finally $\underline{g} \leq 0 on [0,1]$ except for finitely many points, hence $\int_0^1 \underline{g} \leq 0$ and therefore, $I_-(g) \leq 0$.

Simliarly, let $\overline{g}$ be a step function in $E(g)$ and $\sigma = (x_0, x_1, \ldots, x_n)$ be a partition of $[0;1]$ adapted do $\overline{g}$. For any i in $\{0, 1, \ldots, n-1\}$, $]x_i, x_{i+1}[$ contains an element of $[0;1] \cap \mathbb{Q}$, so that $\overline{g} \geq 1$ on $]x_i; x_i + 1[$ and finally $\overline{g} \leq 1$ on $[0,1]$ except for finitely many points, hence $\int_0^1 \overline{g} \geq 1$ and therefore, $I_+(g) \geq 1$.

This proves that $I_-(g) \neq I_+(g)$ and finally that g is not integrable.

2.2. Properties. Recall that $f\colon [a;b] \to \mathbb{R}$ is a function that we assume bounded below and above by m and M respectively.

Proposition 2.6. *If f is integrable then:*

$$m \leq \frac{1}{b-a} \int_a^b f \leq M.$$

PROOF. The function $c_m\colon [a;b] \ni t \mapsto m$ is in $e(f)$ so that $I_-(f) \geq \int_a^b c_m = m(b-a)$. Similarly, function $c_M\colon [a;b] \ni t \mapsto M$ is in $E(f)$ so that $\int_a^b f = I_+(f) \leq \int_a^b c_M = M(b-a)$. $\square$

Proposition 2.7. *Let $f, g\colon [a;b] \to \mathbb{R}$ be two integrable functions.*

1. *The function $f + g$ is integrable and*
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$
2. *If λ is a real number, then λf is integrable and:*
$$\int_a^b (\lambda f) = \lambda \int_a^b f.$$
3. *If $f \geq g$, then:*
$$\int_a^b f \leq \int_a^b g.$$
4. *For all $c \in [a;b]$, $f_{|[a;c]}$ and $f_{|[c;b]}$ are integrable and:*
$$\int_a^b f = \int_a^c f + \int_c^b f$$
5. *If f is integrable and h is equal to f everywhere but on a finite number of points, then h is integrable and*
$$\int_a^b f = \int_a^b g.$$

Remark 2.8. (1) The points (1) and (2) say that the space of integrable functions is a sub $\mathbb{R}$-vector space of the space $\mathbb{R}^{[a;b]}$ of function from $[a;b]$ to $\mathbb{R}$ and that $f \mapsto \int_a^b f$ is a linear form on the space of step functions.

(2) The point (3) says that $f \mapsto \int_a^b f$ is increasing.
(3) In point 4, we slightly abuse notations: $\int_a^c f$ should read $\int_a^c f_{|[a;c]}$ and $\int_c^b f$ should read $\int_a^c f_{|[c;b]}$. This relation is called the Chasles relation.

Proof. The proof makes extensive use of Propositions 1.11 and 2.4.

Point (1): Let $\epsilon > 0$ and let $\underline{f}$ in $e(f)$, $\underline{f}$ in $E(f)$, $\underline{g}$ in $e(g)$ and $\overline{g}$ in $E(g)$ such that:

$$\int_a^b (\overline{f} - \underline{f}) \leq \frac{\epsilon}{2} \quad \text{and}$$

$$\int_a^b (\overline{g} - \underline{g}) \leq \frac{\epsilon}{2}.$$

Then $\underline{f} + \underline{g} \in e(f+g)$, $\overline{f} + \overline{g} \in E(f+g)$ and:

$$\int_a^b ((\overline{f} + \overline{g}) - (\underline{f} + \underline{g})) = \int_a^b (\overline{f} - \underline{f}) + \int_a^b (\overline{g} - \underline{g}) \leq \epsilon$$

and $f + g$ is therefore integrable.

Moreover, for any $\underline{f}$ in $e(f)$ and $\underline{g}$ in $e(g)$, $\underline{f} + \underline{g}$ is in $e(f+g)$, so that one has:

$$\int_a^b \underline{f} + \int_a^b \underline{g} = \int_a^b (\underline{f} + \underline{g}) \leq I_-(f+g) = \int_a^b f + g.$$

Taking suprema on the left-and side, this implies that:

$$\int_a^b f + \int_a^b g = I_-(f) + I_-(g) \leq \int_a^b f + g.$$

Similarly, for any $\overline{f}$ in $E(f)$ and $\overline{g}$ in $E(g)$, $\overline{f} + \overline{g}$ is in $E(f+g)$, so that one has:

$$\int_a^b \overline{f} + \int_a^b \overline{g} = \int_a^b (\overline{f} + \overline{g}) \geq I_+(f+g) = \int_a^b f + g.$$

Taking infima on the left-and side, this implies that:

$$\int_a^b f + \int_a^b g = I_+(f) + I_+(g) \geq \int_a^b f + g.$$

Finally

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Point (2): We treat three cases: $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$.

If $\lambda = 0$, λf is the null function, which is a step function and therefore integrable, moreover, $\int_a^b \lambda f = 0 = \lambda \int_a^b f$.

If $\lambda > 0$. Let $\epsilon > 0$ and let $\underline{f}$ in $e(f)$ and $\underline{f}$ in $E(f)$ such that:

$$\int_a^b (\overline{f} - \underline{f}) \leq \frac{\epsilon}{\lambda}.$$

Then $\lambda \underline{f}$ is in $e(\lambda f)$, $\lambda \overline{f}$ is in $\in E(f)$ and:

$$\int_a^b (\lambda \overline{f} - \lambda \underline{f}) = \lambda \int_a^b (\overline{f} - \underline{f}) \leq \epsilon$$

and λf is therefore integrable.

Moreover, for any $\underline{f}$ is in $e(f)$, $\lambda \underline{f}$ is in $e(\lambda f)$, so that one has:

$$\lambda \int_a^b \underline{f} = \int_a^b (\lambda \underline{f}) \leq I_-(\lambda f) = \int_a^b \lambda f.$$

Taking supremum on the left-and side, this implies that:

$$\lambda \int_a^b f = \lambda I_-(f) \leq \lambda \int_a^b f.$$

Similarly, for any $\overline{f} \in E(f)$, $\lambda\overline{f} \in E(f+g)$, so that one has:

$$\lambda \int_a^b \overline{f} = \int_a^b \lambda\overline{f} \geq I_+(\lambda f) = \int_a^b \lambda f.$$

Taking infimum on the left-and side, this implies that:

$$\lambda \int_a^b f = \lambda I_+(f) \geq \int_a^b \lambda f.$$

Finally

$$\lambda \int_a^b = \int_a^b \lambda f.$$

If $\lambda < 0$. Let $\epsilon > 0$ and let $\underline{f}$ in $e(f)$ and $\underline{f}$ in $E(f)$ such that:

$$\int_a^b (\overline{f} - \underline{f}) \leq -\frac{\epsilon}{\lambda}.$$

Then $\lambda\underline{f}E(\lambda f)$, $\lambda\overline{f} \in e(f)$ and:

$$\int_a^b (\lambda\underline{f} - \lambda\overline{f}) = -\lambda \int_a^b (\overline{f} - \underline{f}) \leq \epsilon$$

and λf is therefore integrable. Since $\lambda f + (-\lambda) f$ is equal to 0, we can conclude using point 1, that

$$\int_a^b \lambda f = -\int_a^b (-\lambda f) = -(-\lambda)\int_a^b f = \lambda \int_a^b f.$$

Point 3: Because of the two previous points, it is enough to prove that if f is integrable and nonnegative, then $\int_a^b f \geq 0$. This is follows from Proposition 8.

Point 4: Let $\epsilon > 0$, $\underline{f}$ be in $e(f)$, $\overline{f}$ be in $E(f)$ such that

$$\int_a^b (\overline{f} - \underline{f}) \leq \epsilon.$$

The function $\underline{f}_{|[a;c]}$ is in $e(f_{|[a;c]})$, the function $\overline{f}_{|[a;c]}$ is in $E(f_{|[a;c]})$, the function $\underline{f}_{|[c;b]}$ is in $e(f_{|[c;b]})$, and the function $\overline{f}_{|[c;b]}$ is in $E(f_{|[c;b]})$. Moreover:

$$\epsilon \geq \int_a^b (\overline{f} - \underline{f}) = \int_a^c (\overline{f} - \underline{f}) + \int_c^b (\overline{f} - \underline{f})$$

and the two terms on the right-hand side are nonnegative, so that:

$$\int_a^c (\overline{f} - \underline{f}) \leq \epsilon \quad \text{and} \quad \int_c^b (\overline{f} - uf) \leq \epsilon.$$

This proves that $f_{|[a;c]}$ and $f_{[c;b]}$ are integrable.

Using the same trick as for point 1 (proving two inequalities), we obtain that:

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Point 5: Because of points (1) and (2) it is enough to prove that $\int_a^b f = 0$ whenever f is a function which equals 0 except for a finite number of points, but we already prove this in the proof of Proposition 1.11(5). □

Given a any function $f\colon [a;b] \to \mathbb{R}$, we define $f_+, f_-\colon [a;b] \to \mathbb{R}$ as follows:

$$\begin{array}{rrcl} f_+\colon & [a;b] & \to & \mathbb{R} \\ & x & \mapsto & \max(0, f(x)) \end{array} \quad \text{and} \quad \begin{array}{rrcl} f_-\colon & [a;b] & \to & \mathbb{R} \\ & x & \mapsto & \max(0, -f(x)) \end{array}$$

In particular, one has $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Lemma 2.9. *If $f\colon [a;b] \to \mathbb{R}$ is integrable, then both f_+ and f_- are integrable and*

$$\int_a^b f = \int_a^b f_+ - \int_a^b f_- \qquad \int_a^b |f| = \int_a^b f_+ + \int_a^b f_- \qquad .$$

Proof. It is enough to prove that f_+ is integrable, the rest follows from Proposition 2.7. Let $\epsilon > 0$, $g \in e(f)$ and $h \in E(f)$ such that

$$\int_a^b (h-g) \leq \epsilon.$$

We us the same decomposition for g and h: $g = g_+ - g_-$ and $h = h_+ - h_-$. One has $g_+ \in e(f_+)$ and $h_+ \in E(f_+)$, so that $g_+ \leq h_+$ and therefore $h_- \leq g_-$. This implies that:

$$\int_a^b (h_+ - g_+) = \int_a^b (h-g) + \int_a^b (h_- - g_-) \leq \int_a^b (h-g) \leq \epsilon$$

which proves that f_+ is integrable. □

Proposition 2.10 (triangular inequality)**.** *If $f\colon [a;b] \to \mathbb{R}$ is integrable, then so is $|f|$ and*

$$\int_a^b |f| \geq \left| \int_a^b f \right|$$

Proof. With the notations introduce earlier, if f is integrable, then so are f_+ and f_- and therefore as well $|f|$. Moreover, one has:

$$\int_a^b |f| = \int_a^b f_+ + \int_a^b f_- \geq \begin{cases} \displaystyle\int_a^b f_+ \geq \int_a^b f \\ \displaystyle\int_a^b f_- \geq -\int_a^b f \end{cases} .$$

□

Proposition 2.11. *If $f, g\colon [a;b] \to \mathbb{R}$ are integrable then so is fg.*

Proof. Using the decompositions $f = f_+ - f_-$ and $g = g_+ - g_-$, we may suppose that f and g are both nonnegative. Since both f and g are integrable, they are bounded. Let $M \geq 1$ such that $f(x)$ and $g(x)$ are smaller than M for all x in $[a;b]$. Let $\epsilon > 0$ and define $\epsilon' = \frac{\epsilon}{2M}$. Let $\underline{f}$ in $e(f)$, $\overline{f}$ in $E(f)$, $\underline{g}$ in $e(g)$ and $\overline{g}$ in $E(g)$ such that:

$$\int_a^b (\overline{f} - \underline{f}) \leq \epsilon' \quad \text{and} \quad \int_a^b (\overline{g} - \underline{g}) \leq \epsilon'.$$

Replacing $\underline{f}$ and $\underline{g}$ by $\underline{f}_+$ and $\underline{g}_+$ respectively, we may suppose that $\underline{f}$ and $\underline{g}$ are both nonnegative. Similarly, replacing $\overline{f}$ and $\underline{g}$ by $\min(\underline{f}, M)$ and $\min(ug, M)$ respectively, we may suppose that $\underline{f}$ and $\underline{g}$ are both bounded by M.

The function $\underline{fg}$ is in $e(fg)$ and the function $\overline{f}\overline{g}$ is in $E(fg)$. One has:

$$\begin{aligned}
\int_a^b (\overline{f}\overline{g} - \underline{fg}) &= \int_a^b (\overline{f}\overline{g} - \overline{f}\underline{g} + \overline{f}\underline{g} - \underline{fg}) \\
&= \int_a^b \overline{f}(\overline{g} - -\underline{g}) + \int_a^b \underline{g}(\overline{f} - \underline{f}) \\
&\leq \int_a^b M(\overline{g} - \underline{g}) + \int_a^b M(\overline{f} - \underline{f}) \\
&\leq M\epsilon' + M\epsilon' = \epsilon.
\end{aligned}$$

This proves that fg is integrable. ☐

Remark 2.12. In general, $\int_a^b fg \neq \left(\int_a^b f\right)\left(\int_a^b g\right)$.

Proposition 2.13. *If f is integrable and that $\inf_{x\in[a;b]} |f(x)| > 0$, then $\frac{1}{f}$ is integrable.*

Proof. Using the decomposition, $f = f_+ - f_-$, one can suppose that f is positive. Let m strictly positive such that $m \leq f(x)$ for all x in $[a;b]$. Let $\epsilon > 0$ and define $\epsilon' = m^2\epsilon$. Let $\underline{f}$ be in $e(f)$ and $\overline{f}$ in $E(f)$ such that

$$\int_a^b (\overline{f} - \underline{f}) \leq \epsilon'$$

The function $\frac{1}{\underline{f}}$ is in $E(\frac{1}{f})$ and the function $\frac{1}{\overline{f}}$ is in $e(\frac{1}{f})$. One has:

$$\int_a^b (\frac{1}{\underline{f}} - \frac{1}{\overline{f}}) = \int_a^b \frac{\overline{f} - \underline{f}}{\underline{f}\overline{f}} =\leq \int_a^b \frac{\overline{f} - \underline{f}}{m^2} \leq \frac{\epsilon'}{m^2} = \epsilon.$$

which proves that $\frac{1}{f}$ is integrable. ☐

3. Two classes of integrable functions

We will show that continuous function and monotone functions are integrable.

3.1. Intergrability of continuous function. Before proving that continuous function on a closed interval are integrable, we briefly recall an important concept.

Definition 3.1. A function $f\colon [a;b] \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists δ_ϵ, such that for all x, y in $[a;b]$, if $|x - y| \leq \delta_\epsilon$, then $|f(x) - f(y)| \leq \epsilon$.

Theorem 3.2 (Heine, see first semester)**.** *If $f\colon [a;b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.*

The "hidden hypothesis" in this theorem, is that the domain of f is a closed interval.

Theorem 3.3. *If $f\colon [a;b] \to \mathbb{R}$ is continuous, then it is integrable.*

The converse is obviously not true, since most step functions are not continuous, yet they are all integrable.

