# Classical mechanics

## (1) Newton law.

in $\mathbb{R}^3$

ex 0 : $q$ position of an object

$q = (x, y, z) \in \mathbb{R}^3$ , $v = \dot{q}$ $\left(= \frac{dq}{dt}\right)$

$\ \ \overset{\shortparallel}{q(t)}$ $\qquad\qquad = (\dot{x}, \dot{y}, \dot{z})$

speed


$v(t) = v_0$ = constant

$q(t) = q_0 + t\, \underset{\overset{\shortparallel}{v(t)}}{\dot{q}(t)}$

Denote by $\vec{F}$ the force acting on the particle

ex 1 : $\vec{F} = \vec{g} = (0, 0, -1)$

Newton law: $\begin{cases} m\vec{a} = \vec{F} \\ \vec{a} = \dot{\vec{v}} = \ddot{\vec{q}} \end{cases}$

$m$ mass, $\vec{a}$ acceleration

$$v(t) = v_0 + t\,\dot{v}(t)$$
$$= -t\,(0,0,1)$$
$$= \dot{q}(t)$$
$$\Rightarrow q(t) = q_0 - \frac{t^2}{2}(0,0,1)$$

$$v(t) = (1, 0, -t)$$
$$q(t) = q_0 + \left(t, 0, -\frac{t^2}{2}\right)$$

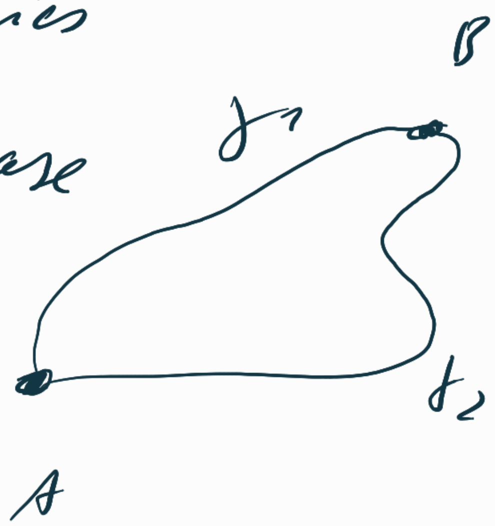
ex 2: $\vec{F} = -\lambda\,\dot{\vec{q}}$ (friction)

$\lambda > 0$

$$\dot{q}(t) = \dot{q}_0\, e^{-\lambda t}$$
$$q(t) = q_0 - \frac{\dot{q}_0}{\lambda} e^{-\lambda t}$$

2) Conservative force / system

**Def:** $\vec{F}$ is conservative if for any trajectories $\gamma_1, \gamma_2$, one has

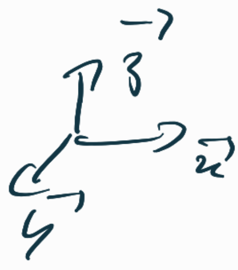



$$\overbrace{\int_{\gamma_1} \langle \vec{F}, \dot{\gamma}_1(t) \rangle \, dt}^{\text{work of } F} = \int_{\gamma_2} \langle \vec{F}, \dot{\gamma}_2(t) \rangle \, dt$$

**exo:** conservative

**Prop:** $\vec{F}$ conservative iff $\vec{F} = -\vec{\nabla} V$ for some function $V : \mathbb{R}^3 \to \mathbb{R}$ (potential)

$$\left[ \vec{\nabla} V = \partial_x V \, \vec{x} + \partial_y V \, \vec{y} + \partial_z V \, \vec{z} \right]$$



Proof: Need to show

$\forall \gamma$ closed trajectory

$$(*) \quad \int_\gamma \langle \vec F, \dot\gamma(t) \rangle \, dt = 0$$

Define $F^\flat : \mathbb{R}^3 \to \mathbb{R}$

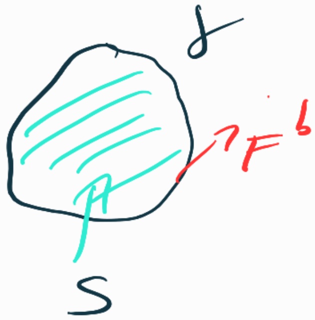
linear form $\vec v \longmapsto \langle \vec F, \vec v \rangle$

one form

$$(*) \iff \int_\gamma F^\flat = 0$$

$\quad \| \quad$ Stokes

$$\int_{S,\, \partial S = \gamma} dF^\flat$$




$$\iff dF^\flat = 0$$

$\iff$ ($\mathbb{R}^3$ contractible) $\exists\, V : \mathbb{R}^3 \to \mathbb{R}$, $dV = F^\flat$

$V$ is potential ∎

ex 1 is conservative since $\vec F = -\nabla V$, $V = g$

Define the energy of a conservative system

$$E = \underset{\substack{\uparrow \\ \text{kinetic} \\ \text{energy}}}{T} + \underset{\substack{\uparrow \\ \text{potential}}}{V}$$

$$\frac{1}{2} m \|v\|^2 \quad v = \dot{q}$$

$$\|\dot{q}\|^2$$

Prop: $E$ is constant along trajectories of the conservative systems.

