Topology Between The Waves IV: *Heegaard–Floer homology*

12–16 Février 2024

The aim of this fourth edition is to study a specific version of Floer homologies, namely Heegaard–Floer homology, which applies to knots in 3–dimensional manifolds.

The workshop will take place at the *Station Biologique de Besse*, University Clermont–Auvergne.

0. Part 0: online pre-talks

0.1. **Overview of the week [Léo Bénard].**

0.2. **Euler and Spin^c–structures [Louis-Hadrien].** [Tu02] (with target[Juh08, Section 3] and [Juh06, Section 4], [MS74]). Prove [Lip06, Lemma 2.2]/[Juh06, Corollary 4.8], which explain why we care about Spin^c-structures.

0.3. **Index theorems [Paolo Ghiggini].** Express the index of the $\bar{\partial}$ operator as a topological quantity ([Lip06, equation (4)]). The reference for the actual proof is [Bou, Section 5.2]. This talk requires some knowledge in analysis.

1. Part 1 – Construction of (cylindrical) $\widehat{HF}$

Constructing only $\widehat{HF}$.
Main reference: Lipshitz : "A cylindrical reformulation of HF homology", [Lip06].

1.1. **Preliminary definitions. [Gaëtan Simian].** Mainly follows the reference [Lip06, Section 1] and the references therein (in particular [BEHWZ] for cylindrical symplectic theory). Explain what a symplectic form and an almost complex structure are, and how they are related (tamed, cf [BEHWZ]). Explain carefully the requirements in the definitions, define the cylindrical manifold W, and the characteristic manifolds of the symplectic form.

1.2. **Define Pseudo-Holomorphic curves [Giovanni Framba].** In the context of [Lip06, Section 1]. Compare with the "usual" definition of a curve in $\mathrm{Sym}^g\Sigma$, see for example the introduction of [Lip06], and [Lip14], in particular the example in Figure 9 there. Define $\pi_2(\vec{x}, \vec{y})$, and for $A \in \pi_2(\vec{x}, \vec{y})$, say something about the compactification $\widehat{\overline{\mathcal{M}^A}}$, as defined in [BEHWZ, Section 7] (no proof expected here, just an idea of why it is not compact, and what one shall add to compactify).

1.3. **Transversality [Nikolas Adaloglou].** State and explain the strategy of proof of the transversality statement that is needed here [Lip06, Section 3], as much as possible.

1.4. **Maslov index (2 talks) [Gregor Masbaum].** Be careful that there is an erratum to the paper, mainly concerning Section 4. The two volunteers should cut [Lip06, Section 4] into two pieces, admitting [Bou, Corollary 5.3] which gives the topological flavour to the index of the $\bar{\partial}$ operator (it will be proved in pre-talks). Then explain how to obtain formulas (4), (5), (6), and finally how to obtain a topological invariant which only depends on the homology class of curve.

1.5. **Definition of HF homology [Emmanuel Graff].** Discuss admissibility of Heegard diagrams, prove [Lip06, Lemmas 5.4]. Definition of $\widehat{CF}$. Describe the H_1-action. Compute examples. [Lip06, Section 8]

1.6. **Invariance (2 talks: [Hugo Zhou and Renaud Detcherry]).** The volunteers should work together and deal with sections 9, 10, 11 and 12. It is clearly too much, so they might make choices and present what they find relevant here. Note that for the second part of the week, it is not really necessary to know that HF is an invariant, but it would be a pity to avoid it.

- Proof of Isotopy invariance ([Lip06, Section 9]).
- Introduce triangle, triangle maps ([Lip06, Section 10]). This section is long and technical, one should decide what is most important and what is less. In particular, bring the material for the next talk.
- Prove Handleslide invariance ([Lip06, Section 11]).
- Prove Stabilization invariance ([Lip06, Section 12]).

2. Part 2 – Sutured Floer

2.1. **Sutured Floer homology [Romain Saunier].** Define (balanced) sutured manifolds, give examples, define (balanced) sutured Heegaard diagrams, explain how to associate a (balanced) sutured manifold to a (balanced) sutured Heegaard diagram (it seems that there is a typo just above [Juh06, Proposition 2.9], where ∂M should be $\partial\Sigma$, and also in [Juh06, Lemma 2.10], where injective should be surjective), state [Juh06, Propositions 2.14 & 2.15, Definitions 7.1, 7.3, 7.4, & 7.6, Theorems 7.2 & 7.5]. The main reference is[Juh06, Sections 2 & 7]. Explain how this is compatible with the cylindrical reformulation of [Lip06].

2.2. Spin^c**-structures in this new context [Alessio Di Prisa].** This is essentially [Juh08, Section 3] (a look at [Juh06, Section 4] might be useful). Recall the definition of Spin^c structures (actually Euler structures) in this context (generalities about that should be already dealt with before) and their associated relative Chern (or Euler) classes. Define outer Spin^c-structures w.r.t. a surface. Define the index and rotational (?) of a surface in a sutured manifold and explain how they can be partially computed (Lemma 3.9). Finally explain how to check that $\mathfrak{s}$ is outer (Lemma 3.10).