Proof. Consider $f\colon [a;b] \to \mathbb{R}$ a continuous function, it is therefore uniformly. Let $\epsilon > 0$ and apply Heine's theorem with $\epsilon' := \frac{\epsilon}{b-a}$, we can then pick δ such that for all x, y in $[a;b]$, if $|x-y| \leq \delta$, $|f(x)-f(y)| \leq \epsilon'$. Let $\sigma = (x_0, x_1, \dots, x_n)$ be a partition of $[a;b]$ such that for all i in $\{0, 1, \dots, n-1\}$, $x_{i+1} - x_i \leq \delta$. Define

$$\begin{array}{rrcl} g\colon & [a;b] & \to & \mathbb{R} \\ & t & \mapsto & \begin{cases} \min_{u\in[x_i;x_{i+1}]} f(u) & \text{if } t \in [x_i; x_{i+1}[\\ f(b) & \text{if } t = b. \end{cases} \quad \text{and} \\ h\colon & [a;b] & \to & \mathbb{R} \\ & t & \mapsto & \begin{cases} \max_{u\in[x_i;x_{i+1}]} f(u) & \text{if } t \in [x_i; x_{i+1}[\\ f(b) & \text{if } t = b. \end{cases} \end{array}$$

By their very definition, $g \in e(f)$ and $h \in E(f)$. For all $i \in \{0, 1, \dots, n\}$, one has:

$$\max_{u\in[x_i;x_{i+1}]} f(u) - \min_{u\in[x_i;x_{i+1}]} f(u) \leq \frac{\epsilon}{b-a},$$

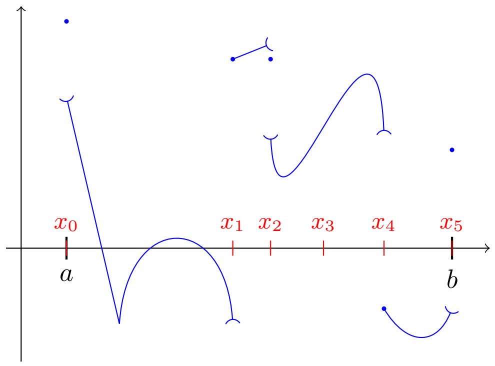
so that

$$\int_a^b (h-g) \leq \sum_{i=0}^{n-1} \frac{\epsilon}{b-a}(x_{i+1}-x_i) = \frac{\epsilon}{b-a} \sum_{i=0}^{n-1} (x_{i+1}-x_i) = \epsilon$$

which proves that f is integrable. □

Definition 3.4. A function $f\colon [a;b] \to \mathbb{R}$ is piecewise continuous if their exists a partition $\sigma = (x_0, x_1, \dots, x_{n+1})$ of $[a;b]$ and continuous functions $f_i\colon [x_i; x_{i+1}] \to \mathbb{R}$ for i in $\{0, 1, \dots, n-1\}$, such that $f_{i|]x_i;x_{i+1}[} = f_{|]x_i;x_{i+1}[}$.

Example 3.5.



Remark 3.6. If $f\colon [a;b] \to \mathbb{R}$ is piecewise continuous, it is the sum of a step function and of a continuous function.

Corollary 3.7. *If $f\colon [a;b]$ is piecewise linear, then it is integrable.*

3.2. Integrability of monotone functions.

Proposition 3.8. *If $f\colon [a;b] \to \mathbb{R}$ is monotone, then f is integrable.*

Proof. Let $f\colon [a;b] \to \mathbb{R}$ be a monotone function. Eventually replacing f by $-f$, one can assume that f is increasing. Let $\epsilon > 0$ and n a positive integer greater than or equal to $\frac{(b-a)(f(b)-f(a))}{\epsilon}$. Define

$$\begin{array}{rrcl} g\colon & [a;b] & \to & \mathbb{R} \\ & t & \mapsto & \begin{cases} f(a+i\frac{b-a}{n}) & \text{if } t\in[x_i;x_{i+1}[\\ f(b) & \text{if } t=b. \end{cases} \quad \text{and} \\ h\colon & [a;b] & \to & \mathbb{R} \\ & t & \mapsto & \begin{cases} f(a+(i+1)\frac{b-a}{n}) & \text{if } t\in[x_i;x_{i+1}[\\ f(b) & \text{if } t=b. \end{cases} \end{array}$$

By construction, $g\in e(f)$ and $g\in E(f)$ (because f is increasing). For both of them the partition $(a, a+\frac{b-a}{n}, a+2b-an, \ldots, b)$ is adapted and

$$\begin{aligned} \int_a^b h-g &= \sum_{i=0}^{n-1}\frac{b-a}{n}\left(f\left(a+(i+1)\frac{b-a}{n}\right)-f\left(a+i\frac{b-a}{n}\right)\right) \\ &= \frac{b-a}{n}\sum_{i=0}^{n-1}\left(f\left(a+(i+1)\frac{b-a}{n}\right)-f\left(a+i\frac{b-a}{n}\right)\right) \\ &= \frac{(b-a)(f(b)-f(a))}{n}\le\epsilon, \end{aligned}$$

which proves that f is integrable. □

4. Riemann sums

As before, $a<b$ are two reals and $f\colon [a;b]\to\mathbb{R}$ is a function.

Definition 4.1. (1) A *pointed partition* of $[a;b]$ is a pair (σ,t) with $\sigma=(x_0,x_1,\ldots,x_n)$ a partition of $[a;b]$ and $t=(t_0,t_1,\ldots,t_{n-1})$ an n-tuple of reals such that for all i in $\{0,1,\ldots,n-1\}$, t_i is in $[x_i;x_{i+1}]$.

(2) The *Riemann sum* of f associated with a pointed division (σ,t) is the quantity, denoted by $\Sigma(f,\sigma,t)$ defined by:

$$\Sigma(f,\sigma,t)=\sum_{i=1}^{n-1}f(f_i)(x_{i+1}-x_i)$$

Remark 4.2. In practice, we will often deal with regular partition, that is partitions $\sigma=(x_0,x_1,\ldots,x_n)$ for which $x_{i+1}-x_i=\frac{b-a}{n}$, namely:

$$x_0=a,\quad x_1=a+\frac{b-a}{n},\ldots\quad ,x_i=a+i\frac{b-a}{n},\ldots,\quad x_n=b.$$

Pour the "pointing data" t, we will often use $t_i=x_i$, $t_i=x_{i+1}$ or $t_i=\frac{x_i+x_{i+1}}{2}$.

Recall that if $\sigma=(x_0,x_1,\ldots,x_n)$ is a partition of $[a;b]$, the mesh of σ, denoted $\delta(\sigma)$ is the quantity $\max_{1\le i\le n-1}x_{i+1}-x_i$.

Proposition 4.3. *Suppose that $f\colon [a;b]\to\mathbb{R}$ is integrable, then for all $\epsilon>0$, there exists $\delta>0$ such that for any pointed partition (δ,t) for which $\delta(\sigma)<\delta$,*

$$\left|\int_a^b f-\Sigma(f,\sigma,t)\right|\le\epsilon.$$

Proof. Let $\epsilon>0$ and $f\colon [a;b]\to\mathbb{R}$ an integrable function. Since f is integrable, $|f|$ and denote M a positive upper bound of $|f|$. Let $\underline{f}$ be in $e(f)$ and $\overline{f}$ be in $E(f)$ such that

$$\int_a^b(\overline{f}-\underline{f})\le\frac{\epsilon}{2}$$

and denote $\sigma' = (x'_0, x'_1, \ldots, x'_{n'})$ a partition of $[a;b]$ adapted to both $\underline{f}$ and $\overline{f}$. One has:

$$\int_a^b (f - \underline{f}) \leq \frac{\epsilon}{2} \quad \text{and} \quad \int_a^b (\overline{f} - f) \leq \frac{\epsilon}{2}$$

Define $\delta = \frac{\epsilon}{8n'M}$. We will prove that this δ, any pointed partition (δ, t) for which $\delta(\sigma) < \delta$,

$$\left| \int_a^b f - \Sigma(f, \sigma, t) \right| \leq \epsilon.$$

Let (σ, t) be such a pointed partition, denote $\sigma'' = (x''_0, x''_1, \ldots, x''_{n''})$, the partition obtained by inserting in σ the points of σ' which are missing. Note that there are at most $n' - 1$ such missing points and that at most $2n'$ intervals $[x''_i, x''_{i+1}]$ for which one (or two) of the two ends are in σ'. Define the step function

$$\begin{array}{rccc} g\colon & [a;b] & \to & \mathbb{R} \\ & x & \mapsto & \begin{cases} g(t_i) & \text{if } x \in [x_i; x_{i+1}[, \\ g(b) & \text{if } x = b. \end{cases} \end{array}$$

By its very construction, one has $\Sigma(f, \sigma, t) = \int_a^b g$. The partition σ'' is adapted to g and $\underline{f}$. If none of the two ends of $[x''_i, x''_{i+1}]$ is in σ', then $g_{|[x''_i;x''_{i+1}} \geq \underline{f}_{|[x''_i;x''_{i+1}[}$. Otherwise, $g_{|[x''_i;x''_{i+1}} \geq \underline{f}_{|[x''_i;x''_{i+1}[} - 2M$. Since σ'' is finer than σ, its mesh is smaller than δ. One has:

$$\begin{aligned} \Sigma(f, \sigma, t) = \int_a^b g &\geq \int_a^b \underline{f} - 4\delta M n' \\ &\geq \int_a^b \underline{f} - \frac{\epsilon}{2} \\ &\geq \int_a^b f - \epsilon \end{aligned}$$

Arguing similarly with $\overline{f}$ (one needs to change the direction of the inequality), one gets

$$\begin{aligned} \Sigma(f, \sigma, t) = \int_a^b g &\leq \int_a^b \overline{f} + 4\delta M n' \\ &\leq \int_a^b \overline{f} + \frac{\epsilon}{2} \\ &\leq \int_a^b f + \epsilon \end{aligned}$$

So that finally

$$\left| \int_a^b f - \Sigma(f, \sigma, t) \right| \leq \epsilon.$$

$\square$

Corollary 4.4. *If $f\colon [a;b] \to \mathbb{R}$ is integrable, then:*

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(a+i\frac{b-a}{n}) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(a+(i+1)\frac{b-a}{n})\\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(a+\frac{(2i+1)(b-a)}{2n})\\ &= \int_a^b f.\end{aligned}$$

Proof. Each sequence corresponds to Riemann sums of partitions of meshes $\frac{b-a}{n}$, since this quantity converges to 0 as n goes to ∞, we can apply 4.3 ☐

Example 4.5. Applying Corollary to $f\colon [1,2] \ni x \mapsto \frac{1}{x} \in \mathbb{R}$, we get immediately that:

$$\lim_{n\to 0} \frac{1}{n}\sum_{i=0} \frac{1}{1+\frac{i}{n}} = \int_1^2 f = \int_1^2 \frac{\mathrm{d}x}{x} = \ln 2.$$

The last identity should not be a surprise, but at this stage we do not have the necessary results to assert it. See Section 5.

5. Fundamental theorem of analysis

Proposition 5.1 (Average realization)**.** *Let $f\colon [a;b] \to \mathbb{R}$ a continuous function. Then there exists c in $[a;b]$ such that:*

$$f(c) = \frac{1}{b-a}\int_a^b f.$$

Proof. Since f is continuous it is bounded and reaches its minimum and maximum. Let $m = \min_{[a;b]} f$ and $M = \max_{[a;b]} f$. Recall (from Proposition 2.6) that:

$$m \le \frac{1}{b-a}\int_a^b f \le M.$$

The intermediate value theorem applied to f, asserts that there exists a c in $[a;b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f.$$

☐

Definition 5.2. Let $f\colon [a;b] \to \mathbb{R}$ be a function, a function $F\colon [a;b] \to \mathbb{R}$ is a *primitive* of f if, F is continuous on $[a;b]$, differentiable on $]a;b[$ and if for all x in $]a;b[$, $F'(x) = f(x)$.

Remark 5.3. If $F\colon [a;b] \to \mathbb{R}$ is a primitive of $f\colon [a;b] \to \mathbb{R}$ and c is a real number, $F + c$ is another primitive of f. Conversely, if F and G are two primitive of f, then $F - G$ is constant.

Theorem 5.4 (Fundamental theorem of analysis, part 1)**.** *Let $f\colon [a;b] \to \mathbb{R}$ be a continuous function, then the function:*

$$\begin{array}{cccc} F\colon & [a;b] & \to & \mathbb{R}\\ & x & \mapsto & \int_a^b f \end{array}$$

is a primitive of f for which $F(a) = 0$.

PROOF. The fact that $F(a) = 0$ is clear. Fix x in $[a; b]$, for y in $[a; b]$, one has:

$$F(y) = \int_a^y f = \int_a^x f + \int_x^y f = F(x) + (x - y)f(c_y)$$

for a given c_y in $[x; y]$ because of Proposition 5.1. Hence

$$F(y) = F(x) + (x - y)f(x) + (x - y)(f(c_y) - f(x)).$$

When y tends to x, c_y tends to x and, since f is continuous, $f(c_y) - f(x)$ tends to 0. This proves that F is differentiable in x and that $F'(x) = f(x)$. □

Corollary 5.5 (Fundamental theorem of analysis, part 2)**.** *Let* $f\colon [a; b] \to \mathbb{R}$ *be a continuous function and* F *be a primitive of* f*, then,*

$$\int_a^b f = F(b) - F(a).$$

PROOF. Since

$$\begin{array}{rccc} G\colon & [a; b] & \to & \mathbb{R} \\ & x & \mapsto & \int_a^x f \end{array}$$

is a primitive of f, $G - F$ is constant, so that $(G - F)(b) = (G - F)(a)$, and hence $F(b) - F(a) = G(b) - G(a) = G(b) = \int_a^b f$. □

CHAPTER 2

Computing primitives

Primitives are an essential tool for computing integrals.

1. Primitives of usual functions

In the following table, we give a list of primitives of classical functions. Of course primitives are all defined up to an additive constant. In this tables, a and b are arbitrary real numbers (sometimes statisfying conditions).

Function	Primitive	Definition domain
$t \mapsto 1/t$	$t \mapsto \log(\lvert t\rvert)$	$\mathbb{R}^*$
$t \mapsto (t+a)^{-1}$	$t \mapsto \log\lvert t+a\rvert$	$\mathbb{R}$ or $\mathbb{R} \setminus \{-a\}$
$t \mapsto (t+a)^b, b \neq -1$	$t \mapsto \dfrac{(t+a)^{b+1}}{b+1}$	$\mathbb{R} \setminus \{-a\}$
$t \mapsto \cos(at), a \neq 0$	$t \mapsto \frac{1}{a}\sin(at)$	$\mathbb{R}$
$t \mapsto \sin(at), a \neq 0$	$t \mapsto -\frac{1}{a}\cos(at)$	$\mathbb{R}$
$t \mapsto e^{at}, a \neq 0$	$t \mapsto e^{at}/a$	$\mathbb{R}$
$t \mapsto \cosh(at), a \neq 0$	$t \mapsto \sinh(at)/a$	$\mathbb{R}$
$t \mapsto \sinh(at), a \neq 0$	$t \mapsto \cosh(at)/a$	$\mathbb{R}$
$t \mapsto a^t, a > 0, a \neq 1$	$t \mapsto a^t/\log(a)$	$\mathbb{R}$
$t \mapsto 1+\tan^2(t) = 1/\cos^2(t)$	$t \mapsto \tan(t)$	$\mathbb{R} \setminus \{\pi/2 + k\pi \mid k \in \mathbb{Z}\}$
$t \mapsto 1+\cot^2(t) = 1/\sin^2(t)$	$t \mapsto -\cot(t)$	$\mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$
$t \mapsto 1/\cosh^2(t)$	$t \mapsto \tanh(t)$	$\mathbb{R}$
$t \mapsto 1/\sinh^2(t)$	$t \mapsto -\text{cotanh}(t)$	$\mathbb{R}^*$
$t \mapsto \dfrac{1}{1+t^2}$	$t \mapsto \arctan(t)$	$\mathbb{R}$
$t \mapsto \dfrac{1}{\sqrt{1-t^2}}$	$t \mapsto \arcsin(t)$	$[-1,1]$
$t \mapsto \dfrac{1}{\sqrt{t^2-1}}$	$t \mapsto \log\lvert t+\sqrt{t^2-1}\rvert$	$]-\infty,-1[\cup]1,+\infty[$
$t \mapsto \dfrac{1}{\sqrt{1+t^2}}$	$t \mapsto \log(t+\sqrt{1+t^2})$	$\mathbb{R}$
$t \mapsto \ln\lvert t\rvert$	$t \mapsto t\ln\lvert t\rvert - t$	$\mathbb{R}$

2. Integration by parts

Proposition 2.1 (Integration by part). *Let $u, v\colon [a;b] \to \mathbb{R}$ be two function of class C^1 on $[a;b]$, then*

$$\int_a^b uv' = uv(b) - uv(a) - \int_a^b u'v \tag{1}$$

Proof. The function $uv\colon [a;b] \to \mathbb{R}$ is a primitive of $uv' + u'v$, hence

$$uv(b) - uv(a) = \int_a^b (uv' + u'v).$$

□

Notation 2.2. If $f\colon [a;b] \to \mathbb{R}$, we write $[f]_a^b := f(b) - f(a)$, with this notation, identity (2) becomes:

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v. \tag{2}$$

Example 2.3. For $n \in \mathbb{Z}$, define:

$$\begin{array}{cccc} f_n\colon & \mathbb{R}_{>0} & \to & \mathbb{R} \\ & t & \mapsto & t^n \ln t. \end{array}$$

For $n \neq -1$, one has:

$$\begin{aligned} \int_1^b f_n = \int_1^b t^n \ln t \mathrm{d}t = \left[\frac{t^{n+1}}{n+1}\ln(t)\right]_1^b - \int_1^b \frac{t^{n+1}}{(n+1)t}\mathrm{d}t \\ = \frac{b^{n+1}}{n+1} - \frac{b^{n+1}}{(n+1)^2} + \frac{1}{(n+1)^2} \end{aligned}$$

If $n = -1$, one has:

$$\int_1^b f_{-1} = \int_1^b \frac{\ln(t)}{t}\mathrm{d}t = \left[\ln(t)^2\right]_1^b - \int_1^b \frac{\ln(t)}{t}\mathrm{d}t$$

so that $\int_1^b \frac{\ln(t)}{t}\mathrm{d}t = \frac{1}{2}\ln(b)^2$.

