1. Motivation:

$Df = g$ (1) If solution exists

(2) How large is the solution space given g, initial condition

(3) What ppty does your solns share?

} key ppty — Elliptic operator

How is Elliptic Regularity (ER) are going to be used?

1. In Morse homology:

$\mathcal{M}(p,q)$

$M(p,q) = \{$trajectories of flow from p to $q\}$

$= \{$Soln of Morse eqn: $\frac{\partial}{\partial t} + \nabla f(u) = 0\}$

Soln of Floer eqn. (Fillipo)

$\mathcal{M}(p,q)$

$\{M(p,q)/_{\mathbb{R}} : \varphi_s(x) \mapsto \varphi_{t+s}(x)\} \longrightarrow$ This defines a cpt mfd (Gromov cpt thm fin-dim mfd)

Also it's fin-dimensional (Transversality condition. How to intrepret $\mathcal{M}(p,q)/_{\mathbb{R}}$ as a

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In proving GCT: (AD. Prop 6.6.2) $W^{1,p}$-Soln $\xRightarrow{\text{Rellich Lemma}}$ Cont soln $\xRightarrow{\text{Elliptic Regularity}}$ Smooth Soln

C^0_{loc}, C^1_{loc}, C^∞_{loc}-topology coincide $\mathcal{M}$.

To prove transversality. See Differential Floer map extends to a Fredholm operator

Elliptic Operator.

$\sigma_\xi(D) =$

(Non-example) $D = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$ 1-dim wave equation. $Du(t,x) = 0$

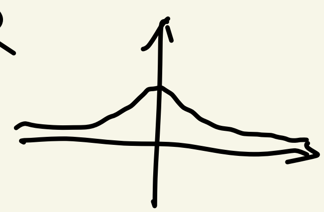
$u(t,x) = F(t-x) + G(t+x)$. F, G are arbitrary function.

u can be as bad as possible. & Large solution space

(Example) $D = \dfrac{\partial^2}{\partial t^2} + \dfrac{\partial^2}{\partial x^2}$ 1-dim heat equation $Du(t,x) = 0$

$$u(t,x) = \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} u(0,y)\,dy$$

$u(t,x)$ is very regular function.



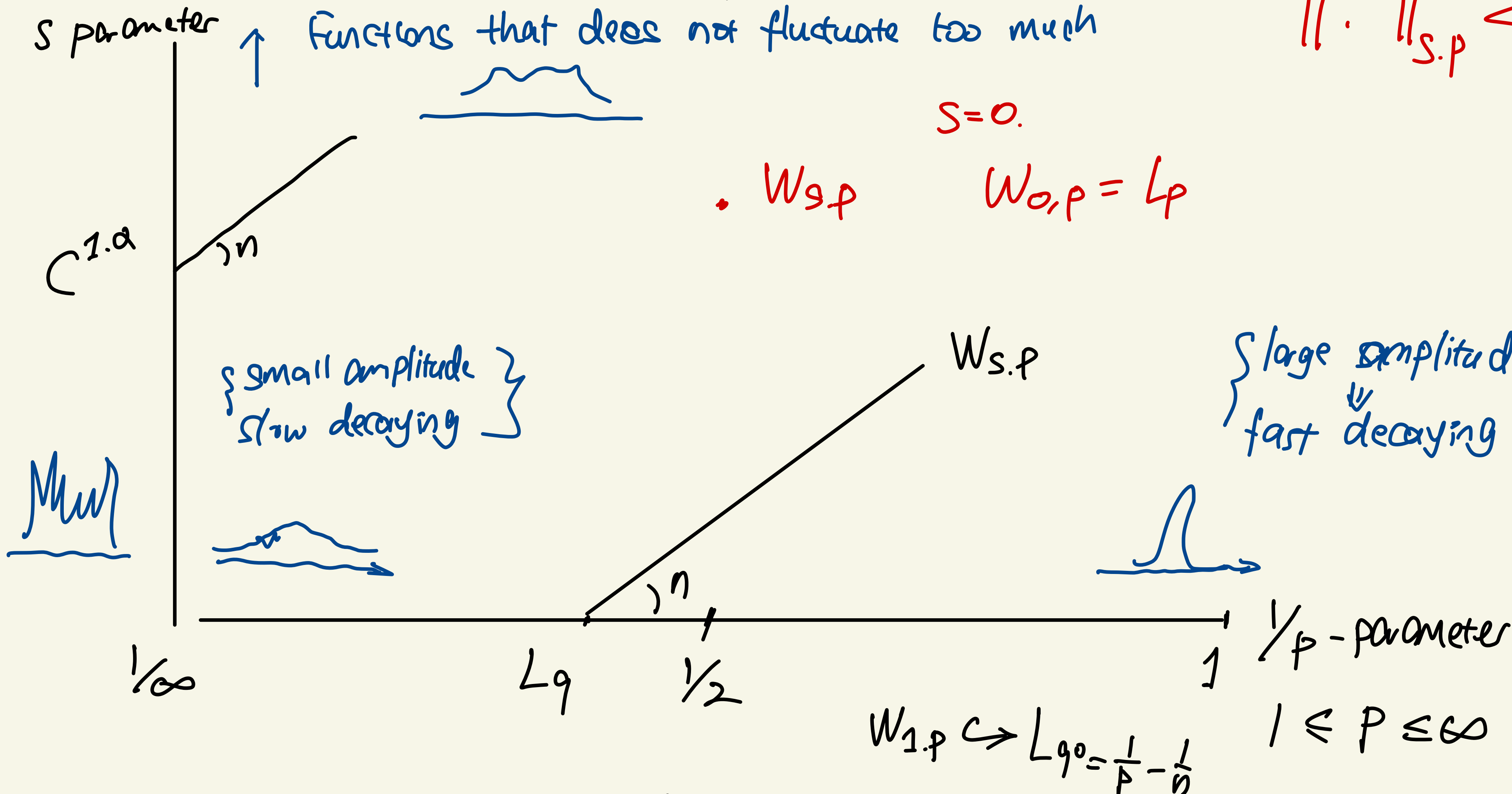
Functional Analysis Preparation.