Proof: compute $\frac{\partial E}{\partial t}$ along a trajectory $\gamma$

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t} \frac{1}{2} m \langle \dot{\gamma}, \dot{\gamma} \rangle$$

$$= m \langle \ddot{\gamma}, \dot{\gamma} \rangle$$

$$\frac{\partial V(\gamma(t))}{\partial t} = d_{\gamma(t)} V(\dot{\gamma}(t)) \quad \text{chain rule}$$

$$= \langle \nabla V, \dot{\gamma} \rangle$$

$$= \langle -F, \dot{\gamma} \rangle$$

Newton Law $= \langle -m\ddot{\gamma}, \dot{\gamma} \rangle$

$$= -\frac{\partial T}{\partial t} \qquad \blacksquare$$

3) Hamiltonian pt of view

$\mathbb{R}^n \times \mathbb{R}^n$ think as $T\mathbb{R}^n$

$(q, p)$ $\qquad (p = \dot{q})$

Hamiltonian

$$H(q,p) = \frac{1}{2}\|p\|^2 + V(q)$$

induces a classical mechanical system on $\mathbb{R}^n \times \mathbb{R}^n = T\mathbb{R}^n$

Level sets $= \{ H(q,p) = E \}$ are trajectories

$-\ddot{\gamma}$

One has

$$\begin{cases} \dfrac{\partial H}{\partial q_i} = \dfrac{\partial V}{\partial q_i} = -\ddot{q}_i = -\dot{p}_i \\[2ex] \dfrac{\partial H}{\partial p_i} = p_i = \dot{q}_i \quad (= \dot{\gamma}) \end{cases}$$

$$\nabla H : T\mathbb{R}^n \longrightarrow T\mathbb{R}^n$$

$$\begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} -\dot p \\ \dot q \end{pmatrix}$$

along trajectories of the system.

4) Symplectic structure

The curve

$$t \longmapsto \begin{pmatrix} \gamma(t) \\ \dot\gamma(t) \end{pmatrix} \text{ in } T\mathbb{R}^n$$

is an integral curve for the vector field

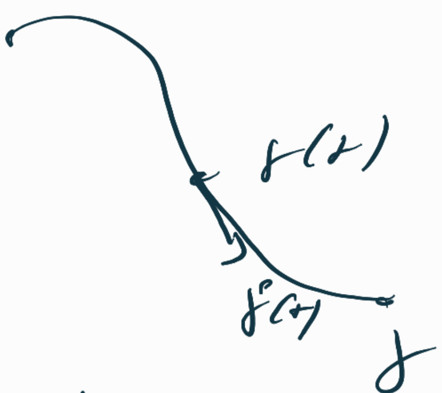
$= \dot q$

$$X_H = J \nabla H : \begin{pmatrix} \gamma(t) \\ \dot\gamma(t) \end{pmatrix} \longmapsto \begin{pmatrix} \dot\gamma(t) \\ \ddot\gamma(t) \end{pmatrix}$$

$= \dot p$

with

$$J = \underbrace{\begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}}_{\text{is a symplectic matrix}}$$




→ endows $T\mathbb{R}^n$ with

a symplectic structure

$$\omega = \sum_i dp_i \wedge dq_i$$

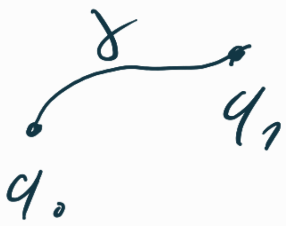
5) Action functional and Lagrangian formalism

The action functional is

$$\mathcal{A}_H : \mathcal{P}(q_0, q_1) \longrightarrow \mathbb{R}$$

$$\gamma \longmapsto \boxed{\int_0^1 |\dot\gamma|^2 - H(\gamma, \dot\gamma)\, dt}$$

$\gamma(0) = q_0$
$\gamma(1) = q_1$




$$\mathcal{P}(q_0; q_1) = \left\{ \begin{array}{l} \gamma : [0,1] \to \mathbb{R}^n \\ \gamma(0) = q_0 \\ \gamma(1) = q_1 \end{array} \right\}$$