(1) One ca compute:

$$\begin{aligned} \int_0^b \arctan(t)\mathrm{d}t &= [t\arctan(t)]_0^b - \int_0^b \frac{t}{1+t^2}\mathrm{d}t \\ &= b\arctan(b) - \frac{1}{2}\ln(1+b^2). \end{aligned}$$

3. Change of variables

Proposition 3.1 (Change of variables)**.** *Let I and J be two intervals of positive length and $\varphi\colon J \to I$ a function of class C^1. Let $f\colon I \to \mathbb{R}$ be a caontiuous function and $a \neq b$ two element of J, then:*

(1)

$$\int_{\varphi(a)}^{\varphi(b)} f = \int_a^b ((f \circ \varphi)\varphi').$$

(2) If φ is bijective and a', b' are two elements of I, then:

$$\int_{a'}^{b'} f = \int_{\varphi^{-1}(a')}^{\varphi^{-1}(b')} (f \circ \varphi)\varphi'.$$

Proof. Let F be a primitive of f. Then $\mathcal{F} \circ \varphi$ is a primitive of $(f \circ \varphi)\varphi'$, so that:

$$\int_a^b ((f \circ \varphi)\varphi') = (F \circ \varphi)(b) - (F \circ \varphi)(a) = F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f.$$

The second statement is obvious, when setting $a' = \varphi(a)$ and $b' = \varphi(b)$. □

In practice, one often writes “$u = \psi(t)$” and one replaces “$\mathrm{d}u$” by “$\psi'(t)\mathrm{d}t$”.

Example 3.2. Let us try to find a primitive of:

$$\begin{array}{cccc} f\colon & \mathbb{R} & \to & \mathbb{R} \\ & t & \mapsto & \frac{1}{(1+t^2)\sqrt{1+t^2}}. \end{array}$$

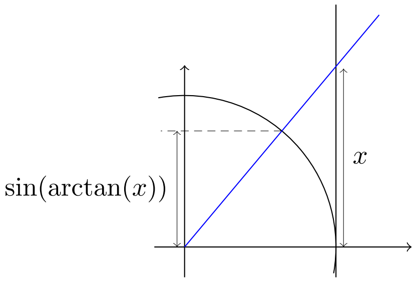
We set "$t = \tan(u)$", so that "$\mathrm{d}t = (1+\tan(u)^2)\mathrm{d}u = (1+t^2)\mathrm{d}u$" and therefore $\frac{\mathrm{d}t}{1+nt^2} = \mathrm{d}u$, using this change of variables, we get:

$$\begin{aligned} \int_a^b f &= \int_a^b \frac{1}{(1+t^2)\sqrt{1+t^2}}\mathrm{d}t = \int_{\arctan(a)}^{\arctan(b)} \frac{\mathrm{d}u}{\frac{1}{1+\tan(u)^2}} \\ &= \int_{\arctan(a)}^{\arctan(b)} |\cos(u)|\mathrm{d}u \int_{\arctan(a)}^{\arctan(b)} \cos(u)\mathrm{d}u \\ &= \sin(\arctan(b)) - \sin(\arctan(a)). \end{aligned}$$

Finally, a primitive of f is given by

$$\begin{array}{cccc} F\colon & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & \sin\circ\arctan(x) = \frac{x}{\sqrt{1+x^2}}. \end{array}$$

The last identity can obtained geometrically.



4. Integration and Taylor approximation

Let I be a non empty open interval.

Theorem 4.1 (Taylor formula, integral form)**.** *Let a and b in I and $f\colon I \to \mathbb{R}$ a function of class C^n for $n \in \mathbb{N}^*$, then:*

$$f(b) = \sum_{k=1}^{n-1} f^{(k)}(a)\frac{(b-a)^k}{k!} + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\mathrm{d}t.$$

Proof. Let us fix a and $b \in I$. We prove this by induction. For $n \in \mathbb{N}^*$, we set:

$$\mathcal{H}_n = \text{“}\forall f \in C^k(I,\mathbb{R}), \quad f(b) = \sum_{k=1}^{n-1} f^{(k)}(a)\frac{(b-a)^k}{k!} + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\mathrm{d}t.\text{”}$$

Initialization: For $n = 1$, $\mathcal{H}_1$ means:

$$\text{“}\forall f \in C^1(I,\mathbb{R}), \quad f(b) = f(a) + \int_a^b f\text{”}$$

which is true, since f is a primitive of f'.

Induction step: Let $n \in \mathbb{N}^*$ such that $\mathcal{H}_n$ is true. Let $f\colon I \to \mathbb{R}$ of class C^{n+1}. In particular f is of class C^n and we can use $\mathcal{H}_n$, so that one has:

$$f(b) = \sum_{k=1}^{n-1} f^{(k)}(a)\frac{(b-a)^k}{k!} + \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\mathrm{d}t. \tag{3}$$

The function $f^{(n)}$ is of class C^1, so that we can integrate by part the last term:

$$\begin{aligned}\int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\mathrm{d}t &= -\left[\frac{(b-t)^n}{n!} f^{(n)}(t)\right]_a^b + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t \\ &= \frac{(b-a)^n}{n!} f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t\end{aligned}$$

Plugging this into (3), we obtain:

$$f(b) = \sum_{k=1}^{n} f^{(k)}(a)\frac{(b-a)^k}{k!} + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t.$$

so that $\mathcal{H}_n$ is true. Finally, by induction, for all $n \in \mathbb{N}^*$, $\mathcal{H}_n$ is true. □

We can recover the Taylor formula we already know:

Corollary 4.2. *If $f\colon I \to \mathbb{R}$ is a function of class C^n and $b \in I$, then for $a \to b$:*

$$f(b) = \sum_{k=1}^{n-1} f^{(k)}(a)\frac{(b-a)^k}{k!} + o((b-a)^{n-1}).$$

This means that there exists a function $\epsilon\colon I \to \mathbb{R}$ such that $\epsilon(a) \overset{a\to b}{\longrightarrow} 0$ for which the following holds

$$f(b) - \sum_{k=1}^{n-1} f^{(k)}(a)\frac{(b-a)^k}{k!} = (b-a)^{n-1}\epsilon(a).$$

Proof. We use Theorem 4.1 and obtain that:

$$f(b) - \sum_{k=1}^{n-1} f^{(k)}(a)\frac{(b-a)^k}{k!} = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\mathrm{d}t = (b-a)\frac{(b-c_a)^{n-1}}{(n-1)!} f^{(n)}(c_a)$$

for a given c_a between a and b. We obtain the result by defining:

$$\epsilon(a) = (b-a)\frac{(b-c_a)^{n-1}}{(b-a)^n - 1} f^{(n)}(c_a).$$

□

Definition 4.3. Let a in I and $f\colon I \to \mathbb{R}$ a function. We say that f admits a *Taylor approximation of order n at a* if there exists reals $c_0, \dots, c_n$ such that for $t \longrightarrow a$,

$$f(t) = \sum_{k=0}^{n} c_k(t-a)^k + o((t-a)^n).$$

Remark 4.4.
(1) A function f admits a Taylor approximation of order 0 in a if and only if it is continuous.
(2) a function f admits a Taylor approximation of order 1 in a if and only if it is differentiable.
(3) If f is of class C^n, then f admits a Taylor approximation of order n (the converse it not necessarily true, see Remark 4.6).

Proposition 4.5. *Let a in I and $f\colon I \to \mathbb{R}$ a continuous function which admits a Taylor approximation around a at the order n: there exists $c_0, c_1, \dots, c_n$ such that for $t \longrightarrow a$,*

$$f(t) = \sum_{k=0}^{n} c_k(t-a)^k + o((t-a)^n).$$

Then for any primitive F of f, one has for $t \longrightarrow a$:

$$F(t) = F(a) + \sum_{k=0}^{n} c_k \frac{(t-a)^{k+1}}{k+1} + o((t-a)^{n+1}).$$

In other words, one can integrate Taylor approximation.

Proof. One has for $t \longrightarrow a$,

$$f(t) - \sum_{k=0}^{n} c_k (t-a)^k = o((t-a)^n) = (t-a)^n \epsilon(t)$$

for a continuous function ϵ with $\epsilon(a) = 0$. For each t, we can find c_t between a and t such that:

$$\int_a^t (u-a)^n \epsilon(u) \mathrm{d}u = (t-a)(c_t - a)^n \epsilon(c_t).$$

Defining

$$\widetilde{\epsilon}(t) = \begin{cases} \left(\frac{c_t - a}{t-a}\right)^n \epsilon(c_t) & \text{if } t \neq a, \\ 0 & \text{if } t = a, \end{cases}$$

we obtain:

$$\int_a^t \left(f(t) - \sum_{k=0}^{n} c_k (t-a)^k \right) \mathrm{d}t = (t-a)^{n+1} \widetilde{\epsilon}(t)$$

and $\widetilde{\epsilon} \overset{t \to a}{\longrightarrow} 0$. $\square$

Remark 4.6. One cannot differentiate Taylor approximations, indeed, for $n \geq 1$, define:

$$\begin{array}{rccc} f_n \colon & \mathbb{R} & \to & \mathbb{R} \\ & t & \mapsto & \begin{cases} t^n \sin(\frac{1}{t^n}) & \text{if } t \neq 0 \\ 0 \text{if } t = 0. \end{cases} \end{array}$$

The function f_n admits a Taylor approximation at the order $n-1$ in 0: for all $t \longrightarrow 0$, one has

$$f_n(t) = o(t^{n-1}).$$

On the other hand, f_n is differentiable on $\mathbb{R}$ (there is something to be checked on 0) and:

$$f_n'(t) = \begin{cases} n t^{n-1} \sin(\frac{1}{t^{n-1}}) - \frac{n}{t} \cos(\frac{1}{t^n}) & \text{if } t \neq 0, \\ 0 & \text{if } n = 0. \end{cases}$$

So that f_n' do not admit a limit (and therefore any Taylor approximation) in 0.

5. Integration of rationnal fraction

Definition 5.1. Let $X \subset \mathbb{R}$, a function $f \colon X \to \mathbb{R}$ is a *rational fraction* if there exists two polynomial $P, Q \in \mathbb{R}[X]$ such that for all $t \in X$, $Q(t) \neq 0$ and $f(t) = \frac{P(t)}{Q(t)}$.

We know how to find primitives of rational fractions, we will see how in this section.

5.1. About polynomials.

Definition 5.2. A *polynomial* with coefficient in a field $\mathbb{K}$ ($\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{C}$) is an almost null sequence[1] of elements of $\mathbb{K}$. If $P := (a_i)_{i\in\mathbb{N}}$ is a polynomial, there exists an n in $\mathbb{N}$ such that for all integer $k > n$, $a_k = 0$. The minimal such n is called[2] the *degree* of P. In that case, we write: $P = \sum_{i=0}^{n} a_i X^i$. The set of all polynomial with coefficients in $\mathbb{K}$ is denoted $\mathbb{K}[X]$.

Example 5.3. If $P = (3, 1, \frac{1}{2}, 0, -2, 0, 0, \dots)$, then one writes:

$$P = -2X^4 + \frac{X^2}{2} + X + 3.$$

One can add polynomials and multiply them according to the usual rules. This turns $\mathbb{K}[X]$ into a ring, which has very nice properties which will be explored in the algebra lectures.

Theorem 5.4 (D'Alembert–Gauß theorem, admitted)**.** *Let $P \in \mathbb{C}[X]$ be a non-zero polynomial, then there exists $\alpha_1, \dots, \alpha_p$ pairwaise distinct element of $\mathbb{C}$, $n_1, \dots, n_p \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$ such that:*

$$P(= \lambda \prod_{k=1}^{p} (X - \alpha_k)^{n_k}.$$

This decomposition is unique up to permutation of the factors.

Corollary 5.5. *Let $P \in \mathbb{R}[X]$ be a non-zero polynomial, then there exists*

- $a_1, \dots, a_p \in \mathbb{R}$,
- $n_1, \dots, n_p \in \mathbb{N}^*$,
- $b_1, \dots, b_q \in \mathbb{R}$,
- $c_1, \dots, c_q \in \mathbb{R}_{>0}$,
- $m_1, \dots, m_q \in \mathbb{N}^*$,
- $\lambda \in \mathbb{R}$ *such that:*

$$P(= \lambda \prod_{k=1}^{p} (X - a_k)^{n_k} \prod_{\ell=1}^{q} ((X - b_\ell)^2 + c_\ell^2)^{m_\ell}.$$

This decomposition is unique up to permutation of the factors (the a_k are pairwise distinct and so are the pairs (m_ℓ, c_ℓ)).

Definition 5.6. The polynomial of the form $\lambda(X - a)$ and $\lambda((X - b)^2 + c^2)$ in $\mathbb{R}[X]$ are the *irreducible elements* of $\mathbb{R}[X]$.

The polynomial of the form $\lambda(X - \alpha)$ $\mathbb{C}[X]$ are the *irreducible elements* of $\mathbb{C}[X]$.

5.2. Decomposition in simple elements.

Proposition 5.7 (Admitted, but not very hard)**.** *Let $f = \frac{P}{Q}$ a rational fraction such that P and Q have common irreducible factor. Let us write $Q = \prod_{i=1}^{p} (X - \alpha_i)^{n_i}$, for $\alpha_1, \dots, \alpha_p$ in $\mathbb{C}$ pairwise distinct and $n_1, \dots n_p$ in $\mathbb{N}*$, then there exists $R \in \mathbb{C}[X]$ and $(\lambda_{ij})_{\substack{1\leq i\leq p\\ 1\leq j\leq n_i}}$ such that for all t for which it makes sense:*

$$f(t) = R(t) + \sum_{i=1}^{p} \sum_{j=1}^{n_i} \frac{\lambda_{ij}}{(t - \alpha_i)^j},$$

Moreover, this decomposition is unique (up to permutation of the terms).

---

[1] An *almost null sequence* is a sequence for which all but finitely many terms are 0.

[2] If $P = (0, 0, \dots)$ then by convention, the degree of P is $-\infty$

Corollary 5.8. *Let* $f = \frac{P}{Q}$ *a rational fraction such that* P *and* Q *have common irreducible factor. Let us write* $Q = \prod_{i=1}^{p}(X - a_k)^{n_k} \prod_{\ell=1}^{q}((X - b_\ell)^2 + c_\ell^2)^{m_\ell}$*, for* $a_1, \ldots, a_p$ *in* $\mathbb{R}$ *pairwise distinct,* $n_1, \ldots n_p$ *in* $\mathbb{N}*$*, pairs* (b_j, c_j) *in* $\mathbb{R} \times \mathbb{R}_{>0}$ *pairwise distinct) and* $m_1, \ldots m_p$ *in* $\mathbb{N}*$*, then there exists* $R \in \mathbb{R}[X]$ *and* $(\lambda_{ij})_{\substack{1 \leq i \leq p \\ 1 \leq j, \leq n_i}}$*,* $(\mu_{k\ell})_{\substack{1 \leq k \leq q \\ 1 \leq \ell, \leq m_k}}$ *and* $(\nu_{ij})_{\substack{1 \leq k \leq n \\ 1 \leq \ell, \leq m_k}}$ *such that for all t for which it makes sense:*

$$f(t) = R(t) + \sum_{k=1}^{p} \sum_{\ell=1}^{n_k} \frac{\lambda_{kj}}{(t - a_i)^\ell}, + \sum_{k=1}^{q} \sum_{\ell=1}^{m_\ell} \frac{\mu_{k\ell} t + \nu_{k\ell}}{((t - b_k)^2 + c_k^2)^k},$$

Moreover, this decomposition is unique (up to permutation of the terms).

Decompostions appearing in the previous results are know as *simple elements decompositions.*

Example 5.9. We want to decompose $\frac{3X^5 - 2}{X^4 - 1}$ in simple elements.

