BTW3 pretalk 2: Symplectic Geometry

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Def: (M,ω) is a symplectic manifold if

- M: smooth mfd,
- $\omega \in \Omega^2(M)$ is a closed non-degenerate 2-form.

↳ $d\omega = 0$

↳ $\forall x$, $\ker \omega_x = \{0\} \subset T_x M$
$(\omega_x : T_xM \to T_xM^*)$

$\Leftrightarrow$ $\dim M = 2n$, and ω^n is a volume form.

Rks:
- * $\dim M$ even
- * M is oriented (by ω^n)
- * if M compact, then

$\forall i = 0, \dots, n$, $0 \neq H^{2i}(M;\mathbb{R}) \ni [\omega]^i \neq 0$
(since $[\omega]^n = [\omega]^i \wedge [\omega]^{n-i} \neq 0$)

Ex:

- oriented surf. with volume form
- $\mathbb{R}^{2n}_{std} = \left(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i\right)$
- Q smooth mfd, $(T^*Q, \omega = -d\lambda)$

canonical 1-form: $\lambda_{(q,p)} \cdot v = p(d\pi \cdot v)$ with $\pi : T^*Q \to Q$

in coord: $= -d\left(\sum_i p_i \, dq_i\right)$

$= \sum_i dq_i \wedge dp_i$

- Coadjoint orbits $(G \curvearrowright \mathfrak{g}^*)$

semi-simple Lie group

dual Lie algebra

- Character varieties of surfaces

$$\chi(\Sigma, G) = \mathrm{Hom}(\pi_1 \Sigma, G) / G^{ad} \quad \text{(when smooth)}$$

- $\mathbb{CP}^n$, smooth projective varieties / $\mathbb{C}$.
- ...

Def: A submfd $S \xhookrightarrow{\iota} (M,\omega)$ is:

- isotropic if $\iota^*\omega = 0$ i.e. $TS \subset TS^{\perp\omega} = \{ v \in TM \,/\, \omega(v,\xi) = 0 \ \forall \xi \in TS \}$ "symplectic orthogonal"

  ($\Rightarrow \dim S \leq n$)

- coisotropic if $TS^{\perp\omega} \subset TS$

  ($\Rightarrow \dim S \geq n$)

- Lagrangian if isotropic & coisotropic

  $\Leftrightarrow$ isotropic with $\dim = n$

  $\Leftrightarrow$ coisotropic with $\dim = n$.

are the important submfd to look at, and will form the objects of $\mathcal{F}uk(M)$ (with extra structure)

Q: Why are Lagrangians submfd important?

A: "Everything is Lagrangian" (Weinstein)

Examples:

- any curve in an oriented surface
- $\mathbb{R}^n, i\mathbb{R}^n \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n$
- $\mathbb{RP}^n \subset \mathbb{CP}^n$, $\mathbb{T}^n \subset \mathbb{R}^{2n}$
- Conormal bundles: $S \subset Q$ smooth submfd

$$N_S^*Q = \left\{ (q,p) \;/\; \begin{array}{l} \cdot\, q \in S \\ \cdot\, p|_{T_qS} = 0 \end{array} \right\} \subset T^*Q$$

↳ $S = Q \to$ zero section

↳ $S = \{q_0\} \to T^*_{q_0}Q$ cotangent fibre.

* Hamiltonian group actions

"moment map"

Def: $G \circlearrowright (M,\omega)$ is <u>Hamiltonian</u> if $\exists\ \mu : M \longrightarrow \mathfrak{g}^*$ G-equivariant s.t. $\forall \xi \in \mathfrak{g}$,

$$X_\xi(m) = \nabla^\omega(\mu,\xi) \longleftarrow \text{symplectic gradient}$$

infinitesimal action:

$$X_\xi(m) = \frac{d}{dt}\Big|_{t=0}\left(e^{t\xi} m\right)$$

$$\iff \iota_{X_\xi}\omega = d(\mu,\xi)$$

"Weinstein Lagrangian":

$$\Lambda_G(M) := \left\{ ((g,p), m, m') \;\middle|\; \begin{array}{l} m \in M \\ m' = gm \\ R^*_{g^{-1}} p = \mu(m) \end{array} \right\}$$

$$\subset T^*G \times M^- \times M$$

* . . .