A good notion of Regularity (Smoothness, Oscillation)

Exponent p: How spread the fct is (L^p-norm) $\quad \|\cdot\|_{s,p}$

Regularity s: How sensitive $\|\cdot\|_{s,p}$ is to oscillation

$\|\cdot\|_{s,p} < C$



Fix $p = 2$ for most of the time.

Recall Fourier transf: $\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} f(x)\, dx$

$$= \{ f \in L^2(\mathbb{R}^n) : (1+\Delta)^{k/2} f \in L^2(\mathbb{R}^n) \}$$

$$W_{k,2} := H_k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : (1+\|\xi\|^2)^{k/2} \hat{f} \in L^2(\mathbb{R}^n) \}$$

Sobolev space $k \in \mathbb{R}$

$$\overset{k\in\mathbb{N}}{=} \{ f \in L^2(\mathbb{R}^n) : \int |Df|^2 + |f|^2 < \infty \ \forall D \text{ diff operator of order} \le k \}$$

This is a Hilbert space:

$$\langle f, g\rangle_k = \int_{\mathbb{R}} \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, (1+\|\xi\|^2)^k \, d\xi$$

Rmk: $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ $\quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$

$$\hat{\hat{f}} = f$$

$$\widehat{Df} = \xi \hat{f}$$

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

$$\widehat{D^\alpha f}(\xi) = \overbrace{\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}}^{\xi^\alpha} \cdot \hat{f}(\xi)$$

$$\int_{\mathbb{R}^n} |D^\alpha f(x)|^2 dx = \int_{\mathbb{R}^n} |\widehat{D^\alpha f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\xi^\alpha \cdot \hat{f}(\xi)|^2 d\xi$$

$$\le \int_{\mathbb{R}^n} \|\xi\|^{2d} |\hat{f}(\xi)|^2 d\xi$$

Ppty of Sobolev spaces

(1) $H_k(\mathbb{R}^n) \subset H_l(\mathbb{R}^n)$ $\quad k > l \geq 0$ $\quad (1+\|\xi\|)^k > (1+\|\xi\|)^l$

(2) $\bigcap_k H_k(\mathbb{R}^n) = \lim_{k\to\infty} H_k(\mathbb{R}^n) =: S(\mathbb{R}^n)$ schwartz space.

$$C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset H_k(\mathbb{R}^n)$$ $\forall k$ dense is H_k

(3) There is a perfect pairing $H_k \times H_{-k} \longrightarrow \mathbb{C}$

$$\int u \cdot \bar{v} = \int \hat{u} \cdot \overline{\hat{v}} = \int \hat{u} (1+\|\xi\|)^k \hat{v} (1+\|\xi\|)^{-k} d\xi$$

in particular $(H_k)^* = H_{-k}$

(4) **Rellich Lemma** U bdd open subset of $\mathbb{R}^n$. Let $p > n$ Then.

$$W^{1,p}(U;\mathbb{R}^m) \hookrightarrow \mathcal{C}^0(U;\mathbb{R}^m)$$ cont even cpt embedding

Rmk: Extend the idea to cpt mfd. Need suff regular patching

(Good presentation)

Sobolev sections of vector bundles $\begin{matrix} E \\ | \\ X \end{matrix}$ extending $\Gamma(E)$

FACT: $D: E \to F$ a diff operator of order m. this extends to

$$D_s: H_s(E) \longrightarrow H_{s-m}(F) \quad \forall s \quad \text{Continuously}$$

Principal symbol: $D: E \to F$ $\quad \sum_{\alpha \leq m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial^\alpha}$ $\quad A^\alpha(x) \in \mathrm{Hom}(E_x, F_x)$