We have $X^4 - 1 = (X - 1)(X + 1)(X^2 + 1) = (X - 1)(X + 1)(X - i)(X + i)$. We start by making the Euclidean division of $3X^5 - 2$ by $X^4 - 1$:

$$3X^5 - 2 = 3X(X^4 - 1) + 1.$$

Hence it remains to decompose $\frac{1}{X^4 - 1}$, We expect something of the form:

$$\frac{a}{X - 1} + \frac{b}{X + 1} + \frac{c}{X^2 + 1}$$

To get the coefficients of $\frac{1}{X-1}$ and $\frac{1}{X+1}$ we can multiply by $(X - 1)$ and $(X + 1)$ and evaluate on $X = 1$ and $X = -1$ respectively. This gives:

$$a = \frac{1}{1 + 1} 1^2 + 1 = \frac{1}{4} \quad \text{and} \quad b = \frac{1}{-1 - 1} (-1)^2 + 1 = -\frac{1}{4}$$

To get the coefficient of $\frac{1}{X^2 + 1}$, we can evaluate in $X = 0$. This gives:

$$\frac{a}{0 - 1} + \frac{b}{0 + 1} + \frac{c}{0^2 + 1} = \frac{1}{0^4 - 1}.$$

so that $c = a - b - 1 = -\frac{1}{2}$. We could as well have decomposed over $\mathbb{C}$ and (with terms in $\frac{1}{X-i}$ and $\frac{1}{X+i}$, find coefficients for the two resulting fractions to get the term in $\frac{1}{X^2+1}$) In the end, we obtain:

$$\frac{1}{X^4 - 1} = \frac{1}{4(X - 1)} - \frac{1}{4(X + 1)} - \frac{1}{2(X^2 + 1)}$$

and

$$\frac{3X^5 - 2}{X^4 - 1} = 3X + \frac{1}{4(X - 1)} - \frac{1}{4(X + 1)} - \frac{1}{2(X^2 + 1)}.$$

It is easy to check the computation by simply putting all terms of the right-hand side on the same denominator and add them up.

There are many recipes to find the coefficients:

- One always should start by taking the Euclidean division from P by Q. This reduces to the case where $\deg(Q) < \deg(R)$.
- Tout find the terms $\frac{1}{(X - \alpha)^{\max}}$, we can multiply by $(X - \alpha)^{\max}$ and evaluating in α (as we have done for $\alpha = -1$ and $\alpha = 1$ in the previous example).
- One can look at the limit in $+\infty$ after having multiplied by an appropriate X, this gives a linear relation between terms of the form $\frac{1}{X - \alpha}$.
- One can evaluate for various $X = t$, this gives linear relations between the coefficients.

5.3. Primitives of rational fractions. From previous subsection, we know that for computing primitives of rational fractions, it is enough to understand the following cases:

- $f\colon t \mapsto R(t)$, where R is a polynomial. This is easy.
- $f\colon t \mapsto \frac{1}{t-\alpha}$ where $\alpha \in \mathbb{R}$. $F\colon t \mapsto \ln|t-\alpha|$ is a primitive of f.
- $f\colon t \mapsto \frac{1}{(t-\alpha)^n}$ where $\alpha \in \mathbb{R}$ and $(n \geq 2)$. $F\colon t \mapsto -\frac{1}{(n-1)(t-\alpha)}$ is a primitive of f.
- $f\colon t \mapsto \frac{2t+2b}{(t+b)^2+c^2}$ where $b, c \in \mathbb{R}$. $F\colon t \mapsto \ln((t+b)^2+c^2)$ is a primitive of f.
- $f\colon t \mapsto \frac{1}{(t+b)^2+c^2}$ where $b, c \in \mathbb{R}$. $F\colon t \mapsto \frac{1}{c}\arctan(\frac{t+b}{c})$ is a primitive of f.
- $f\colon t \mapsto \frac{2t+2b}{((t+b)^2+c^2)^n}$ where $b, c \in \mathbb{R}$ and $n \geq 2$. $F\colon t \mapsto -\frac{1}{(n-1)((t+b)^2+c^2)^{n-1}}$ is a primitive of f.
- $f\colon t \mapsto \frac{1}{((t+b)^2+c^2)^n}$ where $b, c \in \mathbb{R}$ and $n \geq 2$. This case is a little bit more complicated and we shall explain how to proceed below.

Fix b, c in $\mathbb{R}$ and for $n \geq 1$, define

$$f_n\colon t \mapsto \frac{1}{(t+b)^2+c^2}.$$

We look for primitives of f_n for all $n \geq 2$. We already know that

$$F_1\colon t \mapsto \frac{1}{c}\arctan(\frac{t+b}{c})$$

is a primitive of f_1. We define by induction F_n as follows[3]:

$$F_{n+1}\colon t \mapsto \frac{1}{c^2}\left(\frac{2n-1}{2n}F_n(t) + \frac{t+b}{2n}f_n(t)\right).$$

We now show by induction, that $F_n' = f_n'$. For $n = 1$, this is clear. Let $n \in \mathbb{N}^*$ for which this is true.

$$\begin{aligned}
F_{n+1}'(t) &= \frac{1}{c^2}\left(\frac{2n-1}{2n}f_n(t) + \frac{1}{2n}f_n(t) + \frac{t+b}{2n}\left(\frac{-n(2t+2b)}{((t+b)^2+c^2)^{n+1}}\right)\right) \\
&= \frac{1}{c^2}\left(\frac{(t+b)^2+c^2-(t+b)^2}{((t+b)^2+c^2)^{n+1}}\right) \\
&= f_{n+1}'(t)
\end{aligned}$$

So that we have indeed found primitives of f_n for all $n \in \mathbb{N}^*$.

For instance, one has:

$$\begin{aligned}
F_2\colon t \mapsto \frac{1}{c^2}\left(\frac{1}{2c}\arctan(\frac{t+b}{c}) + \frac{t+b}{2((t+b)^2+c^2)}\right) \\
= \frac{1}{c^3}\arctan(\frac{t+b}{c}) + \frac{t+b}{2c^2((t+b)^2+c^2)}
\end{aligned}$$

$$F_3\colon t \mapsto \frac{1}{c^2}\left(\frac{3}{8c^3}\arctan(\frac{t+b}{c}) + \frac{3(t+b)}{8c^2((t+b)^2+c^2)} + \frac{t+b}{4((t+b)^2+c^2)}\right).$$

---

[3] This formula comes from the fact that $f_{n+1}(t)((t+b)^2+c^2) = f_n(t)$ which can be can integrate by part.

6. Functions which reduce to rational fractions

(1) $t \mapsto f(e^t)$ where f is a rational fraction. In that case, we can do the change of variable "$e^t = u$", so that "$\mathrm{d}t = \frac{\mathrm{d}u}{u}$".

(2) $g\colon t \mapsto f(\cos(t), \sin(t))$ where f is a rational fraction (in two variables!). In that case, we can do the change of variable "$\tan(t/2) = u$", so that "$\cos(t) = \frac{1-u^2}{1+u^2}$", "$\sin(t) = \frac{2-u}{1+u^2}$" and "$\mathrm{d}t = \frac{2\mathrm{d}u}{1+u^2}$". Sometimes, some other change of variables are actually more efficient, this is sum up in the so-called Bioche's rules (but they don't always work):
   - If $g(-t) = -g(t)$, then the change of variable "$u = \cos(t)$" might be a good idea.
   - If $g(\pi - t) = -g(t)$, then the change of variable "$u = \sin(t)$" might be a good idea.
   - If $g(\pi + t) = g(t)$, then the change of variable "$u = \tan(t)$" might be a good idea.

(3) $t \mapsto f(t, \sqrt[n]{\frac{at+b}{ct+d}})$ where f is a rational fraction. In that case one can do the change of variable "$u = \sqrt[n]{\frac{at+b}{ct+d}}$", so that we have "$t = \frac{du^n-b}{cu^n+a}$" and "$\mathrm{d}u = \frac{1}{n}u^{1-n}\frac{ad-bc}{(ct+d)^2}\mathrm{d}t$".

(4) $t \mapsto f(t, \sqrt{at^2+bt+c})$ with a first change of variable one reduces to one of the following three cases (depending on a, b and c):
   (a) $t \mapsto f(t, \sqrt{t^2+1})$. In that case we use the change of variable "$t = \sinh(u)$", so that "$\sqrt{t^2+1} = \cosh(u)$" and $\mathrm{d}t = \cosh(u)\mathrm{d}u$.
   (b) $t \mapsto f(t, \sqrt{t^2-1})$. In that case we use the change of variable "$t = \cosh(u)$", so that "$\sqrt{t^2-1} = |\sinh(u)|$" and $\mathrm{d}t = \sinh(u)\mathrm{d}u$.
   (c) $t \mapsto f(t, \sqrt{1-t^2})$. In that case we use the change of variable "$t = \cos(u)$", so that "$\sqrt{1-t^2} = |\sin(u)|$" and $\mathrm{d}t = -\sin(u)\mathrm{d}u$.

   So that we are reduced to cases (1) and (2).

CHAPTER 3

Ordinary differential equations

In all this chapter, I is an interval of $\mathbb{R}$ with non empty interior.

1. Definition, generalities

Definition 1.1. Let $n \geq 1$ be an integer and $F\colon I \times \mathbb{R}^{n+1} \to \mathbb{R}$ and $f\colon I \to \mathbb{R}$ be continuous functions. The equation

$$F(x, y, y', \dots, y^{(n)}) = f(x). \tag{E}$$

is called an *(ordinary) differential equation of order n.* It is *homogeneous* if f is identically equal to 0. A *solution* of (E) is a function $\varphi\colon J \to \mathbb{R}$ of class C^n with J an interval contained in I such that for all t in J,

$$F(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n)}(t)) = f(t).$$

A solution $\varphi\colon J \to \mathbb{R}$ is *maximal* if for any other solution $\psi\colon J' \to \mathbb{R}$ such that $\varphi_{|J\cap J'} = \psi_{|J\cap J'}$, then $J' \subseteq J$.

To solve a differential equation is to find all its maximal solutions, sometimes satisfying extra conditions

Example 1.2. Consider a weight attached to a spring and denote $y(t)$ the height of the weight at time t. Suppose that the spring react in an elastic way[1], then y satisfy the following equation:

$$y''(t) + ay(t) = 0, \tag{4}$$

where a is a real constant depending on the spring (and called the elasticity of the spring). This is an *homogeneous linear differential equation of order 2.* If we include air resistance (proportional to speed), in our model, we obtain a new equation:

$$y''(t) + by'(t) + ay(t) = 0, \tag{5}$$

where b is a constant that depends on the aerodynamics of the weight. If the spring reacts not completely elastically, then we get:

$$y''(t) + by'(t) + f(y(t)) = 0, \tag{6}$$

for some function f. If further more one acts on the weight with an external force depending on time and encoded by a function g, one obtains

$$y''(t) + by'(t) + f(y(t)) = g(t). \tag{7}$$

Sometimes we can solve differential equation explicitly, sometimes we can only prove that solutions satisfy some properties.

Definition 1.3. A differential equation if *linear* of it can be written as:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f \tag{8}$$

for some functions $a_i\colon I \to \mathbb{R}$ and $f\colon I \to \mathbb{R}$. We will be mostly interested in such equation especially when the functions a_i, called *coefficients* are constant.

---

[1] This means that its strength is proportional to its elongation.

The case of differential equation of order 1 can be interpreted geometrically. Suppose that (E) can be written as follows

$$y' = F(x,y) \tag{E1}$$

for a function $F\colon\ I \times \mathbb{R} \to \mathbb{R}$. Then for any point (x,y) of the plane, with x in I, define $D_{(x,y)}$ to be the straight line through (x,y) with slope given by $F(x,y)$ — this kind of data is a line field (or vector field) on $I \times \mathbb{R}$. Then a solution $\varphi\colon\ J \to \mathbb{R}$ of (E1) is a function whose corresponding curve is tangent to $D_{(x,\varphi(x))}$ in $(x,\varphi(x))$ for any $x \in J$.

2. Linear differential equation of order 1

Definition 2.1. A *linear differential equation of order 1* is a differiential equation of the form

$$y' = ay + b, \tag{L1}$$

where $a\colon\ I \to \mathbb{R}$ and $b\colon\ I \to \mathbb{R}$ are two functions (which we assume to be continuous). The *homogenous associated equation* is

$$y' = ay. \tag{HL1}$$

To solve (L1), we proceed in three steps:

1. We solve find the general form of solutions of the homogeneous equation (HL1).
2. We find a specific solution of (L1) using the so-called “variation of constant” method.
3. We sum up and give the general form of the solutions of (L1).

2.1. Solving the homogeneous equation.

Proposition 2.2. *Let $a\colon\ I \to \mathbb{R}$ a continuous function and $A\colon\ I \to \mathbb{R}$ a primitive of a on I. Then the solution of* (HL1) *are the functions*

$$\begin{array}{rccl} \varphi_\lambda\colon & I & \to & \mathbb{R} \\ & t & \to & \lambda \exp(A(t)). \end{array}$$

for $\lambda \in \mathbb{R}$.

Remark 2.3. In particular, the set of solution is a vector space of dimension 1.

Proof. The function A being differentiable on I, φ_λ is differentiable on I for any λ, and $\varphi_\lambda' = A'\lambda \exp \circ A = a\varphi_\lambda$ so that φ_λ is indeed a solution of (HL1).

Conversely, if φ is a solution of (HL1), one has: $\varphi' = a\varphi$. If $\varphi = 0$, the $\varphi = \varphi_0$. Otherwise, let J be a maximal interval on which φ does not vanish. For all $x \in J$, one has:

$$\frac{\varphi'(x)}{\varphi(x)} = a(x).$$

Taking primitives, we obtain that there exists a constant $C \in \mathbb{R}$, such that $\log|\varphi(x)| = A(x) + C$ for all $x \in J$, so that for all $x \in J$, $\varphi(x) = \lambda \exp(A(x))$ for some fixed λ in $\mathbb{R}$. This implies in particular that φ does not vanish and therefore that $I = J$ and that $\varphi = \varphi_\lambda$. □

Corollary 2.4. *If $a \in \mathbb{R}$ is a real constant, then the solutions of*

$$y' = ay \tag{LC1H}$$

are the function $(\mathbb{R} \ni x \mapsto \lambda \exp(ax))$ *for* $\lambda \in \mathbb{R}$.

That the non-trivial solutions of (L1) do not vanish will be widely used, we won’t always recall it.

2.2. Solutions of the original equation.

Proposition 2.5. *Let φ_0 be a solution of* (L1)*, then φ is solution of* (L1) *if and only if $\varphi - \varphi_0$ is solution of* (HL1)*.*

PROOF. Suppose that φ is solution of (L1), then one has:

$$(\varphi - \varphi_0)' = (a\varphi + b) - (a\varphi_0 + b) = a(\varphi - \varphi_0),$$

so that $\varphi - \varphi_0$ is solution of (HL1).

Conversly, suppose that $\psi := \varphi - \varphi_0$ is solution of (HL1), then

$$\varphi_0' = (\varphi - \psi)' = a\varphi + b - a\psi = a\varphi_0 + b$$

so that φ_0 is solution of (L1). ☐

2.3. Variation of constant. The idea is to look for a solution φ_0 of (L1) of the form $\varphi_0 = f\varphi$ with φ a non-trivial solution of (HL1) and f a differentiable function (rather than a constant).

With $\varphi_0 = f\varphi$, we can differentiate:

$$\begin{aligned} \varphi_0' &= f\varphi' + f'\varphi = af\varphi + f'\varphi \\ &= a\varphi_0 + f'\varphi. \end{aligned}$$

Hence φ_0 is a solution of (L1) if and only if $f' = \frac{b}{\varphi}$, that is if and only if f is a primitive of $b \exp \circ (-A)$.

2.4. Cauchy problem.

Theorem 2.6. *Let $x_0 \in I$ and $y_0 \in \mathbb{R}$, there exists a unique solution $\varphi\colon I \to \mathbb{R}$ of* (L1) *such that $\varphi(x_0) = y_0$.*

Remark 2.7. The system consisting of (L1) with the condition $y(x_0) = x_0$ is called a *Cauchy problem.*

PROOF. We have shown already that solutions of (L1) are of the form $\varphi = \varphi_0 + \lambda \exp \circ A$ with φ_0 a solution of (L1), λ a real and $A\colon I \to \mathbb{R}$ a primitive of a, then we have $\varphi(x_0) = y_0$ if and only if $\lambda = \frac{y_0 - \varphi_0(x_0)}{\exp(A(x_0))}$. ☐

2.5. An example. Consider the differential equation

$$y' = \frac{2y}{x} + 1 \tag{ExL1}$$

either on $]-\infty; 0[$ or $]0, +\infty[$.

