WORKSHOP TOPOLOGY BTW 3: MORSE THEORY AND FLOER HOMOLOGY

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Comments on the bibliography. In addition to the references provided for each talk, the following bibliography might be helpful:

- Books on Fukaya categories and Lagrangian Floer homology: [Oh15a, Oh15b] can be found (merged) online for free at [Oh], see also [Sei08] for Fukaya categories in the exact case, and [FOOO09a, FOOO09b] for a more general setting.
- Surveys on Fukaya categories and Lagrangian Floer homology: [Aur14, Smi15].
- Lecture notes: Pascaleff's lecture notes [Pas] are good. [Caz] are sketchy, but follow roughly the program of the school (or vice versa). Guillem is happy to chat and explain more details if needed (g.cazassus@gmail.com).
- For Morse homology [AD14, Hut].
- For symplectic geometry and pseudoholomorphic curves: [MS17, MS12].

1. Pretalks (online, January)

1.1. Classical mechanics.

1.2. Symplectic geometry.

1.3. Elliptic regularity.

2. Morse Theory

2.1. Morse functions. (Clément)

The aim of this talk is to present the basics of Morse theory: definition of Morse functions, existence and genericity, Morse lemma and index of a critical point, pseudo-gradients, stable and unstable manifolds, Morse charts, topology of the sub-level sets and handle gluing associated to a critical point, Smale condition. Examples and pictures are expected (proofs are not).

Explain in a picture that for a perfect Morse function, the unstable manifolds give a cell decomposition of the manifold, and the cell differential counts flowlines.

[AD14, Chap. 1, 2]

2.2. Morse complex. (Rudy)

This talk aims at explaining the construction of the Morse complex. Define trajectories and broken trajectories, construct the Morse complex and prove that it is a complex, briefly define the complex over $\mathbb{Z}$. In the proof of $\partial^2 = 0$, use the identity

$$\partial\overline{\mathcal{M}}(x,y) = \coprod_z \mathcal{M}(x,z)\times\mathcal{M}(z,y), \tag{2.1}$$

explain it in a picture, but leave its proof to the next speaker.

[AD14, Chap. 3]

2.3. *Morse complex, $\partial^2 = 0$. (Quentin)

Outline a proof strategy for (2.1) that can be later adapted to the Floer setting (in the Morse setting, a more direct proof can be found in [AD14]). Highlight the key ingredients:

- transversality: first form universal moduli space, then apply the Sard-Smale theorem;
- compactness: line breaking;
- gluing: construct a pre-gluing map, then make it a gluing map (use the implicit function theorem (IFT) or the fixed point theorem);
- elliptic regularity: for the Sard-Smale theorem and the IFT, it requires working in Banach spaces, but actual solutions are smooth.

[Caz, Sec. 2.4–2.6] [AD14, Chap. 8, 9]

2.4. Morse homology. (Pierre)

The main goal of this talk is to prove that the homology of the Morse complex constructed in the previous talk does not depend on the input Morse function nor on the pseudo-gradient [AD14, Sec. 3.4]. As an application, state and prove the Morse inequalities [AD14, Sec. 4.4]. State without proof that it computes the homology of the manifold.

AudinDamian

2.5. *Fukaya-Morse category of a smooth manifold. (Benjamin)

Define the "Fukaya-Morse category" $\mathcal{FM}(X)$, introduced in [Fuk93, Chap. 1], and denoted $Ms(X)$ there.

In particular, define the moduli space of metric rooted ribbon trees (associahedron), its compactification, and explain how the structure of its codimension 1 boundary dictates the algebraic structure of $\mathcal{FM}(X)$ (A_∞-category).

Define the higher composition operations μ^k, the A_∞ relations they satisfy, and explain the meaning of μ^k for $k \leqslant 3$.

[Fuk93, Chap. 1], [Caz, Sec. 2.8, 2.9], [FO97], [Mes18], [AL22], [Abo11]

3. Floer Theory

3.1. Pseudo-holomorphic curves and Gromov compactness. (Filippo)

Present the Arzelà-Ascoli theorem on compactness when $|du|$ is uniformly bounded for a pseudo-holomorphic curve u. When it is not the case, explain what *bubbling* is. Explain the necessary rescaling, and removal of singularity. Give the example of $xy = \epsilon$ in $\mathbb{C}P^2$. Give an intuitive statement of Gromov compactness theorem (without defining stable maps in full details)

[Oh, Chap. 9], [Pan94].