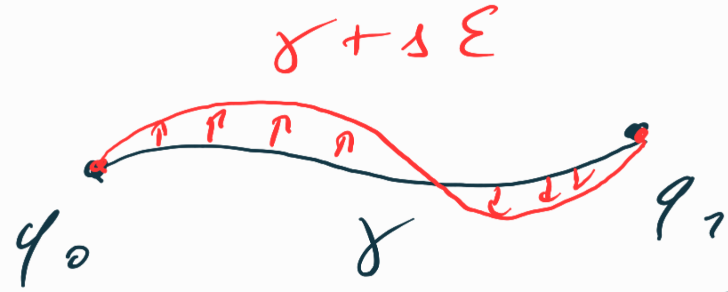
**Theorem** (Least action principle)
$\gamma$ is a <u>trajectory</u> iff it is a critical point of $\mathcal{A}_H$ i.e. $D_\gamma \mathcal{A}_H = 0$

$$\boxed{L(\gamma) = \int_0^1 |\dot\gamma|\, dt \quad \text{length}}$$

Differential:

$$\gamma \in \mathcal{P}(q_0, q_1)$$

$$T_\gamma \mathcal{P}(q_0, q_1) = \left\{ \varepsilon : [0,1] \to T\mathbb{R}^n \atop \varepsilon(0) = \varepsilon(1) = 0 \right\}$$




$$D_\gamma \mathcal{A}_H(\varepsilon) = \lim \left. \frac{\mathcal{A}_H(\gamma + s\varepsilon) - \mathcal{A}_H(\gamma)}{s} \right|_{s=0}$$

Proof of the Thm:

$$\mathcal{A}_H(\gamma + s\varepsilon)$$

$$= \int_0^1 \left\| \dot{\overline{\gamma + s\varepsilon}}(t) \right\|^2 - H\left(\gamma + s\varepsilon, \dot{\overline{\gamma + s\varepsilon}}\right) dt$$

$$H(\gamma + s\varepsilon, \dot\gamma + s\dot\varepsilon)$$

$$= H(\gamma, \dot\gamma) + s\left[\left(\overrightarrow{\frac{\partial H(\gamma, \dot\gamma)}{\partial q}} \cdot \vec{\varepsilon}(t)\right)\right.$$

$$\Longrightarrow \qquad \sum_i \frac{\partial H}{\partial q_i} \cdot \varepsilon_i \in \mathbb{R}$$

$$+ \frac{\partial \vec{H}(\gamma,\dot\gamma)}{\partial \dot q} \cdot \dot{\vec{\varepsilon}}(t) \Big]$$

Hamil. eq.

$$\Rightarrow A_H(\gamma + s\varepsilon) = A_H(\gamma) + s\int_0^1 \left( \cancel{2\langle\dot\gamma,\dot\varepsilon\rangle} - \langle\ddot\gamma,\varepsilon\rangle - \cancel{\langle\dot\gamma,\dot\varepsilon\rangle} \right) dt$$

$$= A_H(\gamma) + s\int_0^1 \underbrace{\cancel{\langle\dot\gamma,\dot\varepsilon\rangle - \langle\ddot\gamma,\varepsilon\rangle}}_{=0}\, dt$$

$$\left( \int_0^1 \langle\dot\gamma,\dot\varepsilon\rangle\, dt \overset{\text{by parts}}{=} \cancel{\langle\dot\gamma,\varepsilon\rangle\Big|_0^1} - \int_0^1 \langle\ddot\gamma,\varepsilon\rangle\, dt \right)$$

$$= A_H(\gamma)$$

$$\Rightarrow D_\gamma A_H(\varepsilon) = 0 \text{ for all } \varepsilon$$

$$\Rightarrow D_\gamma A_H = 0$$

Note: $L(q,\dot q) = |\dot q|^2 - H(q,\dot q)$ is called the Lagrangian.