The solutions of the homogeneous differential equation are of the form

$$\varphi_\lambda\colon\ x \mapsto \lambda x^2 \qquad \text{for } \lambda \in \mathbb{R}.$$

Using previous notation, we get $A\colon\ x \mapsto 2\ln|x|$. We then look for a primitive of $x \mapsto \exp(-2\ln|x|) = \frac{1}{x^2}$. We can for instance take $x \mapsto -\frac{1}{x}$, so that $x \mapsto -\frac{1}{x}x^2 = -x$ is a solution of (ExL1). Finally the solution of (ExL1) are the function of the form

$$x \mapsto \lambda x^2 - x \qquad \text{for } \lambda \in \mathbb{R}.$$

2.6. Bernouilli's and Ricatti's differential equations. We'll study two kinds of differential equations which are not linear but can be "reduced" to linear ones and therefor solved using the techniques we have seen in the previous sections.

2.6.1. *Bernoulli's equation.* Consider the differential equation

$$y' = \alpha y + \beta y^n \tag{B}$$

where $\alpha, \beta \colon I \to \mathbb{R}$ are two continuous function and $n \geq 2$ is an integer. Suppose that $\varphi \colon J \to \mathbb{R}$ is a solution of (B) which does not vanish, then the function $\psi \colon \frac{1}{\varphi^{n-1}}$ satisfies:

$$\psi' = -(n-1)\frac{\varphi'}{\varphi^n} = -(n-1)\left(\frac{a\varphi}{+}\frac{\beta\varphi^n}{\varphi^n}\right) = -(n-1)\psi - (n-1)\beta.$$

In other word, ψ is then solution of the differential equation

$$y' = -(n-1)\alpha y - (n-1)\beta,$$

which is linear of order 1.

2.6.2. *Ricatti's equation.* Consider the differentiation equation

$$y' = \alpha y^2 + \beta y + \gamma \tag{R}$$

where $\alpha, \beta, \gamma \colon I \to \mathbb{R}$ are continuous functions. Suppose that φ_0 is a solution (R). If φ is another solution of (R), then one has:

$$\begin{aligned}
(\varphi - \varphi_0)' &= \alpha(\varphi^2 - \varphi_0^2) + \beta(\varphi - \varphi_0) \\
&= \alpha(\varphi - \varphi_0)^2 + 2\alpha\varphi\varphi_0 - 2\alpha\varphi_0^2 + \beta(\psi - \varphi_0) \\
&= \alpha(\varphi - \varphi_0)^2 + (2\alpha\varphi_0 + \beta)(\varphi - \varphi_0).
\end{aligned}$$

In other words, $\psi = \varphi - \varphi_0$ is solution of the differential equation

$$y' = \alpha y^2 + (2\alpha\varphi_0 + \beta)y$$

which is a Bernoulli's equation which we can solve.

3. Linear differential equation of order 2

3.1. Cauchy problem. Consider the following differential equation:

$$y'' + py' + qy = r \tag{L2}$$

where $p, q, r \colon I \to \mathbb{R}$ are continuous functions. The associated homogeneous differential equation is:

$$y'' + py' + qy = 0 \tag{HL2}$$

Definition 3.1. Let $x_0 \in I$ and $y_0, y_1 \in \mathbb{R}$. A *solution* to the *Cauchy problem*

$$\left\{\begin{array}{l} y'' + py' + qy = r, \\ y(x_0) = y_0, \\ y'(x_0)) = y_1 \end{array}\right. \tag{9}$$

is a maximal solution $\varphi \colon J \to \mathbb{R}$ of (L2) such that $x_0 \in J$, $\varphi(x_0) = y_0$ and $\varphi'(x_0) = y_1$.

Theorem 3.2. *Let $x_0 \in I$, $y_0, y_1 \in \mathbb{R}$, there exists a unique solution to the Cauchy problem* (9).

The proof of this result will take a while and we will need to admit a small part of the argument.

3.2. Homogeneous equation and Wronskian.

Proposition 3.3. *The set of solutions of* (HL2) *is a sub-vector space of* $\mathbb{R}^I$.

Proof. Let us denote S the set of solutions of (HL2). The zero function is clearly solution of (HL2), so that S is not empty. Moreover, if $f, g \in S$ and $\lambda, \mu \in \mathbb{R}$, the function $\lambda f + \mu g$ is clearly solution. This proved that S is a sub-vector space of $\mathbb{R}^I$. □

Definition 3.4. Let $\varphi_1, \varphi_2 \colon I \to \mathbb{R}$ be two solutions of (HL2). The *Wronskian* associated with (φ_1, φ_2) is the function:

$$\begin{array}{rccc} W_{\varphi_1,\varphi_2} \colon & I & \to & \mathbb{R} \\ & t & \mapsto & \det \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{pmatrix} \end{array}$$

Proposition 3.5. *Let* $\varphi_1, \varphi_2 \colon I \to \mathbb{R}$ *two solutions of* (L2) *and* $x_0 \in I$*, then for all* $x \in I$*,*

$$W_{\varphi_1,\varphi_2}(x) = \exp\left(-\int_{x_0}^{x} p\right) W_{\varphi_1,\varphi_2}(x_0).$$

Proof. The function W_{φ_1,φ_2} is differentiable and for all x in I, one has:

$$\begin{aligned} W'_{\varphi_1,\varphi_2} &= \det \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{pmatrix} + \det \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_1'' & \varphi_2'' \end{pmatrix} \\ &= \det \begin{pmatrix} \varphi_1 & \varphi_2 \\ -p\varphi_1' - q\varphi_1 & -p\varphi_2' - q\varphi_2 \end{pmatrix} \\ &= \det \begin{pmatrix} \varphi_1 & \varphi_2 \\ -p\varphi_1' & -p\varphi_2' \end{pmatrix} \\ &= -pW_{\varphi_1,\varphi_2}. \end{aligned}$$

Hence, satisfy the differential equation

$$y' = -py$$

which is homogeneous of degree 1. The result follows. □

Corollary 3.6. *Let* φ_1 *and* φ_2 *be two solutions of* (HL2)*. One has the following dichotomy:*

- *Either for all* $x \in I$, $W_{\varphi_1,\varphi_2}(x) = 0$,
- *Or for all* $x \in I$, $W_{\varphi_1,\varphi_2}(x) \neq 0$.

Proof. This follows from the fact that non-zero solutions of homogeneous linear differential equations do not vanish. □

3.3. Fundamental system of solutions.

Definition 3.7. Two solutions $\varphi_1, \varphi_2 \colon I \to \mathbb{R}$ of (HL2) form a *fundamental system of solutions* of (HL2). If there exists $x_0 \in I$ such that $W_{\varphi_1,\varphi_2}(x_0) \neq 0$ (and therefore $W_{\varphi_1,\varphi_2}(x) \neq 0$ for all $x \in I$).

Proposition 3.8. *Let* (φ_1, φ_2) *a fundamental system of solutions of* (HL2)*. Then for any solution* $\psi \colon I \to \mathbb{R}$ *of* (HL2)*, there exist* $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that* $\psi = \lambda_1\varphi_1 + \lambda_2\varphi_2$.

Proof. Let $\varphi \colon I \to \mathbb{R}$ be a solution of (HL2). For all x in I, there exists two unique reals $c_1(x)$ and $c_2(x)$ such that

$$\begin{pmatrix} \varphi(x) \\ \varphi'(x) \end{pmatrix} = c_1(x) \begin{pmatrix} \varphi_1(x) \\ \varphi_1'(x) \end{pmatrix} + c_2(x) \begin{pmatrix} \varphi_2(x) \\ \varphi_2'(x) \end{pmatrix}. \tag{10}$$

In fact one has:

$$\begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} = \frac{1}{W_{\varphi_1,\varphi_2}(x)} \begin{pmatrix} \varphi_2'(x) & -\varphi_2(x) \\ -\varphi_1'(x) & \varphi_1(x) \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \varphi'(x) \end{pmatrix},$$

so that $c_1, c_2 \colon I \to \mathbb{R}$ are differentiable. Differentiating (10), we get:

$$\begin{pmatrix} \varphi' \\ \varphi'' \end{pmatrix} = c_1' \begin{pmatrix} \varphi_1 \\ \varphi_1' \end{pmatrix} + c_2' \begin{pmatrix} \varphi_2 \\ \varphi_2' \end{pmatrix} + c_1 \begin{pmatrix} \varphi_1' \\ \varphi_1'' \end{pmatrix} + c_2 \begin{pmatrix} \varphi_2' \\ \varphi_2'' \end{pmatrix}$$

However, since, φ, φ_1 and φ_2 are solutions of (HL2), one has $\varphi'' = c_1\varphi_1'' + c_2\varphi_2''$. This implies that

$$\begin{pmatrix} \varphi' \\ \varphi'' \end{pmatrix} = c_1' \begin{pmatrix} \varphi_1 \\ \varphi_1' \end{pmatrix} + c_2' \begin{pmatrix} \varphi_2 \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This, in turns, implies that c_1' and c_2' are identically 0 and finally that the function c_1 and c_2 are constant. Their values are the λ_1 and λ_2 whose existence was claimed in the statement. □

Remark 3.9. This shows that the space of solution is at most of dimension 2. To prove that it is indeed of dimension 2, we need to prove existence of a fundamental system of solution.

We need to admit the following result:

Theorem 3.10 (Cauchy)**.** *There exist a non-trivial solution to* (HL2)

Remark 3.11. In practice, we will always aim for an “obvious” (but non-trivial) solution.

Lemma 3.12. *Let φ a solution of* (HL2) *and $\lambda \colon I \to \mathbb{R}$ of class C^2 such that $\lambda'\varphi^2$ is a non-zero solution of*

$$y' + py = 0 \tag{11}$$

then $\lambda\varphi$ is a solution of (HL2) *(in particular, φ is non-vanishing).*

Proof. We compute:

$$(\lambda\varphi)' = \lambda'\varphi + \lambda\varphi' = \frac{\lambda'\varphi^2}{\varphi} + \lambda\varphi'$$

$$(\lambda\varphi)^{(2)} = \lambda''\varphi + 2\lambda'\varphi' + \lambda\varphi'' = \frac{(\lambda'\varphi^2)'}{\varphi} + \lambda\varphi''$$

so that:

$$\begin{aligned}(\lambda\varphi)^{(2)} + p(\lambda\varphi)' + q\lambda\varphi &= \frac{(\lambda'\varphi^2)'}{\varphi} + \lambda\varphi'' + p\frac{\lambda'\varphi^2}{\varphi}\varphi + p\lambda\varphi' + q\lambda\varphi \\ &= 0 + 0.\end{aligned}$$

□

Remark 3.13. This gives a method to find another solution of (HL2) and therefore a fundamental system of solution of (HL2).

Example 3.14. Consider the equation

$$y'' - \frac{2y}{x^2} = 0 \tag{12}$$

on $]0; +\infty[$. The function $\varphi \colon x \mapsto x^2$ is an obvious solution. We now look for another solution ψ of the form $\lambda\varphi$. For all $x \in]0; +\infty[$, one has:

$$\begin{aligned}\psi'(x) &= (\lambda'\varphi + \lambda\varphi')(x) = \lambda'(x)x^2 + 2\lambda(x)x, \\ \psi''(x) &= \lambda''(x)x^2 + 4x\lambda'(x) + 2\lambda(x),\end{aligned}$$

so that ψ is solution of (12) if and only if for all $x \in]0;+\infty[$,

$$\lambda''(x) + 4x\lambda'(x) + 2\lambda(x) - \frac{2\lambda(x)x^2}{x^2} = 0$$

that is, if and only if, for all $x \in]0;+\infty[$,

$$\lambda''(x) - \frac{4\lambda'(x)}{x} = 0.$$

in other word, if λ' is solution of

$$y' - \frac{4y}{x} = 0$$

A solution of this differential equation is $x \mapsto \frac{1}{x^4}$. Hence we can choose $\lambda : x \mapsto \frac{-1}{3x^3}$. Finally we obtain that $x \mapsto x^2$ and $x \mapsto \frac{1}{x}$ (because we can multiply solutions by a constant) are solutions to (12). One easily check that these form a fundamental system of solutions (either by computing the Wronksian, or by noticing that these two function are not co-linear). Hence we obtain that the general solution of (12) are the functions

$$x \mapsto \mu x^2 + \nu\frac{1}{x} \qquad \text{for } \mu, \nu \in \mathbb{R}.$$

3.4. Solutions of the non-homogeneous equation.

Lemma 3.15. *Let φ_0 be a solution of* (L2)*, then φ is solution of* (L2) *if and only if $\varphi - \varphi_0$ is solution of* (HL2).

The proof is the same as that of Proposition 2.5.

As for the linear differential equation of order 1, there is a method to obtain a solution of (L2) by varying the constants of the general solution of (HL2) (see Proposition 3.8).

Proposition 3.16. *Let (φ_1, φ_2) be a fundamental system of solutions of* (HL2)*. There exists two unique function $c_1, c_2 \colon I \to \mathbb{R}$ such that*

$$\begin{cases} c_1\varphi_1 + c_2\varphi_2 = 0 \\ c_1\varphi_1' + c_2\varphi_2' = r. \end{cases}$$

These functions are continuous and for all $x_0 \in I$, the function:

$$\begin{array}{rccc} \varphi_0\colon & I & \to & \mathbb{R} \\ & x & \mapsto & \left(\int_{x_0}^x c_1\right)\varphi_1 + \left(\int_{x_0}^x c_2\right)\varphi_2 \end{array}$$

is a solution of (L2).

Proof. The fact that these functions exist and are unique directly follows from the fact that for all $x \in I$ the matrix

$$M(x) := \begin{pmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{pmatrix}$$

is invertible. In fact, one has for all $x \in I$:

$$\begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} = M^{-1}(x) \begin{pmatrix} 0 \\ r(x) \end{pmatrix}.$$

The function r as well as all functions defining M are continuous, so that c_1 and c_2 are themselves continuous. Let us fix x_0 in I and denote $C_i \colon x \mapsto \int_{x_0}^x c_i$ for $i = 1, 2$. Finally define $\varphi_0 = C_1\varphi_1 + C_2\varphi_2$. One has:

$$\begin{aligned} \varphi_0' &= c_1\varphi_1 + c_2\varphi_2 + C_1\varphi_1' + C_2\varphi_2' \\ &= C_1\varphi_1' + C_2\varphi_2' \end{aligned}$$

and

$$\begin{aligned}\varphi_0'' &= c_1\varphi_1' + c_2\varphi_2' + C_1\varphi_1'' + C_2\varphi_2'' \\ &= r + C_1\varphi_1' + C_2\varphi_2'\end{aligned}$$

so that:

$$\varphi_0'' + p\varphi_0' + q\varphi_0 = r + C_1(\varphi_1'' + p\varphi_1' + q\varphi_1) + C_2(\varphi_2'' + p\varphi_2' + q\varphi_2).$$

And φ_0 is indeed solution of (L2). □

4. Linear differential equation of order 2 with constant coefficients

Consider the following differential equation:

(L2C) $$ay'' + by' + cy = f$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$ and $f\colon\ I \to \mathbb{R}$ are continuous functions. The associated homogeneous differential equation is:

(HL2C) $$ay'' + by' + cy = 0$$

4.1. Solutions of the homogeneous equation.

Definition 4.1. The *characteristic polynomial* associated with (HL2C) is the polynomial $\chi = aX^2 + bX + c$.

Lemma 4.2. *If $s \in \mathbb{R}$ is a root of χ, then*

$$\varphi\colon\ x \mapsto \exp(sx)$$

is a solution of (HL2C).

Proof. One has $a\varphi'' + b\varphi' + c\varphi = (as^2 + bs + c)\varphi = 0$. □

Lemma 4.3. *If $s \in \mathbb{R}$ is a double root of χ, then*

$$\psi\colon\ x \mapsto x\exp(sx)$$

is a solution of (HL2C).

Proof. If s is a double root of χ', then it is also a root of $\chi' = 2aX + b$. Keeping $\varphi\colon\ x \mapsto \exp(sx)$, one has:

$$a\psi'' + b\psi' + c\psi = (as^2 + bs + c)\psi + (2as + b)\varphi = 0.$$

□

Lemma 4.4. *If $r \pm i\omega$ are two complex conjugate root ($r \in \mathbb{R}$ and $\omega \in \mathbb{R}^*$) of χ, then*

$$\varphi_1\colon\ x \mapsto \exp(rx)\cos(\omega x) \quad \textit{and} \quad \varphi_2\colon\ x \mapsto \exp(rx)\sin(\omega x)$$

are solutions of (HL2C).