Def: An **almost-complex structure** (acs) on a $\mathbb{R}$-v.b. $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is $J \in \operatorname{End} E \simeq \Gamma\left(\begin{array}{c} E^* \otimes E \\ \downarrow \\ B \end{array}\right)$

s.t. $J^2 = -\mathrm{Id}$.

($\Leftrightarrow$ a structure of $\mathbb{C}$-v.b. on E)

* An acs on a manifold M is an acs on $TM \to M$.

* Assume (M, ω) symplectic. An acs J on M is:

- **ω-tame** if $\forall v \neq 0, \omega(v, Jv) > 0$.

$\Leftrightarrow g_J(v,w) = \frac{1}{2}\left[\omega(v,Jw) + \omega(w,Jv)\right]$ Riem. metric

- **ω-compatible** if ω-tame + $\forall v,w$, $\omega(Jv,Jw) = \omega(v,w)$

$\Leftrightarrow g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$ J-invariant Riem. metric.

Ex: * $\mathbb{C}$ - structure $\rightsquigarrow$ acs structure.

* Riem. metric on Q $\rightsquigarrow$ $T^*Q \simeq TQ$

$\rightsquigarrow$ ω- compat. acs on T^*Q.

Th: (Gromov) (M,ω) symplectic. The spaces $\mathcal{J}_\tau(M,\omega)$ and $\mathcal{J}(M,\omega)$ of ω-tame and ω - compat. acs are nonempty and contractible.

- $J \in \mathcal{J}(M,\omega)$ compatible acs

$(\Sigma, j, dvol_{\Sigma})$ Riemann surface (say without boundary first)

$u : \Sigma \longrightarrow M$ smooth map.

$\hookrightarrow$ Energy: $E(u) = \| du \|^2_{L^2_{g_J}} = \frac{1}{2} \int_{\Sigma} |du|^2_{g_J} \cdot dvol_{\Sigma}$

$\left(|L|_{g_J} = |\xi|^{-1} \sqrt{|L(\xi)|^2 + |L(j\xi)|^2} \right)$

Rk: * $E(u) \geq 0$, $= 0$ iff u loc. constant

* sensitive to perturbations of u.

$\hookrightarrow$ Area $A(u) = \int_{\Sigma} u^* \omega$

Rk: * Only depends on $u_*[\Sigma] \in H_2(M;\mathbb{Z})$ (say Σ closed)

(since $d\omega = 0$, Stokes formula ...)

$$E(u) = \|du\|^2_{L^2_{g_J}} \qquad A(u) = \int_\Sigma u^*\omega$$

Proposition: $E(u) = A(u) + \int_{\Sigma} |\bar\partial_J u|^2_{g_J} \, d\mathrm{vol}_{\Sigma}$,

with $\bar\partial_J$ the *Cauchy-Riemann* operator:

$$\bar\partial_J u := \frac{1}{2}\left(du + J \circ du \circ j \right)$$

In particular, $E(u) \geq A(u)$.

Def: u is *pseudo-holomorphic* (or J-holomorphic, or (j,J)-holomorphic) if $E(u) = A(u)$

$\Leftrightarrow \bar\partial_J u = 0$

$\Leftrightarrow du$ is $\mathbb{C}$-linear : $du \circ j = J \circ du$

$$\begin{tikzcd} T_z\Sigma \arrow[r, "du"] \arrow[d, "j"'] & T_{u(z)}M \arrow[d, "J"] \\ T_z\Sigma \arrow[r, "du"] & T_{u(z)}M \end{tikzcd}$$

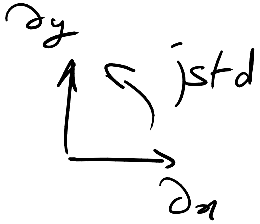
Some obvious, but very important remarks:

- $*$ J-holom. curves are energy local minimizers
- $*$ doesn't depend on $d\mathrm{vol}_{\Sigma}$
- $*$ if u is J-holom, then $A(u) \geq 0$, with $=$ iff loc. constant
- $*$ if $\partial\Sigma \neq \emptyset$, usually ask Lagrangian boundary conditio

$$u : (\Sigma, \partial\Sigma) \longrightarrow (M, L)$$

↳ $A(u)$ only dep.^{t} on $u_*[\Sigma] \in H_2(M, L; \mathbb{Z})$

- $*$ if $\Sigma \subset (\mathbb{C}, j_{std})$



$$\bar{\partial}_J u = 0 \iff 0 = \frac{\partial u}{\partial x} + J \cdot \frac{\partial u}{\partial y} \quad \left(= \bar{\partial}_J u \cdot \frac{\partial}{\partial x} \right)$$

Th: [Gromov compactness] (More details in Talk 3.1).