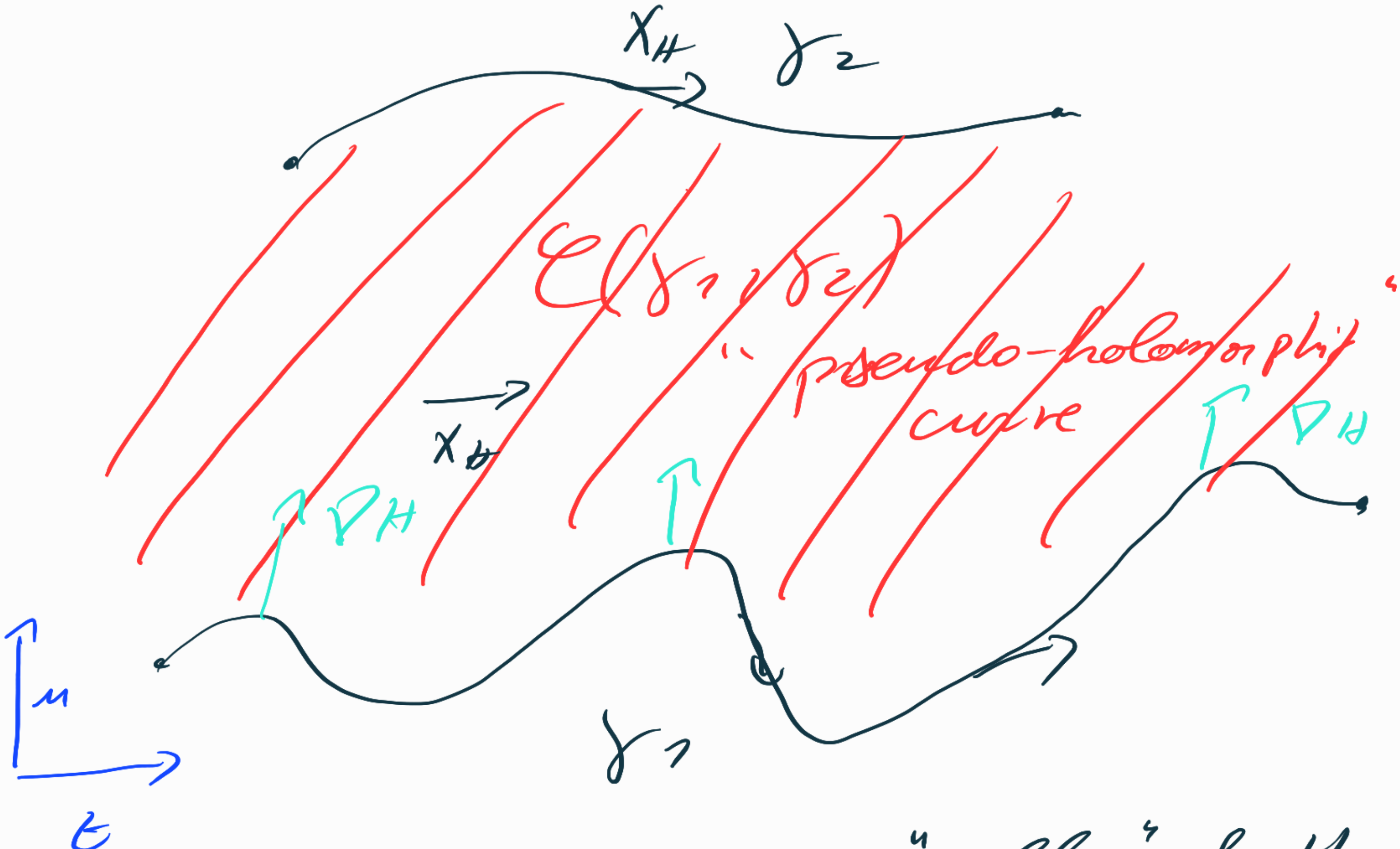
Hamilton equations

$$\Longleftrightarrow \quad \boxed{\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}} = 0}$$

Euler-Lagrange equations

6) What we will do?

consider critical points of $\mathcal{A}_H$ action functional seen as a Morse function:




critical points $\longrightarrow$ "cells" of the Floer complex

Pseudo-holomorphic curves $\rightsquigarrow$ $\partial$ operators of the complex

$\mathcal{E}(\gamma_1, \gamma_2)$

$= \tilde{\gamma}(t, u)$

$\tilde{\gamma}(t, -\infty) = \gamma_1(t)$

$\tilde{\gamma}(t, +\infty) = \gamma_2(t)$

ask that

$$\begin{pmatrix} \frac{\partial \tilde{\gamma}}{\partial t} \\ \frac{\partial \dot{\tilde{\gamma}}}{\partial t} \end{pmatrix} = X_H \quad \text{symplectic gradient}$$

$$\begin{pmatrix} \frac{\partial \tilde{\gamma}}{\partial u} \\ \frac{\partial \dot{\tilde{\gamma}}}{\partial u} \end{pmatrix} = \nabla H$$

but we saw

$X_H = J \cdot \nabla H$

$$\Rightarrow \boxed{\frac{\partial \tilde{\gamma}}{\partial u} + J \cdot \frac{\partial \tilde{\gamma}}{\partial t} = 0}$$

# Cauchy-Riemann equation