Proof. Looking at real and imaginary parts, one gets:

$$a(r^2 - \omega^2) + br + c = 0 \quad \text{and} \quad (-2\omega r - w) = 0.$$

One has:

$$a\varphi_1'' + b\varphi_1' + c\varphi_1 = (a(r^2 - w^2) + br + c)\varphi_1 + (-2\omega r - \omega)\varphi_2 = 0$$

and One has:

$$a\varphi_2'' + b\varphi_2' + c\varphi_2 = (a(r^2 - w^2) + br + c)\varphi_2 + (2\omega r + \omega)\varphi_1 = 0.$$

□

From the previous lemma, we immediately deduce:

Theorem 4.5. *Define* $\Delta = b^2 - 4ac$.

1. *If* $\Delta = 0$, *denote* s *the double root of* χ, *then*
$$\varphi\colon\ x \mapsto \exp(sx) \quad \textit{and} \quad \psi\colon\ x \mapsto x\exp(sx)$$
form a fundamental system of solutions of (HL2C).
2. *If* $\Delta > 0$, *denote* s_1 *and* s_2 *the two real roots of* χ, *then*
$$\varphi_1\colon\ x \mapsto \exp(s_1 x) \quad \textit{and} \quad \psi\colon\ x \mapsto \exp(s_2 x)$$
form a fundamental system of solutions of (HL2C).
3. *If* $\Delta < 0$, *denote* $r \pm i\omega$ *the two complex conjugate roots of* χ, *then*
$$\varphi_1\colon\ x \mapsto \exp(rx)\cos(\omega x) \quad \textit{and} \quad \psi\colon\ x \mapsto \exp(rx)\sin(\omega x).$$
form a fundamental system of solutions of (HL2C).

4.2. Solutions of the non-homogeneous equation. As for general linear equation of order 2, we can apply the variation of constants method.

A variation of that works as well: If φ_0 is a solution of (HL2C). One can look for a solution of (L2C) of the form $\lambda\varphi_0$ with $\lambda\colon I \to \mathbb{R}$ of class C^2. The function λ' then satisfies a linear differential of order 1, so that one can find such a λ' and from this a function λ such that $\lambda\varphi_0$ is solution of (L2C).

If f is a polynomial function:
$$f\colon\ x \mapsto p_n x^n + \cdots + p_1 x + p_0.$$
If $c \neq 0$, one can find a solution $\varphi_0$of (L2C) of the form
$$\varphi_0\colon\ x \mapsto q_n x^n + \cdots + q_1 x + q_0.$$
by identifying the terms degree by degree (one should start with highest degree). If $c = 0$, $b \neq 0$ one can find a solution φ_0 of (L2C) such that
$$\varphi_0'\colon\ x \mapsto q_n x^n + \cdots + q_1 x + q_0.$$
by identifying the terms degree by degree (one should start with highest degree). One obtains φ_0 by finding a primitive of φ_0'. If $c = 0$, $b \neq 0$ one can find a solution φ_0 of (L2C) such that
$$\varphi_0''\colon\ x \mapsto q_n x^n + \cdots + q_1 x + q_0.$$
by identifying the terms degree by degree (one should start with highest degree). One obtains φ_0 by finding a primitive φ_0' of φ_0'' and a primitive of that φ_0'.

If f is a product of an exponential and of a polynomial function:
$$f\colon\ x \mapsto \exp(mx)\left(p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x_1 + p_0\right)$$
one can find a solution of (L2C) of the form
$$\varphi_0\colon\ x \mapsto \exp(mx)\left(q_n x^n + q_{n-1}x^{n-1} + \cdots + q_1 x_1 + q_0\right)$$
For finding the coefficients, one proceeds just as for polynomials.

5. Linear differential equation of higher order

For this section we fix n an integer greater than or equal to 3. Consider the linear differential equation
$$\text{(L}n\text{)} \qquad y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = f.$$
with $f\colon I \to \mathbb{R}$ and $p_1, \ldots, p_n\colon I \to \mathbb{R}$ continuous functions. Consider also the associated homogeneous equation
$$\text{(HL}n\text{)} \qquad y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = 0.$$

The result we have seen for orders 1 and 2 generalize. We will show the following two results (admitting an important point, just like for order 2).

Proposition 5.1. *The set of solutions of* (HLn) *is a vector space of dimension* n.

Remark 5.2. The fact that it is a vector space is obvious, the real content of the statement is the dimension of that space.

Theorem 5.3. *For all* x_0 *in* I *and all* $y_0, \dots, y_{n-1} \in \mathbb{R}$, *there exists a unique solution* φ (Ln) *such that* $\varphi^{(k)}(x_0) = y_k$ *for all* $i \in \{0, \dots, n-1\}$.

5.1. Homogeneous equation.

Definition 5.4. Let $\varphi_1, \dots, \varphi_n \colon I \to \mathbb{R}$ be solutions of (HLn). The *Wronskian* associated with $\varphi_1, \dots, \varphi_n$ is the function

$$\begin{array}{rcl} W_{\varphi_1,\dots\varphi_n}\colon & I & \to & \mathbb{R} \\ & x & \mapsto & \det\begin{pmatrix} \varphi_1(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & \dots & \varphi_n'(x) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(x) & \dots & \varphi_n^{(n-1)}(x) \end{pmatrix}. \end{array}$$

Proposition 5.5. *Let* $\varphi_1, \dots, \varphi_n$ *be solutions of* (HLn). *We have the following dichotomy:*

- *Either for all* $x \in I$, $W_{\varphi_1,\dots,\varphi_n}(x) = 0$,
- *Or for all* $x \in I$, $W_{\varphi_1,\dots,\varphi_n}(x) \neq 0$.

Proof. The Wronskian is derivable because the function $\varphi_1, \dots \varphi_n$ are of class C^n and one has for all $x \in I$.

$$\begin{aligned} W'_{\varphi_1,\dots\varphi_n}(x) &= \det\begin{pmatrix} \varphi_1(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & \dots & \varphi_n'(x) \\ \vdots & & \vdots \\ \varphi_1^{(n)}(x) & \dots & \varphi_n^{(n)}(x) \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & \dots & \varphi_n'(x) \\ \vdots & & \vdots \\ -p_1(x)\varphi_1^{(n-1)}(x) & \dots & -p_1(x)\varphi_n^{(n-1)}(x) \end{pmatrix} \\ &= -p_1 W_{\varphi_1,\dots,\varphi_n}(x) \end{aligned}$$

In other words, $W_{\varphi_1,\dots,\varphi_n}$ is solution of the differential equation

$$y' = -p_1 y.$$

We know that solutions of such equations are either identically 0 or do not vanish. ☐

Definition 5.6. A *fundamental system of solutions* for (HLn) is an n-tuple $(\varphi_1, \dots, \varphi_n)$ of solutions of (HLn) such that $W_{\varphi_1,\dots,\varphi_n}$ does not vanish.

Proposition 5.7. *If* $(\varphi_1, \dots, \varphi_n)$ *is a fundamental system of solutions of* (HLn) *and* φ *is a solution of* (HLn), *there there exists* $c_1, \dots c_n$ *in* $\mathbb{R}$ *such that*

$$\varphi = \sum_{i=1}^{n} c_i \varphi_i.$$

Proof. The Wronskian is by hypothesis not 0 and differentiable since the function φ_i are of class C^n. Hence we obtain by inverting the matrices

$$\begin{pmatrix} \varphi_1(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & \dots & \varphi_n'(x) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(x) & \dots & \varphi_n^{(n-1)}(x) \end{pmatrix}$$

the existence of differentiable functions $c_1, \dots c_n \colon I \to \mathbb{R}$ such that

$$\begin{pmatrix} \varphi \\ \varphi' \\ \vdots \\ \varphi^{(n-1)} \end{pmatrix} = \sum_{i=1}^{n} c_i \begin{pmatrix} \varphi_i \\ \varphi_i' \\ \vdots \\ \varphi_i^{(n-1)} \end{pmatrix}$$

We will show that the functions c_i are constant. One has:

$$\begin{pmatrix} \varphi' \\ \varphi^{(2)} \\ \vdots \\ \varphi^{(n)} \end{pmatrix} = \sum_{i=1}^{n} c_i' \begin{pmatrix} \varphi_i \\ \varphi_i' \\ \vdots \\ \varphi_i^{(n-1)} \end{pmatrix} + \sum_{i=1}^{n} c_i \begin{pmatrix} \varphi_i' \\ \varphi_i^{(2)} \\ \vdots \\ \varphi_i^{(n)} + \end{pmatrix}$$

Since φ and the φ_i satisfy (HLn), one gets that

$$\varphi^{(n)} = \sum_{i=1}^{n} c_i \varphi_i^{(n)}$$

and therefore that

$$\begin{pmatrix} \varphi' \\ \varphi^{(2)} \\ \vdots \\ \varphi^{(n)} \end{pmatrix} = \sum_{i=1}^{n} c_i \begin{pmatrix} \varphi_i' \\ \varphi_i^{(2)} \\ \vdots \\ \varphi_i^{(n)} + \end{pmatrix}$$

This implies that

$$\sum_{i=1}^{n} c_i' \begin{pmatrix} \varphi_i \\ \varphi_i' \\ \vdots \\ \varphi_i^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and therefore that $c_i' = 0$ for all $i \in \{0, \dots, n-1\}$, so that the functions c_i are constant as claimed. ☐

Remark 5.8. This proves that the space of solutions of (HLn) is at most n. In order to prove that the dimension is n, one only need to find a fundamental system of solutions.

We admit the following result.

Proposition 5.9. *All equation (HLn) admits a non-trivial solution $\varphi \colon I \to \mathbb{R}$.*

Idea of the proof. One construct a space of function with an ad hoc metric and a well-chosen operator Ψ such that:

- One can show that Ψ admits a fixed point different from the 0 function.
- The fixed point of this operator are solutions of (HLn). ☐

Sketch of the proof of Proposition 5.1. As said above, it is enough to show that there exists a fundamental system of solutions of (HLn). We proceed by induction and set for $n \geq 1$:

$$\mathcal{H}_n = \begin{array}{l}\text{“Any linear differential equation of order } n\\ \text{admits a fundamental system of solutions.”}\end{array}$$

We have seen that $\mathcal{H}_1$ is true. Let $n \geq 2$ such that $\mathcal{H}_{n-1}$ is true. Proposition 5.9 ensures that we can find a non-trivial solution to (HLn). Denote φ such a solution. We now look for solutions of (HLn) of the form $\lambda\varphi$ with λ of class C^n. Writing what it means, we obtain that λ' satisfies an homogeneous linear differential equation of order $n-1$. We get a fundamental system of solutions $(\lambda_1', \dots, \lambda_{n-1}')$ of that system and taking primitives of this functions, we obtain solutions

$$(\varphi, \lambda_1\varphi, \dots \lambda_{n-1}\varphi)$$

and check that this is a fundamental system of solutions. ☐

5.2. Non-homogeneous equation. As before we have:

Lemma 5.10. *Let φ_0 be a solution of* (Ln)*, then φ is solution of* (L2) *if and only if $\varphi - \varphi_0$ is solution of* (HLn)*.*

The proof is the same as that of Proposition 2.5.

This says that in order to solve (Ln), it is enough to solve (HLn) and to find one solution of (Ln). For finding such a solution, we use the same techniques as for order 1 and 2.

Let $(\varphi_1, \dots, \varphi_n)$ a fundamental system of solution of (HLn). Since the Wronskian associated with these solutions is non-vanishing, we obtain the existence of $c_1, \dots, c_n \colon I \to \mathbb{R}$ such that for all $x \in I$:

$$\begin{cases} \sum_{i=1}^n c_i(x)\varphi(x) & = 0\\ \sum_{i=1}^n c_i(x)\varphi'(x) & = 0\\ \vdots & \\ \sum_{i=1}^n c_i(x)\varphi^{(n-2)}(x) & = 0\\ \sum_{i=1}^n c_i(x)\varphi^{(n-1)}(x) & = f(x) \end{cases}$$

The function c_i are continuous because the Wronskian, the and f are continuous and the function φ_i are of class C^n. Fix an $x_0 \in I$ and consider

$$\psi = \sum_{i=1}^n C_i\varphi_i$$

where for $i \in \{0, \dots, n-1\}$,

$$\begin{array}{rccc} C_i \colon & I & \to & \mathbb{R}\\ & x & \mapsto & \int_{x_0}^x c_i. \end{array}$$

One has:

$$\begin{aligned}
\psi' &= \sum_{i=1}^{n} c_i\varphi_i + \sum_{i=1}^{n} C_i\varphi_i' = \sum_{i=1}^{n} C_i\varphi_i' \\
\psi'' &= \sum_{i=1}^{n} c_i\varphi_i' + \sum_{i=1}^{n} C_i\varphi_i'' = \sum_{i=1}^{n} C_i\varphi_i'' \\
&\vdots \\
\psi^{(n-1)} &= \sum_{i=1}^{n} c_i\varphi_i^{(n-2)} + \sum_{i=1}^{n} C_i\varphi_i(n-1) = \sum_{i=1}^{n} C_i\varphi_i(n-1) \\
\psi^{(n)} &= \sum_{i=1}^{n} c_i\varphi_i^{(n-1)} + \sum_{i=1}^{n} C_i\varphi_i(n-1) = f + \sum_{i=1}^{n} C_i\varphi_i(n-1)
\end{aligned}$$

and therefore Ψ is a solution of (Ln).

6. Linear differential equation of order n with constant coefficients

Consider the linear differential equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f. \tag{LnC}$$

with $f\colon I \to \mathbb{R}$ a continuous function and $a_0, a_1 \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. Consider also the associated homogeneous equation

$$a_n y^{(n)} + a_1 y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{HLnC}$$

As before, we define the *characteristic polynomial* to be

$$\chi := a_n X^n + \cdots + a_1 X + a_0.$$

Proposition 6.1. *(1) If $s \in \mathbb{R}$ is a root of χ of order k, then*

$$x \mapsto \exp(sx), \quad \ldots, \quad x \mapsto x^{k-1}\exp(sx)$$

are solutions of (HLnC).

(2) If $r \pm i\omega$ are complex conjugates roots of χ of order k, then the functions:

$$\begin{aligned}
&x \mapsto \exp(rx)\cos(\omega x), \quad ,\ldots, \quad x \mapsto x^{k-1}\exp(rx)\cos(\omega x) \\
&x \mapsto \exp(rx)\sin(\omega x), \quad ,\ldots, \quad x \mapsto x^{k-1}\exp(rx)\sin(\omega x)
\end{aligned}$$

are solutions of (HLnC).

The proof is analogous to that of the case $n = 2$. One then construct a fundamental system of solutions of (HLnC) by taking all the solutions listed in Proposition 6.1 for the differents roots of χ.

In order to solve (LnC), we can of course use the variation of constants method as well as the techniques explained for the cas $n = 2$ when f is a polynomial or the production of an exponential and a polynomial.

CHAPTER 4

Improper integrals

1. Improper integral

In this section, I denotes an interval with non-empty interior and not necessarily closed.

Definition 1.1. Let J be a subset of I and $f\colon I \to \mathbb{R}$ a function. The function f is *locally integrable on* J if it is integrable on all intervals $[c;d] \subseteq J$.

Definition 1.2.

(1) Let $a < b$ two reals (b can be $+\infty$) such that $[a;b[\subseteq I$ and $f\colon I \to \mathbb{R}$ locally integrable on $[a;b[$. If

$$\lim_{\substack{x\to b\\ x<b}} \int_a^x f$$

exists, then one says that the *improper integral of f from a to b converges* and one writes:

$$\int_a^b f = \int_a^b f(t)\mathrm{d}t = \lim_{\substack{x\to b\\ x<b}} \int_a^x f.$$

If the limit does not exist (or is not finite), we says that the improper integral of f from a to b *diverges*.

(2) Similarly, if $a < b$ are two reals (a can be $-\infty$) such that $]a;b] \subseteq I$ and $f\colon I \to \mathbb{R}$ locally integrable on $]a;b]$. If

$$\lim_{\substack{x\to a\\ x>a}} \int_x^b f$$

exists, then one says that the *improper integral of f from a to b converges* and one writes:

$$\int_a^b f = \int_a^b f(t)\mathrm{d}t = \lim_{\substack{x\to a\\ x>a}} \int_x^b f.$$

If the limit does not exist (or is not finite), we says that the improper integral of f from a to b *diverges*.

Definition 1.3. Let $]a;b[\subseteq I$, $c \in]a;b[$ and $f\colon I \to \mathbb{R}$ a function locally integrable on $]a;b[$. If both $\int_a^c f$ anf $\int_c^b f$ converge, one says that *the improper integral of f from a to b converges* and one writes:

$$\int_a^b f(t)\mathrm{d}t = \int_a^b f = \int_a^c f + \int_c^b f.$$

Otherwise one says that the improper integral of f from a to b *diverges*.

Remark 1.4.