Let $u_n : (\Sigma^i, j_n) \longrightarrow (K, J)$, $K \subset (M, \omega)$ compact

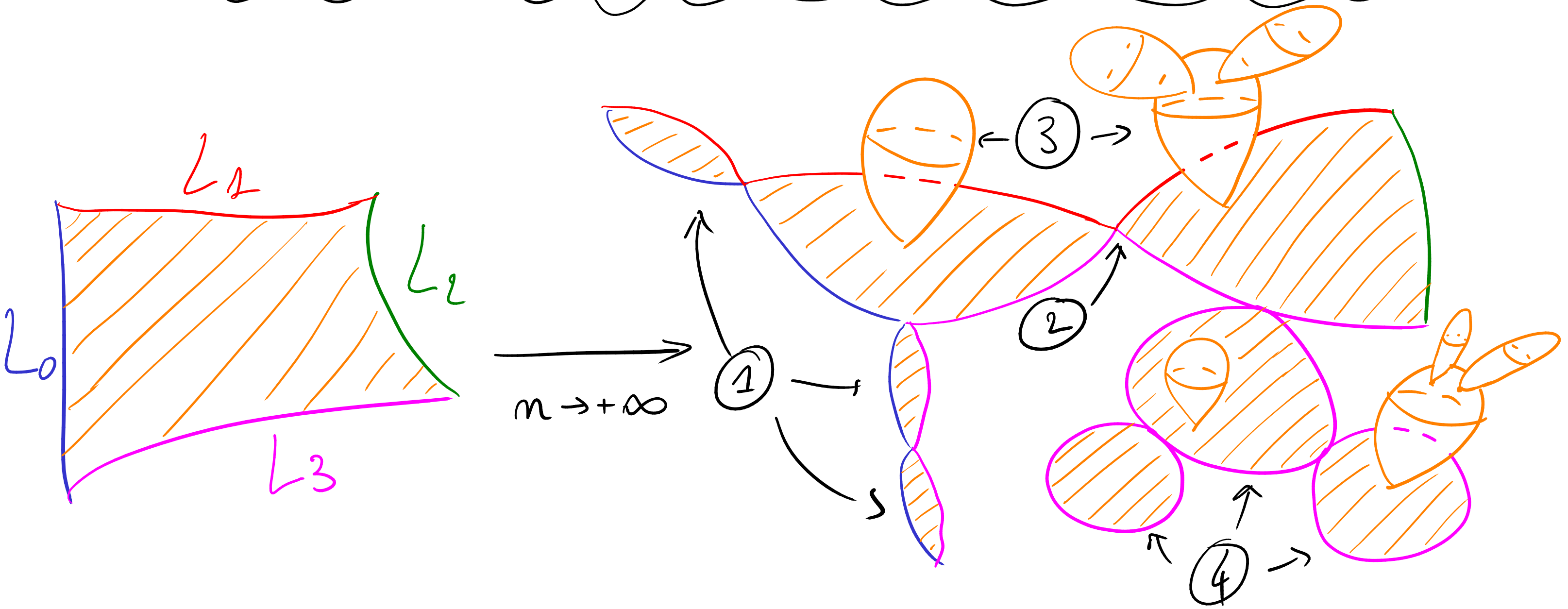
$\partial^i \Sigma^i \longrightarrow L_i$ $J \in \mathcal{J}(M, \omega)$

$L_1, L_2, \ldots \subset M$ Lagrangians.

be (j_n, J)-holomorphic, with bounded energy:

$$E(u_n) \leq C \leftarrow \text{indep}^{t} \text{ on } n.$$

then $\{u_n\}$ converges to a "stable map" in the Gromov topology:

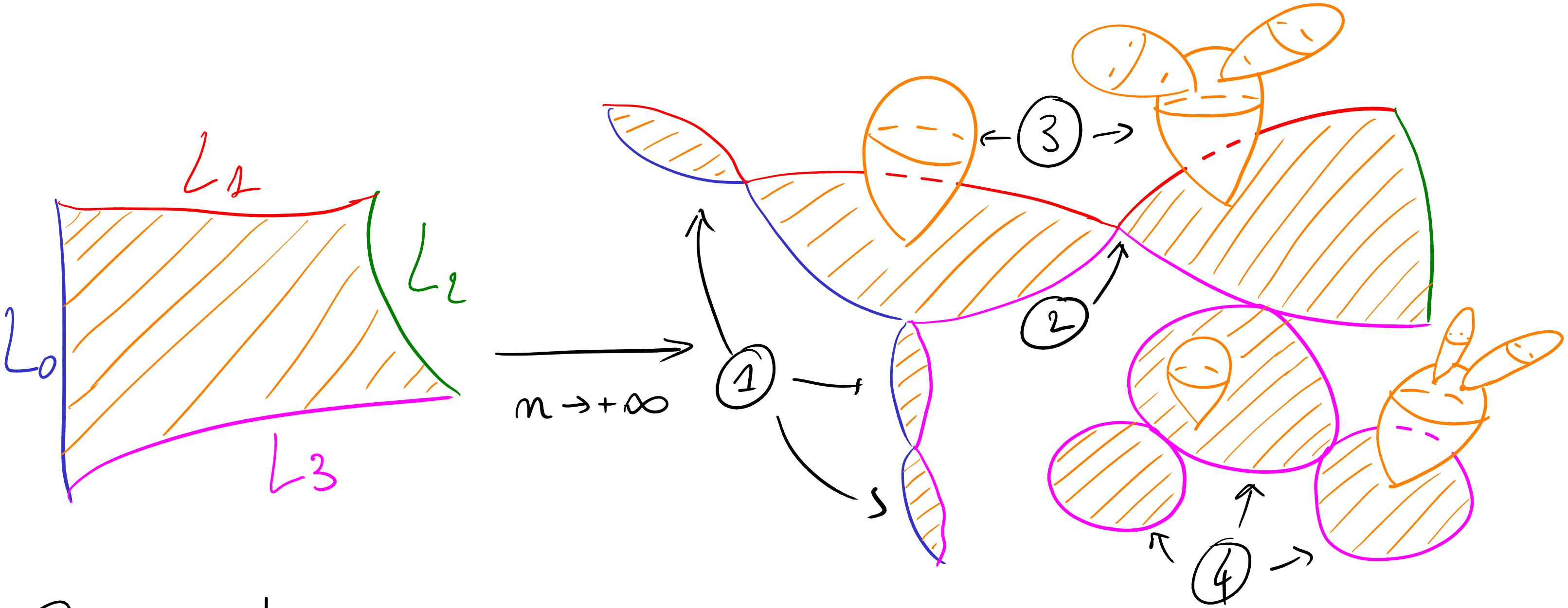


① Strip breaking

② Domain degeneration

③ Sphere bubbling

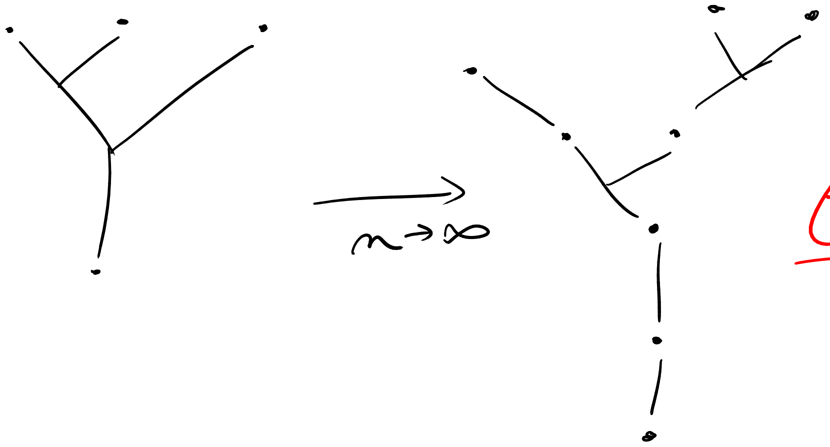
④ Disc bubbling



① Strip breaking
② Domain degeneration
③ Sphere bubbling
④ Disc bubbling

Remarks: * K compact & $E(u_n) \leq C$ are important assumptions

* degenerations ① and ② already happen in Morse theory (see talk 2.5) but ③ and ④ are new features of Floer theory.