2.3. **Decomposition Theorem: overview and setup [Paula Truöl].** ([Juh08, Section 2, 4]) Definition of decomposition along a surface. Statement of Theorem 3.11 (which is 1.3 in a more accurate form). Explain the strategy of the proof. Definition of balanced diagram adapted to a surface and of surface diagram. Statements and proofs of the intermediate results of section 4 (Proposition 4.4, Lemma 4.5 and Proposition 4.8).

2.4. **Sutured manifold hierarchies [Diego Santoro].** The goal is to explain the statement of Theorem 8.2 in [Juh08], originally Theorem 4.2 in [Gab83], see also [Sch90, Theorem 6.5] and [Sch89, Theorem 4.19]. Define sutured manifold hierarchies ([Sch90, Sections 4,6]). Explain (and possibly prove) that the taut property lifts [Sch90, Theorem 6.6]. State Gabai's theorem. Explain the difficulty of the proof [Sch90, 6.5].

2.5. **Nice diagrams [Filippo Bianchi].** Explain the algorithm to get a nice diagram from a Heegard diagram ([SW10, Section 4]): Outline the steps of the proof, explain the complexity on Heegard diagrams, and explain why the algorithm decreases it. If there are too many cases to treat, give some examples using handle slides and isotopies on Heegard diagrams. Give examples of application of the algorithm with the resulting nice diagram. Finally adapt the result to the level of generality of [Juh08, Theorem 6.4] in the sutured case.

2.6. **Playing with diagrams [Quentin Faes].** [Juh08, Section 5,6] and (see also [SW10, Section 4] for the last bit). Explain how to associate a new diagram out of a surface diagram. Prove that this fits well in a surface decomposition of balanced sutured manifold (Proposition 5.2). Explain the bijection between outer point of the first diagram with the generators of the second diagram (Lemma 5.5). Explain why it is interesting (Theorem 7.4, which will be proved afterwards).

2.7. **Nice diagrams are nice indeed [Pierre Godfard].** The aim is to prove [Juh08, Theorem 7.4]. This is the content of [SW10, Section 3]. In order to actually have the statement of Theorem 7.4, one needs to understand why Proposition 7.3 provides the necessary adjustment. Discuss this also. Proof of Theorem 7.4 refers to [SW10]. In this proof main tools are basics on Riemann surfaces: holomorphic maps are branched covers (only disc-based ones are needed), Hurwitz formula, almost complex structure... Recalls on these notions are expected (using your favorite reference on Riemann surfaces!).

2.8. **End of the proof of the surface decomposition. Genus detection [Eric Stenhede].** [Juh08, Sections 7, 8] In this final talk, building up on the whole week, one proves first Propositions 7.5 and 7.6 and then harvest Theorem 1.3 (Surface decomposition reformulated in Theorem 3.11), Theorem 1.4 and Corollary 8.3 (genus detection). If there is enough time, one may want to explain the proof of Proposition 8.6 and Corollary 8.6.

References

[Bou] Frederic Bourgeois – "A Morse–Bott approach to contact homology"

[BEHWZ] Frederic Bourgeois, Yakov Eliashberg, Helmut Hofer, Kris Wysocki and Eduard Zehnder – "Compactness results in Symplectic Field Theory", *Geometry and Topology* **7** (2003) 799–888

[Gab83] David Gabai – "Foliations and the topology of 3–manifolds", *Journal of Differential Geometry* **18** (1983), p. 445–503.

[Juh06] András Juhász – "Holomorphic discs and sutured manifolds", *Algebraic & Geometric Topology* **6** (3) (2006), p. 1429–1457.

[Juh08] András Juhász – "Floer homology and surface decomposition", *Geometry & Topology* **12** (1) (2008), p. 299–350.

[Lip14] Robert Lipshitz – "Heegaard Floer Homologies: lecture notes", *arXiv:1411.4540* (2014).

[Lip06] ROBERT LIPSHITZ – "A cylindrical reformulation of Heegaard Floer homology", *Geom. Topol.* **10**(2): 955-1096 (2006).

[LipErr] ROBERT LIPSHITZ – Errata to "A cylindrical reformulation of Heegaard Floer homology", arXiv: 1301.4919

[MS74] JOHN MILNOR & JAMES STASHEFF – *Characteristic classes*, Princeton University Press, 1974,

[OS04] PETER OZSVATH & ZOLTAN SZABO – "Holomorphic disks and genus bounds", *Geometry & Topology* **8** (1) (2004), p. 311–334.

[Sav11] NIKOLAI SAVELIEV – *Lectures on the topology of 3–manifolds: an introduction to the Casson invariant* (2011).

[Sch89] MARTIN SCHARLEMANN – "Sutured manifolds and generalized Thurston norms", *Journal of Differential Geometry* **29** (3) (1989), p. 557–614.

[Sch90] MARTIN SCHARLEMANN – "Lectures on the theory of sutured 3–manifolds", *Proceedings of Kaist Mathematics Workshop* (1990), p. 25–46.

[SW10] SUCHARIT SARKAR AND JIAJUN WANG – An algorithm for computing some Heegaard Floer homologies, *Annals of Mathematics* (2010).

[Tu02] VLADIMIR TURAEV – Torsions of 3–dimensional manifolds

More References.

- András Juhász, A survey on Heegard Floer homology.
- A webpage with several surveys and references.
- The paper of Baldwin–Sivek.