- This last definition does not depend on the element c chosen in $]a;b[$.

- In what follows, we'll see results about improper integrals for any of the three notions of Definitions 1.2 and 1.3. Since the proof are always similar in these different cases, we'll only deal with one of the three case (the first one).

Proposition 1.5. *Let* α *in* $\mathbb{R}$ *and* c *in* $]0;+\infty[$. *The function*

$$\begin{array}{rccc} f\colon &]0;+\infty[& \to & \mathbb{R} \\ & t & \mapsto & \frac{1}{t^\alpha} \end{array}$$

(1) The improper integral of f *from* 0 *to* c *converges if and only if* $\alpha < 1$.
(2) The improper integral of f *from* c *to* $+\infty$ *converges if and only if* $\alpha > 1$.

Proof. If $\alpha \neq 1$, a primitive of f is given by

$$t \mapsto \frac{-1}{(\alpha-1)t^{\alpha-1}}.$$

One has:

$$\int_x^c f = \frac{1}{(\alpha-1)x^{\alpha-1}} - \frac{1}{(\alpha-1)c^{\alpha-1}} \xrightarrow{x\to 0} \begin{cases} +\infty & \text{if } \alpha > 1, \\ \frac{1}{(1-\alpha)c^{\alpha-1}} & \text{if } \alpha < 1 \end{cases}$$

and

$$\int_c^x f = \frac{1}{(\alpha-1)c^{\alpha-1}} - \frac{1}{(\alpha-1)x^{\alpha-1}} \xrightarrow{x\to +\infty} \begin{cases} \frac{1}{(\alpha-1)c^{\alpha-1}} & \text{if } \alpha > 1, \\ +\infty & \text{if } \alpha < 1. \end{cases}$$

If $\alpha = 1$, a primitive of f is given by

$$t \mapsto \ln t.$$

and one has:

$$\int_x^c \ln(c) - \ln(x) \xrightarrow{x\to 0} +\infty \qquad \text{and} \qquad \int_c^x \ln(x) - \ln(c) \xrightarrow{x\to +\infty} +\infty. \qquad \square$$

Proposition 1.6. *Let* f *and* g *are two functions locally integrable on* $[a;b[$ *(resp.* $]a;b]$, *resp.* $]a;b[$*) such that both the integrals* $\int_a^b f$ *and* $\int_a^b g$ *converge. Then for all* $\lambda, \mu \in \mathbb{R}$, *the improper integral* $\int_a^b (\lambda f + \mu g)$ *converges and one has:*

$$\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g.$$

Proof. This is a direct consequence of the algebraic operations on limits. $\square$

2. Convergence criteria for positive functions.

In this section, we will only consider $\mathbb{R}_{\geq 0}$-valued functions. Using Proposition 1.6, one can translated all statements for $\mathbb{R}_{\leq 0}$-valued functions. Bare in mind however that in this case most inequality has to be changed.

Proposition 2.1. *Let* $f\colon [a;b[\to \mathbb{R}_{\geq 0}$ *(resp.* $]a;b] \to \mathbb{R}$*) locally integrable. Then the improper integral* $\int_a^b f$ *converges if and only if the function* $[a;b[\ni x \mapsto \int_a^x f$ *(resp.* $]a;b] \ni \mapsto \int_x^b f$*) is bounded.*

Proof. The proofs of both statements are analogous, we only prove the case $[a;b[$. By positivity of Riemann integral, the function

$$\varphi\colon [a;b[\ni x \mapsto \int_a^x f$$

is increasing. Hence it is bounded if and only if it admits a finite limit in b if and only if the improper integral $\int_a^b f$ converges. $\square$

Remark 2.2. In the previous proposition (as in many statements), b can be $+\infty$ in the first case, and a can be $-\infty$ in the second case.

Theorem 2.3. *Let* $f, g\colon [a;b[\to \mathbb{R}$ *(resp.* $]a;b] \to \mathbb{R}_{\geq 0}$*, resp.* $]a;b[\to \mathbb{R}$*) to locally integrable function. Suppose that for all* t *in* $[a;b[$ *(resp.* $]a;b]$*, resp.* $]a;b[$*),* $0 \leq f(t) \leq g(t)$*. Then:*

1. *If the improper integral* $\int_a^b g$ *converges then so does* $\int_a^b f$ *and* $\int_a^b f \leq \int_a^b g$*.*
2. *If the improper integral* $\int_a^b f$ *diverges than so does* $\int_a^b g$*.*

Proof. The third case can be deduced from the two first. The two first cases are analogous, we only prove the first one ($[a,b[$).

The second statement is the contrapostion of (first part of) the first one, so that we only prove the first one. Let us suppose that the improper integral $\int_a^b g$ converges. By positivity of the integral, we have for all $x \in [a;b[$:

$$\int_a^x f \leq \int_a^x g \leq \int_a^b g$$

This show that $x \mapsto \int_a^x f$ is bounded so that the improper integral $\int_a^b f$ converges and that:

$$\int_a^b f \leq \int_a^b g. \qquad \square$$

Proposition 2.4. *Let* $\alpha \in \mathbb{R}$*, then:*

1. *The improper integral* $\int_0^1 \frac{-lnt}{t^\alpha}\mathrm{d}t$ *converges if and only if* $\alpha < 1$*.*
2. *The improper integral* $\int_1^+ \infty\frac{lnt}{t^\alpha}\mathrm{d}t$ *converges if and only if* $\alpha > 1$*.*

Proof. We will deduce the second statement from the first one. Note that the function

$$\begin{array}{cccc} f\colon &]0;1] & \to & \mathbb{R} \\ & t & \mapsto & \frac{-\ln(t)}{t^\alpha} \end{array}$$

is positive and continuous and therefore locally integrable.

If $\alpha < 1$, set $\beta := \frac{\alpha+1}{2} < 1$. From Proposition 2.4, we obtain that the improper integral $\int_0^1 \frac{1}{t^\beta}$ converges. Since

$$\log(-\log(t)t^{\beta-\alpha}) \xrightarrow{t\to 0} 0,$$

we can find $1 \leq \eta > 0$, such that for all $t \in]0,\eta]$,

$$0 \leq \log(-\log(t)t^{\beta-\alpha}) \leq 1.$$

Hence for all $t \in]0;\eta]$, one has:

$$\frac{-\ln t}{t} = \frac{1}{t^\beta}(-\ln(t)t^{\beta-\alpha}) \leq \frac{1}{t^\beta}.$$

The previous Proposition ensures that the improper integral $\int_0^1 \frac{-\ln t}{t^\alpha}\mathrm{d}t$ converges.

If $\alpha \geq 1$, for all $t \in]0,\frac{1}{e}]$, $\frac{-\ln t}{t^\alpha} \geq \frac{1}{t^\alpha}$. Since the improper integral $\int_0^1 \frac{\mathrm{d}t}{t^\alpha}$ diverges, the previous proposition implies that the improper integral $\int_0^1 \frac{-\ln t}{t^\alpha}\mathrm{d}t$ diverges.

The second statement is obtained from the first one by making the change of variables $u = \frac{1}{t}$: for all $x \in [1;+\infty[$, one has:

$$\int_1^x \frac{\ln t}{t^\alpha}\mathrm{d}t = \int_{\frac{1}{1}}^{\frac{1}{x}} -\frac{\ln\left(\frac{1}{u}\right)}{u^{-\alpha+2}}\mathrm{d}u = \int_{\frac{1}{x}}^1 \frac{-\ln u}{u^{2-\alpha}}\mathrm{d}u.$$

Hence the improper integral $\int_1^{+\infty} \frac{\ln t}{t^\alpha}\mathrm{d}t$ converges if and only if the improper integral $\int_0^1 \frac{-\ln u}{u^{2-\alpha}}\mathrm{d}u$ converges, that is, if and only if $2-\alpha < 1$ or, in other words, if and only if $\alpha > 1$. $\square$

3. Absolute convergence

Proposition 3.1. *Let $f\colon [a;b[\to \mathbb{R}$ (resp. $]a;b]$, resp. $]a;b[$) be a locally integrable function. Suppose that the improper integral $\int_a^b |f|$ converges, then the improper integral $\int_a^b f$ converges and:*

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof. As usual, we only prove the case $f\colon [a;b[\to \mathbb{R}$. We use the decomposition $f = f_+ - f_-$, with the usual notations. So that we also have $|f| = f_+ + f_-$. We know that f_+ and f_- are positive and locally integrable. Moreover, since $f_+ \leq |f|$, $f_- \leq |f|$ and the improper integeral $\int_a^b |f|$ converges, we deduce that the improper integrals $\int_a^b f_+$ and $\int_a^b f_-$ converge. And therefore that the integral $\int_a^b f$ converge. ☐

Definition 3.2. If $f\colon [a;b[\to \mathbb{R}$ (resp. $]a;b]$, resp. $]a;b[$) be a locally integrable function such that the improper integral $\int_a^b |f|$ converges, one says that the improper integral $\int_a^b f$ *converges absolutely.*

Remark 3.3.

(1) The proposition says that that if an improper integral absolutely converges then it converges.

(2) As we shall see the concept of absolute convergence is better-behaved than that of convergence.

4. Integration and equivalence

Before going back to improper integrals, we'll make a small detour with the notion of equivalence. Which is a weaker tool than the small o and big O notations but still quite useful. However you are advised to deal with small o and big O notation wich are better-behaved with by aspects.

Definition 4.1. Let I be an interval, b in I (b can be an end of I) and $f, g\colon I\setminus\{b\} \to \mathbb{R}$. The function f *is equivalent to g in the neighborhood of b* if there exists $\eta > 0$ and $w\colon I \setminus \{b\} \cap [b-\eta; b+\eta]$ such that:

(1) $w(x) \xrightarrow{x\to b} 1$ and

(2) for all x in $I \setminus \{b\} \cap [b-\eta; b+\eta]$, $f(x) = w(x)g(x)$.

One write $f \sim_b g$ or $f(x) \overset{x\to b}{\sim} g(x)$.

Remark 4.2. The previous definition can be adapted for $b = \pm\infty$, just like for small o and big O notations.

Proposition 4.3. *$\sim_b$ is an equivalence relation.*

Proof. Reflexivity: For all f, $f \sim_b f$ because one can take $\eta = 1$ and $w = 1$.

Symmetry: Let $f, g\colon I \setminus \{b\} \to \mathbb{R}$ with $f \sim_b g$ and η and w as in the definition of $\sim_b$. Since $w(x) \xrightarrow{x\to b} 1$, there exists η' such that if $x \in I \setminus \{b\} \cap [x-\eta; x+\eta] \cap [x-\eta'; x+\eta'[$, $w(x) > 0$. Define $\eta'' = \min(\eta, \eta') > 0$ and

$$\begin{array}{rccc} w''\colon & I \setminus \{b\} \cap [b-\eta''; b+\eta''] & \to & \mathbb{R} \\ & x & \mapsto & \frac{1}{w(x)}. \end{array}$$

For all x in $I \setminus \{b\} \cap [b-\eta; b+\eta]$, one has $f(x) = w(x)g(x)$. And, of course, $w''(x) \xrightarrow{x\to b} \frac{1}{1} = 1$, so that $g \sim_b f$.

Transitivity: Suppose $f \sim_b g$ and $g \sim_b h$. Let us pick $\eta, \eta' > 0$ and $w\colon I\setminus\{b\} \cap [b-\eta; b+\eta] \to \mathbb{R}$ and $w'\colon I\setminus\{b\}\cap[b-\eta'; b+\eta'] \to \mathbb{R}$ such that $f = wg$ and $g = w'h$.

Setting $\eta'' = \min(\eta, \eta')$ and $w'' = ww'$ one has $f = w''h$ on $I \setminus \{b\} \cap [b - \eta''; b + \eta'']$, and $w''(x) \overset{x \to b}{\longrightarrow} 1 \cdot 1 = 1$, so that $f \sim_b h$. $\square$

Example 4.4.

$$x + x^3 \; x \overset{\sim}{\to} 0 \; x \; x \overset{\sim}{\to} 0 \; x + 5x^2.$$

Remark 4.5.

(1) One has $f \sim_b g$ if and only if $f(x) - g(x) \overset{x \to b}{=} o(f(x))$.

(2) If g does not vanish in a neighborhood of b, then $f \sim_b$ if and only if $\frac{f(x)}{g(x)} \overset{x \to b}{\longrightarrow} 1$.

Proposition 4.6. *Let $f, g \colon [a; b[\to \mathbb{R}$ (resp. $]a; b] \to \mathbb{R}$) locally integrable such that the improper integrable $\int_a^b f$ converges absolutely, if $g(x) \overset{x \to b}{=} o(f(x))$ (resp. $g(x) \overset{x \to a}{=} o(f(x))$), then $\int_a^b g$ is converges absolutely.*

Proof. We only deal with the case $[a; b[$, the other case is similar. Since $g(x) \overset{x \to b}{=} o(f(x))$. There exists $\eta > 0$ such that for all x in $[b - \eta; b[$, $|g(x)| \leq |f(x)|$. Since the improper integral $\int_{b-\eta}^b |f|$ converges, the improper integral $\int_{b-\eta} |g|$ converges and therefor the improper integral $\int_a^b g$ absolutely converges. $\square$

Proposition 4.7. *Let $f, g \colon [a; b[\to \mathbb{R}$ (resp. $]a; b] \to \mathbb{R}$) be two locally integrable functions, such that $f \sim_b g$. Then the improper integral $\int_a^b f$ converges absolutely if and only if the improper integral $\int_a^b g$ converges absolutely.*

Proof. Suppose that $\int_a^b f$ converges absolutely. Since $g(x) - f(x) \overset{x \to b}{=} o(f(x))$, the improper integral $\int_a^b (g - f)$ converges absolutely. Moreover, $|g| = |f + (g - f)| \leq |f| + |g - f|$, so that the improper integral $\int_a^b |g|$ converges and finally $\int_a^b g$ converges absolutely. $\square$

CHAPTER 5

Numerical series

1. Definitions and first properties

Definition 1.1. Let $E = \mathbb{R}, \mathbb{C}, \mathbb{R}^d$ or $\mathbb{C}^d$. A sequence $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ *converges to* $\ell \in E$ if for any $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, if $n \geq M_\epsilon$, $|x_n - \ell| \leq \epsilon$, where $|\cdot|$ stands for absolute value if $E = \mathbb{R}$, complex modulus if $E = \mathbb{C}$ or a norm if $E = \mathbb{R}^d$ or $\mathbb{C}^d$.

Definition 1.2. Let $x := (x_n)_{n\in\mathbb{N}}$ be a sequence and define the sequence $S := (S_n)_{n\in\mathbb{N}}$ to be that of partial sums of $(x_n)_{n\in\mathbb{N}}$ by:

$$S_n = \sum_{k=0}^{n} x_k.$$

If the sequence S converges, one says that the *series associated with* x *converges* and in that case the limit of S is denoted

$$\sum_{k=0}^{\infty} x_k$$

and called the *sum of the series associated with* $(x_n)_{n\in\mathbb{N}}$. If the sequence S diverges, one says that the *series associated with* x *diverges.*

Remark 1.3. If the indices of the sequence x start at 1 or 2 (or even someghing else), we adapt the definition consequently.

Example 1.4. Let $q \in \mathbb{C}$. The series associated with $(q^n)_{n\in\mathbb{N}}$ converges if and only if $|q| < 1$, and if $|q| < 1$, one has:

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

Proposition 1.5. *Let* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *be two sequences in* $E^{\mathbb{N}}$ *with* $E = \mathbb{R}, \mathbb{C}, \mathbb{R}^d$ *or* $\mathbb{C}^d$. *Let* λ *and* μ *be two scalars (elements of* $\mathbb{R}$ *if* $E = \mathbb{R}$ *or* $\mathbb{R}^d$, *elements of* $\mathbb{C}$ *if* $E = \mathbb{C}$ *or* $\mathbb{C}^d$*). Suppose that the series associated with* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *converge. Then the series associated with* $(\lambda x_n + \mu y_n)_{n\in\mathbb{N}}$ *converges and one has:*

$$\sum_{n=0}^{\infty} \lambda x_n + \mu y_n = \lambda \sum_{n=0}^{\infty} x_n + \mu \sum_{n=0}^{\infty} y_n.$$

PROOF. This is a direct consequence of algebraic operations on limits of sequences. ☐

Proposition 1.6 (Cauchy criterion for series)**.** *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence. The series associated with* $(x_n)_{n\in\mathbb{N}}$ *converges if and only if for any* $\epsilon > 0$, *there exists* $N_\epsilon \in \mathbb{R}$ *such that if* $N_\epsilon \leq p \leq q$ *are two integers, then*

$$\left| \sum_{k=p}^{q} x_k \right| \leq \epsilon.$$

Proof. This is the Cauchy criterion for the sequence of partial sums of $(x_n)_{n\in\mathbb{N}}$. □

Looking at the previous proposition for $q = p$, one immediately obtains:

Corollary 1.7. *If the series associated with a sequence* $(x_n)_{n\in\mathbb{N}}$ *converges then the sequence sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* 0.