Goal: Put assumptions on M, L, J that:
→ ensure stay in K compact + bounded E
→ rule out ③ and ④

Def: * (M,ω) is exact if ω is exact, i.e. $\omega = d\lambda$, for some $\lambda \in \Omega^1(M)$.

* Let $L \xrightarrow[\iota]{} (M, \omega = d\lambda)$ be a Lagrangian. then $\iota^*\lambda$ is closed $\left(d(\iota^*\lambda) = \iota^* d\lambda = \iota^*\omega \underset{\substack{\uparrow \\ \text{Lagr.}}}{=} 0 \right)$

Say that L is exact if $\iota^*\lambda$ is exact, i.e. $\exists\, f : L \to \mathbb{R}$ s.t. $\iota^*\lambda = df$.

Ex: $M = T^*Q$, $L = \Gamma(df)$, $L = N^*_S Q$

(graph of exact 1-form) (conormal bundle)

Having M & L exact rules out sphere & disc bubbling:

<u>prop</u>: 1- If $(M, \omega = d\lambda)$ exact, $J \in \mathcal{J}(M,\omega)$, any J-holomorphic sphere $u : \mathbb{C}P^1 \to M$ must be constant.

2- If $L \hookrightarrow M$ exact, $J \in \mathcal{J}(M,\omega)$, any J-holomorphic disc $u : (D^2, \partial D^2) \longrightarrow (M, L)$ is constant.

<u>proof</u>: Stokes formula

1- $A(u) = \int_{\mathbb{C}P^1} u^* d\lambda = \int_{\partial \mathbb{C}P^1} u^* \lambda = 0.$

2- $A(u) = \int_{D^2} u^* d\lambda = \int_{\partial D^2} u^* \lambda = \int_{\partial \partial D^2} u^* f = 0 \qquad \square$

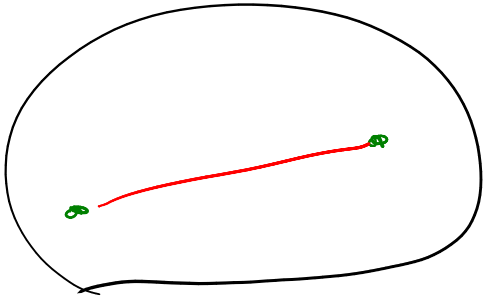
- Sphere bubbling ✓
- Disc bubbling ✓
- Energy bound ✓ ← can express the energy of the curves you'll count using the functions $f: L \to \mathbb{R}$ (see talk 3.2)
- Stay in K compact ✗

How about asking M compact?

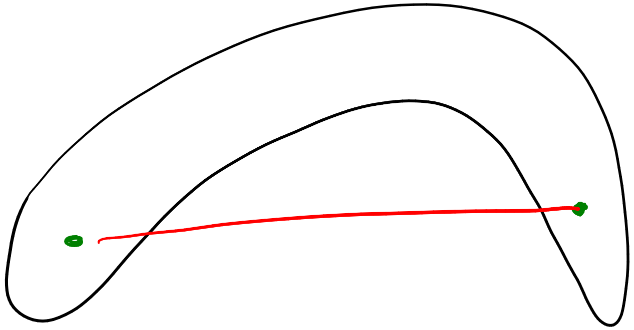
Bad idea: any exact M must be noncompact (otherwise $[\omega] \neq 0 \in H^2(M)$)

⇒ use convexity instead.

Convexity:



convex



not convex

⚠ Really depends on which J...