Define PS: $\sigma_\xi(D) := i^m \sum_{|\alpha|=m} \underbrace{A^\alpha(x)}_{id} \cdot \underbrace{\xi^\alpha}_{\|\xi\|^2} : E_x \longrightarrow F_x$

$$\sigma(D) \in \Gamma((\underset{\xi}{\odot^m} TX) \otimes \mathrm{Hom}(E, F))$$

Def (Elliptic operator) D is elliptic if $\sigma_\xi(D)$ is invertible map.
$\forall \xi \in T_x^*X \setminus \{0\}$

Recall examples: $T_x(X=\mathbb{R}^2)=\mathbb{R}^2$ $\quad (t,x) \xrightarrow[T_x^*X]{} (\xi_1,\xi_2)=\xi\in\mathbb{R}^2$

$$D_w=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2} \qquad \sigma_\xi(D)=\begin{pmatrix}\xi_1^2-\xi_2^2 & \\ & \xi_1^2-\xi_2^2\end{pmatrix} \text{ for } \xi_1=\xi_2\neq 0$$

$\sigma_\xi(D)$ not invertible

NOT Elliptic

$$D_h=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial t^2} \qquad \sigma_\xi(D)=\begin{pmatrix}\|\xi\|^2 & \\ & \|\xi\|^2\end{pmatrix}.$$

non-invertible $\Rightarrow \xi=0$

Elliptic

$$\begin{pmatrix}x\\y\end{pmatrix} \xrightarrow{\sigma_\xi(D)} \begin{pmatrix}x'\\y'\end{pmatrix}$$

$$\begin{tikzcd} TX \arrow[rr] \arrow[dr] & & TX \arrow[dl] \\ & X & \end{tikzcd} \quad x$$

$$\begin{tikzcd} \Gamma(E) \arrow[d, dash] & = & U\times C^\infty(U) \arrow[d, dash] \\ X & & U \end{tikzcd}$$

Thm [Elliptic Regularity]

Ellip

NOT so much soln $\not\longrightarrow$ high reg

$D : \Gamma(E) \to \Gamma(F)$ an elliptic operator

(0) $Df|_U \in C^\infty \Rightarrow f|_U \in C^\infty$

(1) $D_s : H_s(E) \to H_{s-m}(F)$ a bounded Fredholm operator (dim ker, dim coker $< \infty$)

ind D_s = dim ker − dim coker is indep of s

$\leq C_s \|f\|_{s+m} + \|Pf\|_{s+m}$

(2) $\|f\|_s \leq C_s(\|f\|_{s-m} + \|Df\|_{s-m})$

Two norm $\|\cdot\|_s$ and $\|\cdot\|_{s-m} + \|D\cdot\|_{s-m}$ equivalent on H_s

$Df = 0$ Elliptic bootstrapping

$\|f\|_{s-m} < \infty \Rightarrow \|f\|_s < \infty \Rightarrow \|f\|_{s+m} < \infty$

$\cdots$

Pf: (idea) One can construct an "inverse" of D : (parametrix) P

$$PD = \mathrm{id} + K_1 \qquad DP = \mathrm{id} + K_2$$

$P : C^\infty \to C^\infty$: P is ε-local

$$\operatorname{supp} P \subset B(r) \mapsto P(B(r)) \subset B(r+\varepsilon)$$

Fredholm if K_1, K_2 are cpt then. D, P are Fredholm operator

index invariance

$$\begin{tikzcd}
H^{s-m}(F)^* \arrow[r, "(D_s)^*"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & (H^s(E))^* \arrow[d, "\cong"] \\
H^{m-s}(F) \arrow[r, "(D^*)_s"] & H^{-s}(E)
\end{tikzcd}$$

$\operatorname{Ker} D_s \overset{\text{o is smooth}}{=} \operatorname{Ker} D$

$\operatorname{Ker}(D^*)_s = \operatorname{Ker} D^*$

$\quad\ \ \shortparallel \qquad\qquad\quad \shortparallel$

$(\operatorname{Coker} D)_s = \operatorname{Coker} D$

$\operatorname{Im}(d) \oplus \operatorname{Im}(d^*)$

$\qquad\ \shortparallel$

Hodge decomposition $\Gamma(E) = \operatorname{Ker}(\Delta) \oplus \operatorname{Im}(\Delta)$

Important message: Cauchy-Riem operator is Elliptic!

$$f(x+iy) = f(z) = \underset{\operatorname{Re} f}{u(x,y)} + \underset{\operatorname{Im} f}{v(x,y)}$$

$\Downarrow$ C.R.

$$u_x = v_y \qquad u_{xx} = -u_{yy} \Leftarrow \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)u = 0$$

$$\Longrightarrow$$

$$u_y = -v_x \qquad v_{xx} = -v_{yy} \qquad v$$

$$\frac{\partial}{\partial \bar{z}} = \underbrace{\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}}$$

i by J

$$d\mathcal{F}_u = \left(\frac{\partial}{\partial s} + J(u)\frac{\partial}{\partial t}\right)$$