Remark 1.8. The convergence of the series associated with a sequence $(x_n)_{n\in\mathbb{N}}$ does not depend on the first terms of $(x_n)_{n\in\mathbb{N}}$ however its sum does depend on its first terms.

2. Non-negative series

2.1. Comparisons. In this section we will consider series associated with sequences with value in $\mathbb{R}_{\geq 0}$ called *non-negative sequences*

Proposition 2.1. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a non-negative sequence, then its associated series converges if and only if the sequence of its partial sums is bounded*

Proof. Since $(x_n)_{n\in\mathbb{N}}$ is non-negative its sequence of partial sum is increasing. Hence it converges if and only if it is bounded. □

Proposition 2.2. *Let* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *be two non-negative sequences such that for all* n *in* $\mathbb{N}$, $x_n \leq y_n$, *then:*

1. *If the series associated with* $(x_n)_{n\in\mathbb{N}}$ *diverges, then so does that associated with* $(y_n)_{n\in\mathbb{N}}$.
2. *If the series associated with* $(y_n)_{n\in\mathbb{N}}$ *converges, then so does that associated with* $(x_n)_{n\in\mathbb{N}}$ *and:*

$$\sum_{n=0}^{+\infty} x_n \leq \sum_{n=0}^{+\infty} y_n.$$

Proof. Since the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are non-negative, the sequences of their partial sums (denoted $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$) are non-negative and increasing. Therefore they converge if and only if they are bounded from above. Moreover the inequality in the hypothesis implies that for all n in $\mathbb{N}$, $X_n \leq Y_N$.

Suppose that the series associated with $(x_n)_{n\in\mathbb{N}}$ diverges. This means that the sequence $(X_n)_{n\in\mathbb{N}}$ is not bounded from above hence $(Y_n)_{n\in\mathbb{N}}$ is not bounded from above and therefore the series associated with $(y_n)_{n\in\mathbb{N}}$ diverges.

If on the other hand the series associated with $(y_n)_{n\in\mathbb{N}}$ converges. This means that the sequence $(Y_n)_{n\in\mathbb{N}}$ converges and for all n in $\mathbb{N}$ one has:

$$\sum_{k=0}^{n} x_n \leq X_n \leq Y_n = \sum_{k=0}^{n} y_n \leq \sum_{k=0}^{+\infty}.$$

Hence the sequence $(X_n)_{n\in\mathbb{N}}$ converges and its limit $\sum_{k=0}^{+\infty} x_n$ satisfies:

$$\sum_{n=0}^{+\infty} x_n \leq \sum_{n=0}^{+\infty} y_n.$$

□

Corollary 2.3. *Let* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *be two non-negative sequences such that for all* n *big enough,* $y_n > 0$ *and for which*

$$\lim_{n\to+\infty} \frac{x_n}{y_n} = \ell$$

for an $\ell \in \mathbb{R}_{>0}$. *Then the series associated with* $(x_n)_{n\in\mathbb{N}}$ *converges if and only if the series associated with* $(y_n)_{n\in\mathbb{N}}$ *converges.*

Proof. The statement is symmetric in $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ (one only needs to change ℓ in $\frac{1}{\ell}$). Hence it is enough to prove that if the series associated with $(x_n)_{n\in\mathbb{N}}$ converges then so does that associated with $(y_n)_{n\in\mathbb{N}}$.

Suppose then that the series associated with $(x_n)_{n\in\mathbb{N}}$ converges. For n big enough, $y_n \leq (\ell+1)x_n$. However the series associated with $((\ell+1)x_n)_{n\in\mathbb{N}}$ converges, hence so does that associated with $(y_n)_{n\in\mathbb{N}}$. ☐

Remark 2.4. Be careful the previous statement does not give any kind of inequality regarding the sums of the series.

2.2. Relation to integrals.

Theorem 2.5. *Let* $f\colon [0;+\infty[\to \mathbb{R}_{geq0}$ *be a locally integrable continuous function with is decreasing on* $[\eta_0;0[$ *for a given* η_0 *in* $R_{\geq 0}$*. Then the series associated with* $(f(n)_{n\in\mathbb{N}}$ *converges if and only if the improper integral* $\int_0^{+\infty} f$ *converges.*

Proof. Up to an eventual change of the first terms, we may suppose that f is decreasing on $[0;+\infty[$. For all k in $\mathbb{N}$, one has:

$$f(k) \geq \int_k^{k+1} f \geq f(k+1),$$

so that for all n in $\mathbb{N}$, one has:

$$\sum_{k=0}^{n} f(k) \geq \int_0^n f \geq \sum_{k=1}^{n+1} f(k).$$

Hence if $\int_0^{+\infty} f$ converges, then for all n in $\mathbb{N}$,

$$\sum_{k=0}^{n+1} f(k) \leq \int_0^n f + f(0) \leq \int_0^{+\infty} +f(0)$$

and therefore the series associated with $(f(n))_{n\in\mathbb{N}}$ converges.

Reciprocally, if the improper integral $\int_0^{+\infty} f$ diverges, then $\lim_{n\to+\infty}\int_0^n f = +\infty$ so that the sequences of the partial sums of $(x_n)_{n\in\mathbb{N}}$ is not bounded and the series diverges. ☐

Example 2.6. The series associated with $\left(\frac{1}{n^\alpha}\right)_{n\in\mathbb{N}^*}$ converges if and only if $\alpha > 1$.

Proposition 2.7. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a series which is non-negative for* n *big enough. Then:*

1. *If* $\lim_{n\to\infty} n^\alpha x_n$ *exists and is finite for an* $\alpha > 1$*, then the series associated with* $(x_n)_{n\in\mathbb{N}}$ *converges.*
2. *If* $\lim_{n\to\infty} n^\alpha x_n$ *exists and is not equal to* 0 *for an* $\alpha < 1$*, then the series associated with* $(x_n)_{n\in\mathbb{N}}$ *diverges.*

Proof. This is a direct consequence of Example 2.6 and Corollary 2.3. ☐

2.3. Convergence criteria.

Proposition 2.8. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a non-negative sequence. If* $\lim_{n\in\infty}\sqrt[n]{x_n} = \ell < 1$*, then the series associated with* $(x_n)_{n\in\mathbb{N}}$ *converges.*

Proof. For n big enough, one has:

$$\sqrt[n]{x_n} \leq \frac{\ell+1}{2} < 1.$$

so that $x_n \leq \left(\frac{\ell+1}{2}\right)^n$. We already know that the series associated with $\left(\left(\frac{\ell+1}{2}\right)^n\right)_{n\in\mathbb{N}}$ converges, so that the the series associated with $(x_n)_{n\in\mathbb{N}}$ converges. ☐

Remark 2.9. With the same setup but with $\ell > 1$, we can argue similarly and conclude that the series associated with $(x_n)_{n\in\mathbb{N}}$ diverges. If $(\sqrt[n]{x_n})_{n\in\mathbb{N}}$ converges to 1 from above, then the sequence $(x_n)_{n\in\mathbb{N}}$ does not converge to 0, so that the series associated with $(x_n)_{n\in\mathbb{N}}$ diverges. If it converges to 1 from below, we cannot conclude.

Proposition 2.10 (d'Alembert criterion)**.** *Let $(x_n)_{n\in\mathbb{N}}$ a non-negative sequence with such that for n big enough $x_n \neq 0$. If $\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = \ell < 1$, then the series associated with $(x_n)_{n\in\mathbb{N}}$ converges.*

Proof. There exists an integer n_0 such that for all $n \geq n_0$, $\frac{x_{n+1}}{x_n} \leq \frac{\ell+1}{2}$. Hence for all $n \geq n_0$,

$$x_n \leq \left(\frac{\ell+1}{2}\right)^{n-n_0} x_0.$$

We already know that the series associated with $\left(\left(\frac{\ell+1}{2}\right)^{n-n_0} x_0\right)_{n\in\mathbb{N}}$ converges from which we conclude. □

Remark 2.11. With the same setup but with $\ell > 1$, we can argue similarly and conclude that the series associated with $(x_n)_{n\in\mathbb{N}}$ diverges. If $(\frac{x_{n+1}}{x_n})_{n\in\mathbb{N}}$ converges to 1 from above, then the sequence $(x_n)_{n\in\mathbb{N}}$ is increasing (for n big enough) so that it can not converge to 0. Hence the series associated with $(x_n)_{n\in\mathbb{N}}$ diverges. If it converges to 1 from below, we cannot conclude.

Example 2.12. The series associated with $\left(\frac{n!}{n^n}\right)_{n\in\mathbb{N}}$ converges. Indeed, for all $n \in \mathbb{N}$, one has:

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \left(1+\frac{1}{n}\right)^{-n} \overset{n\to\infty}{\longrightarrow} \frac{1}{e}.$$

2.4. Absolute convergeance of series. In this section, $E = \mathbb{R}, \mathbb{C}, \mathbb{R}^d$ or $\mathbb{C}^d$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.13. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $E^{\mathbb{N}}$. The series associated with $(x_n)_{n\in\mathbb{N}}$ *converges absolutely* if the series associated with $(|x_n|)_{n\in\mathbb{N}}$.

Proposition 2.14. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence $E^{\mathbb{N}}$. If the series associated with $(x_n)_{n\in\mathbb{N}}$ converges absolutely, then it converges.*

Proof. We use Cauchy's criterion. Suppose that the series associated with $(|x_n|)_{n\in\mathbb{N}}$ converges. Let $\epsilon > 0$, there exists n_0, such that for all $q \geq p \geq n_0$,

$$\sum_{k=p}^{q} |x_k| = \left|\sum_{k=p}^{q} |x_k|\right| \leq \epsilon.$$

From the triangular inequality, we get that for all $q \geq p \geq n_0$,

$$\left|\sum_{k=p}^{q} x_k\right| \leq \sum_{k=p}^{q} |x_k| \leq \epsilon,$$

so that the series associated with $(x_n)_{n\in\mathbb{N}}$ satisfies Cauchy criterion and therefore converges. □

Proposition 2.15. *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in $E^{\mathbb{N}}$ whose associated series converge absolutely, then for any scalars λ and μ, the series associated with $(\lambda x_n + \mu y_n)_{n\in\mathbb{N}}$.*

Proof. This follows directly from the fact that $|\lambda x + \mu y| \leq |\lambda||x| + |\mu||y|$ for any x, y in E. □

Remark 2.16. Since the series associated with the zero sequence converges absolutely, we obtain that the set of sequences whose series converges absolutely is a sub-vector space of $E^{\mathbb{N}}$.

Definition 2.17. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in $\mathbb{K}^{\mathbb{N}}$. The *Cauchy product* of these two sequences is the sequence $(z_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by:

$$z_n = \sum_{i=0}^{n} x_i y_{n-i} = \sum_{i+J=n} x_i y_j.$$

Theorem 2.18. *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in $\mathbb{K}^{\mathbb{N}}$ whose associated series converge absolutely. Denote $(z_n)_{n\in\mathbb{N}}$ the Cauchy product of $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$. The series associated with $(z_n)_{n\in\mathbb{N}}$ converges absolutely and:*

$$\sum_{k=0}^{+\infty} z_k = \left(\sum_{k=0}^{+\infty} x_k\right)\left(\sum_{k=0}^{+\infty} y_k\right)$$

Proof. We start by proving that the series associated with $(z_n)_{n\in\mathbb{N}}$ converges absolutely by bounding from above the partial sums. For $n \in \mathbb{N}$, one has:

$$\begin{aligned}
\sum_{k=0}^{n} |z_k| &= \sum_{k=0}^{n} \left| \sum_{i+j=k} x_i y_j \right| \\
&\leq \sum_{k=0}^{n} \sum_{i+j=k} |x_i||y_j| \\
&\leq \sum_{i=0}^{n} \sum_{j=0}^{n} |x_i||y_j| \\
&\leq \left(\sum_{i=0}^{n} |x_i|\right)\left(\sum_{j=0}^{n} |y_j|\right) \\
&\leq \left(\sum_{i=0}^{+\infty} |x_i|\right)\left(\sum_{j=0}^{+\infty} |y_j|\right)
\end{aligned}$$

Hence the sequence of partial sums of the $(|z_k|)_{k\in\mathbb{N}}$ is bounded and therefore the series associated with $(z_k)_{k\in\mathbb{N}}$ converges absolutely.

We know wish to prove that

$$\sum_{k=0}^{+\infty} z_k = \left(\sum_{k=0}^{+\infty} x_k\right)\left(\sum_{k=0}^{+\infty} y_k\right)$$

To that end we compute for $n \in \mathbb{N}$:

$$\begin{aligned} a_n := \left| \sum_{k=0}^{n} z_k - \left(\sum_{i=0}^{n} x_i \right) \left(\sum_{j=0}^{n} y_j \right) \right| &= \left| \sum_{i+j \leq n}^{n} x_i y_j - \left(\sum_{i=0}^{n} x_i \right) \left(\sum_{j=0}^{n} y_j \right) \right| \\ &= \left| \sum_{\substack{i+j>n \\ 0 \leq i,j \leq n}}^{n} x_i y_j \right| \\ &\leq \sum_{\substack{i+j>n \\ 0 \leq i,j \leq n}}^{n} |x_i y_j| \\ &\leq \sum_{n<i+j \leq 2n}^{n} |x_i y_j| \\ &\leq \sum_{k=n+1}^{2n} |z_k| \end{aligned}$$

Since the series associated with $(|z_k|)_{k \in \mathbb{N}}$ converges absolutely, the sequence $\left(\sum_{k=n+1}^{2n} |z_k| \right)_{n \in \mathbb{N}}$ converges to 0. Hence, so does the sequence $(a_n)_{n \in \mathbb{N}}$. However, from its very definition, we also know that it converges to

$$\left| \sum_{k=0}^{+\infty} z_k - \left(\sum_{k=0}^{+\infty} x_k \right) \left(\sum_{k=0}^{+\infty} y_k \right) . \right|$$

Hence

$$\sum_{k=0}^{+\infty} z_k = \left(\sum_{k=0}^{+\infty} x_k \right) \left(\sum_{k=0}^{+\infty} y_k \right) .$$

☐

Definition 2.19. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence. A sequence $(y_n)_{n \in \mathbb{N}}$ is a *rearrangement* of $(x_n)_{n \in \mathbb{N}}$ if there exists a bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$ such that for all n in $\mathbb{N}$, $y_n = x_{\sigma(n)}$.

In other words $(y_n)_{n \in \mathbb{N}}$ is a *rearrangement* of $(x_n)_{n \in \mathbb{N}}$ if is obtained by reordering the terms of $(x_n)_{n \in \mathbb{N}}$.

Proposition 2.20. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $E^{\mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ a* rearrangement *of $(x_n)_{n \in \mathbb{N}}$. If the series associated with $(x_n)_{n \in \mathbb{N}}$ converges absolutely, then so does the series associated with $(y_n)_{n \in \mathbb{N}}$ and*

$$\sum_{n=0}^{+\infty} x_n = \sum_{n=0}^{+\infty} y_n .$$

PROOF. Let us denote $\sigma \colon \mathbb{N} \to \mathbb{N}$ a bijection of $\mathbb{N}$ such that $y_n = x_{\sigma(n)}$. Note that such a bijection is not necessarily unique (it is not as soon as some terms in $(x_n)_{n \in \mathbb{N}}$ repeat).

We start with showing that the series associated with $(y_n)_{n \in \mathbb{N}}$ converges: Let $n \in \mathbb{N}$ and set $m_n = \max\{\sigma(i) | 0 \leq i \leq n\}$. One has:

$$\sum_{k=0}^{n} |y_n| = \sum_{k=0} |x_{\sigma(n)}| \leq \sum_{k=0}^{m} |x_k| \leq \sum_{k=0}^{m_n} |x_k| \leq \sum_{k=0}^{+\infty} ,$$

so that the sequence of partial sums of $(|y_k|)_{k\in\mathbb{N}}$ is bounded and therefore the series associated with $(y_k)_{k\in\mathbb{N}}$ converges absolutely.

Let us now prove the identity. Let $n \in \mathbb{N}$ and, as before, set $m_n = \max\{\sigma(i)|0 \leq i \leq n\}$. Note that $m_n \geq n$.

$$\left|\sum_{k=0}^{n} y_k - \sum_{k=0}^{m_n}\right|$$

□