Def $K \subset (M, \omega, J)$ is convex if $\forall u : \Sigma \longrightarrow M$ J-holomorphic and s.t. $u(\partial\Sigma) \subset K$, one has $u(\Sigma) \subset K$.

compact with boundary

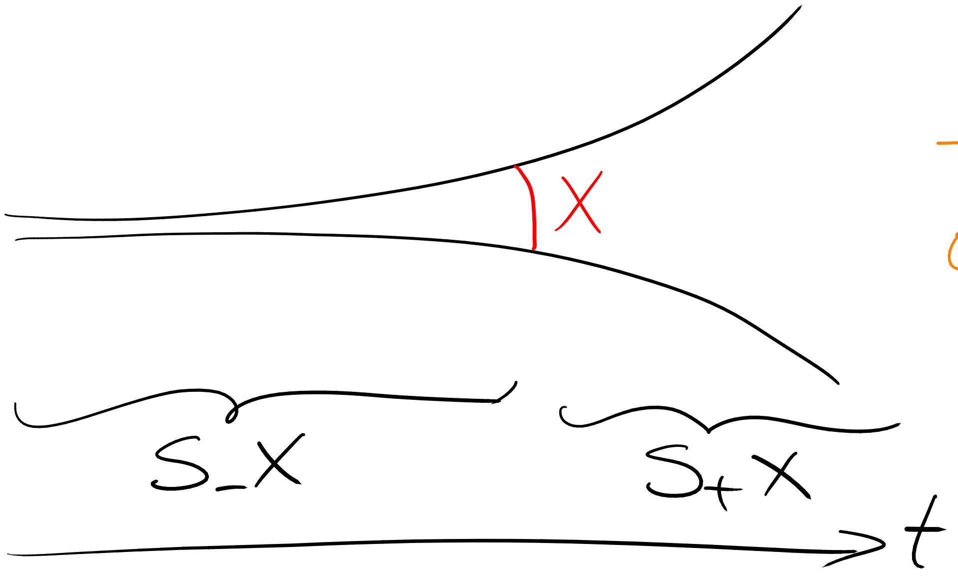
Def: $(X^{2n+1}, \xi = \ker\alpha)$ is a (co-oriented) **contact manifold** if $\alpha \in \Omega^1(X)$ is s.t. $\alpha \wedge (d\alpha)^n$ is a volume form ($\Leftrightarrow d\alpha|_{\xi}$ is symplectic)

. $\xi \subset TX$: "contact distribution", α : "contact form"

* **Symplectization** $(X, \xi) \rightsquigarrow SX = (X \times \mathbb{R}_t, d(e^t\alpha))$

positive/negative symplectizations: $S_+X = X \times \mathbb{R}_{\geq 0} \subset SX$

$S_-X = X \times \mathbb{R}_{\leq 0} \subset SX$



→ We will model the ends of M on S_+X.

⚠ Still need to find good acs J to work with.

Def: An exact $(M, \omega = d\lambda)$ has convex ends if can find (X, ξ) + $\psi : S_+X \xhookrightarrow[\text{exact sympl. embedding}]{} M$ such that $M \setminus \psi(\mathrm{int}\, S_+X)$ compact

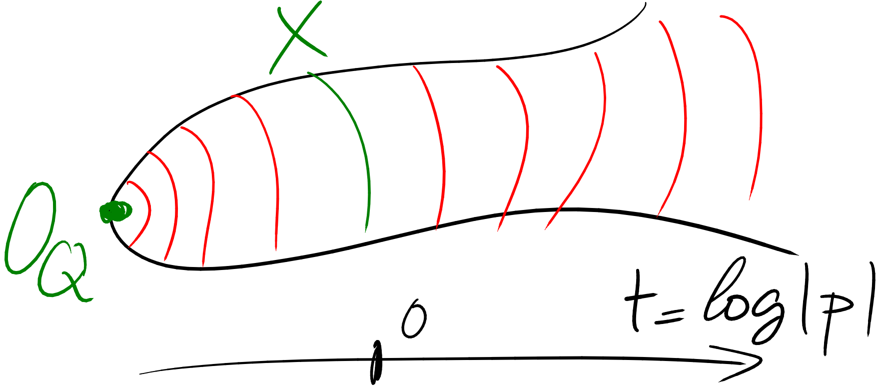
$(\psi^*\lambda = e^t\alpha)$

Ex: (Q, g) Riemannian

$\rightsquigarrow (M = T^*Q, \omega = d\lambda, J)$

$X := \{(q,p) \in T^*Q \mid |p| = 1\}$: unit cotangent bdle

$T^*Q \setminus \underset{\text{zero sect}}{0_Q} \simeq SX \Longrightarrow T^*Q$ has convex ends