3.2. Floer complex: a naive definition. (Diego)

Derive a simple naive definition of the Floer complex (in the exact setting) as the Morse homology of the action functional on the space of arcs $\mathcal{P}(L_0, L_1)$ for given Lagrangians L_0, L_1 (with e.g. a fixed almost complex structure, don't worry about transversality).

Underline the important fact that all strips in $\mathcal{M}(x, y)$ have the same area (using twice the Stokes formula).

Explain why Floer $\neq$ Morse (infinite dimension, flow being not globally defined, no cellular decomposition...). Therefore one gets something different than $H_*(\mathcal{P}(L_0, L_1))$.

State the main expected properties: $\partial^2 = 0$, the Hamiltonian isotopy invariance, or $HF(L, L) \simeq H_*(L)$. Apply these properties to prove the Arnold-Givental conjecture, and the Arnold conjecture as a particular case.

[Aur14, Sec. 1.1], [Smi15, Sec. 3], [Oh15b, Chap. 12]

3.3. **Surfaces.** (Yohan)
Recall the uniformization theorem, and apply it to get explicit descriptions of moduli spaces of strips in a surface. Describe examples and counter examples of the above expected properties.
[Caz, Sec. 3.3], [Sei08, Sec. 13], [Abo08]

3.4. ***Maslov index.** (Arno) Explain the construction of the Maslov index (Lagrangian Grassmanian, canonical short paths, ...), compute it on examples in surfaces (include polygons).
State the Riemann–Roch formula giving the dimension of moduli spaces, check that it gives the right dimension for examples on surfaces.
[Caz, Sec. 3.3] [Sei08, Sec.11e]

3.5. **Cotangent bundles.** (Laura) Outline a correspondence between smooth topology on X and symplectic topology on T^*X. For $HF(L,L) \simeq H_*(L)$ give idea of heuristic proof (adiabatic limit: intuitive but technically involved), and also another proof using Piunikhin–Salamon–Schwarz morphisms (technically easier). Derive from $\mathcal{FM}(X)$ a definition of $\mathcal{F}uk(T^*X)$.
[Caz, Sec. 2.10, 3.9], [Alb08].

3.6. **Fredholm theory.** (Anthony)
Definition and properties of Fredholm operators, Sobolev spaces and Linearized Cauchy–Riemann operator. Index formula for strips (without proof).
[Oh, Sec. 10.1–10.3]

3.7. ***Transversality and failure of transversality.** (Hao)
Explain strategy of proof of transversality, importance of the somewhere injectivity assumption. Describe examples of branched covers, give moduli spaces of wrong dimensions, no matter how you perturb.
[Oh, Sec. 10.4, 10.6]

3.8. ***t-dependent almost complex structures, transversality.** (Léo S.)
Explain why allowing domain-dependent perturbations solves the above problem. How symmetries restrict the choices of such perturbations. [FHS95]

4. Fukaya categories

4.1. **Deligne–Mumford vs associahedron.** (Bangxin)
Define the Deligne–Mumford moduli space $\mathcal{M}_{0,k}$ of discs with k marked boundary points, give its dimension and its compactification, explain that it identifies with the space of trees.
[Sei08, Sec. 9c-9f]

4.2. **The Fukaya category.** (Stavroula) Define the μ^k maps, recall quickly why A_∞ relations hold (just same as for $\mathcal{FM}(X)$), do some examples on surfaces (what polygons count, what don't, according to the dimensions). Define $\mathcal{F}uk(M)$ (objects, morphisms involving perturbations). Briefly explain how to make it an A_∞–category (universal choice of perturbations...).
[Aur14, Sec. 2], [Smi15, Sec. 4]

4.3. ***More general approaches to noncompactness and disc bubbling.** (Renaud)

- Unbounded area: introduce the Novikov ring.
- To avoid bubbling: curved A_∞ structures, bounding cochains.

Recall examples in surfaces when this happens. [Caz, Sec. 3.11], [Smi15].

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