Def: $J \in \mathcal{J}(M,\omega)$ of contact type at the ends of M if:

- J indep^t on t
- $dh \circ J = -\lambda$, with $h : \psi(S_+X) \longrightarrow \mathbb{R}$

$$\psi(x,t) \longmapsto t$$

prop:

- $(M, \omega = d\lambda)$ with cvx ends $E = \psi(S_+X)$
- $J \in \mathcal{J}(M,\omega)$ of ctact type
- $u : \Sigma \longrightarrow M$ J-holomorphic

$\hat{\Sigma} := u^{-1}(E) \subset \Sigma$, $\rho = h \circ u$

$$\begin{tikzcd} \hat{\Sigma} \arrow[r, "u"] \arrow[dr, "\rho"'] & E \arrow[d, "h"] \\ & \mathbb{R} \end{tikzcd}$$

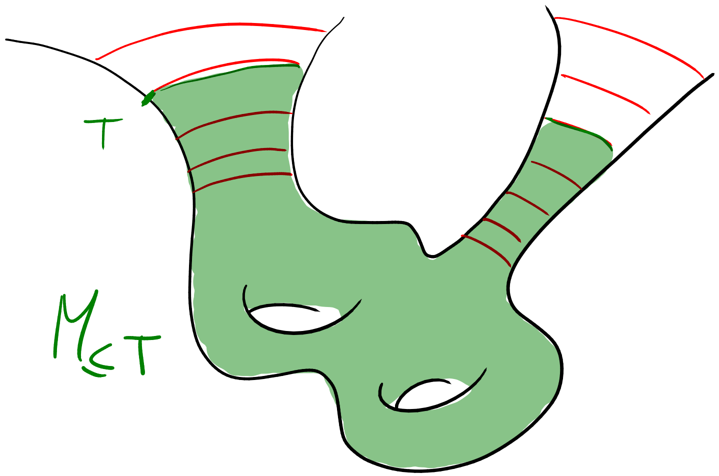
Then ρ can't have loc. max, unless locally constant.

proof:

$$\begin{aligned}(\Delta\rho)\cdot dvol_{\Sigma_1} &= -d(d\rho\circ j)\\ &= -d(dh\circ J\circ du) &&\leftarrow u \text{ is } J\text{-hol.}\\ &= d(\lambda\circ du) &&\leftarrow J \text{ is of ctact type}\\ &= u^*\omega &&\leftarrow \omega = d\lambda\\ &\geq 0.\end{aligned}$$

Maximum principle: local max $\Rightarrow$ locally constant $\square$

Consequence: The compact subsets $M_{\leq T} := M\setminus\{t>T\}$ are convex for all $T\geq 0$.



Conclusion: The working assumptions will be:

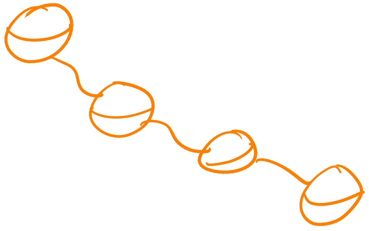
- M exact with convex ends
- $J \in \mathcal{J}(M,\omega)$ of contact type
- $L \subset M$ compact exact Lagrangian

- Sphere bubbling ✓
- Disc bubbling ✓
- Energy bound ✓
- Stay in K compact ✓

Remark: * can work with other assumptions

* failure of checking the four items doesn't mean it's impossible, but you usually need more sophisticated machinery (→ talk 4.3)

- Sphere bubbling X → include "pearl trajectories" in your moduli spaces



- Disc bubbling X → get curved A_∞-structures, use "bounding cochains" (talk 4.3)

→ failure of transversality
⇒ use "Kuranishi structures" (or "stabilizing divisors", or "polyfolds" ...)

- Energy bound X → keep track of the area with a Novikov ring (talk 4.3)

- Stay in K compact X → use "SFT compactness"