What I understand about Spin^c structures

0. Why do we care?

I. Principal G-bundles and Spin^c structures

II. Euler structures

III. A proof of [Lip lemma 2.2].

DISCLAIMER: I'm not a geometer

0. $\widehat{HFK}(K) = \bigoplus_{d,i} \widehat{HFK}_d(K, i)$ → this grading comes from spin^c structures

This is the Alexander grading

Thm (Ozsváth–Szabó, Juhász).

if $K \subseteq S^3$ has genus g then:

- $\widehat{HFK}_*(K, g) \supseteq \mathbb{Z}$.

$\widehat{HFK}_*(K, i) = 0$ for $i > g$

In particular $\widehat{HFK}$ detects the genus

Thm (Juhász).

(M, γ) balanced sutured 3-mfd and $(M, \gamma) \rightsquigarrow^{S} (M, \gamma')$ be a surface decomposition

If S is nice enough,

$$SFH(M', \gamma') = \bigoplus_{\mathfrak{s} \in O_S} SFH((M, \gamma), \mathfrak{s})$$

SFH: Sutured Floer Homology

$$\langle c_1(\mathfrak{s}, t), [S] \rangle = c(S, t)$$

$c_1(\mathfrak{s}, t)$: relative Euler class of spin^c structure $\mathfrak{s}$. $\in H^2(M, \partial M)$

$c(S,t)$: evaluated on $[S]$

I. Spin^c structures

G topological group then a principal G-bundle is a fiber bundle $E \to B$ where G acts on fibers freely and transitively (in particular $E_x \simeq G$ as set). Locally, on as $U \times G \to U$.

faithfully

Example: Let M 3-mfd oriented.

$Fr(M) = \{$ oriented orthonormal basis of $T_x M \}_{x \in M}$

$$\downarrow$$

$$M$$

This is an principal $SO(3)$ bundle.

What will matter to us are principal $U(2)$-bundles which are lifts of $Fr(M)$ on M.

$$SO(3) \simeq U(2)/U(1).$$

if we have $F \to M$ a principal $U(2)$ bundle, we can mod out fiber by $U(1)$

⟶ principal $SO(3)$-bundle denoted $F/U(1)$.

Def A spin^c-structure on M is a pair $(F \to M, \alpha)$ where $F \to M$ is ppal $U(2)$-bundle and α is an iso: $F/U(1) \longrightarrow Fr(M)$.

Notion of iso of such things.

Spin^c structures are regarded up to iso.

$\varphi : (F \to M, \alpha) \longrightarrow (F' \to M, \alpha')$ bundle iso.

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "\pi"'] & F' \arrow[d, "\pi'"] \\
F/U(1) \arrow[r, "\varphi_*"] \arrow[d, "\alpha"'] \arrow[ur, phantom, "\circlearrowleft"] & F'/U(1) \arrow[d, "\alpha'"] \\
Fr(M) \arrow[r, "id"] \arrow[ur, phantom, "\circlearrowleft"] & Fr(M)
\end{tikzcd}$$

condition.

Example: Let v be a ~~non-singular~~ unitary vector field on M.

$$E = \{ (b_0, b_1, b_2) \text{ frame of } T_x M \mid x \in M,\ b_0 = v_x \}.$$

this is a $U(1)$ bundle. (ppal)

$E \times_{U(1)} U(2) \longrightarrow M$ where $1 \oplus U(1) \hookrightarrow U(2)$ is a $U(2)$ bundle. (ppal)

and provide a spin^c structure

$$((b_0, b_1, b_2), \varphi) \in (E \times_{U(1)} U(2))_x$$

$$\downarrow /U(1).$$

$$(b_0, b_1, b_2) \cdot \varphi_{SO(3)}$$

$$\downarrow$$

$$(v_0, v_1, v_2) \in (Fr M)_x.$$

$H_1(M)$ action on spin^c structures

Let $E \to M$ be a prol $U(1)$-bundle $\rightsquigarrow V \to M$.
a rank(2) vector bundle.
let s be a section transverse to 0-section (seen as $M \subseteq V$).
$s \cap M$ is a 1-d submanifold.
$\to$ represent a class in $H_1(M)$.

Fact: any element of $H_1(M)$ can be realized like that.

Prop: $F \to M$ a spin^c structure, and $E \to M$ a $U(1)$ bundle then
$$E \times_{U(1)} F \to M.$$
$*$ is a spin^c structure.

$*$ this defines an action of $H_1(M) \curvearrowright \mathrm{Spin}^c(M)$

$*$ this action is free and transitive ($H_1(M) \underset{\substack{\uparrow \\ \text{set.}}}{\simeq} \mathrm{Spin}^c(M)$)

II - Euler structures

Spoiler: $\mathrm{Eul}(M) \simeq \mathrm{Spin}^c(M)$

① Combinatorial Euler structures

Let X be a finite CW-cx.
an Euler chain is a singular chain θ such that:
$$\partial\theta = \sum_a \underbrace{(-1)^{\dim a}}\, x_a$$
where a runs over all cells in X and x_a's a point in $\mathring{a}$.

Such a thing exists iff, $\chi(X) = 0$.

Let θ and η be two Euler chains,
$$\partial\theta = \sum_a (-1)^{\dim a} x_a$$
$$\partial\eta = \sum_a (-1)^{\dim a} y_a.$$
pick up paths γ_a in $\mathring{a}$. $x_a \to y_a$.
define
$$\text{"}\theta - \eta\text{"} = \theta - \eta + \sum (-1)^{\dim a} \gamma_a.$$
$$\partial(\text{"}\theta - \eta\text{"}) = 0$$

In particular $\theta - \eta \in H_1(X)$.
Two Euler chains are equivalent if
$$\theta - \eta = 0 \in H_1(X)$$
An Euler structure is an equivalent class of Euler chains

If $[\theta]$ is a Euler structure and $[h] \in H_1(X)$. The $[h] + \theta$ is $\underset{\uparrow}{\underline{\underline{\text{the}}}}$ Euler structure $[\eta]$ s.t $\theta - \eta = -[h]$.
in other words $H_1(X)$ acts freely and transitively on Euler structures.

If X' is subdivision of X.
$$\underset{\text{canonical.}}{\mathrm{Eul}(X') \simeq \mathrm{Eul}(X)}$$
$$\rightsquigarrow \mathrm{Eul}(\underset{\substack{\uparrow \\ \text{mfd.}}}{M})$$

Alternative description:
Let $\hat{X}$ be a maximal abelian cover X.
and $\hat{e}$ be a fundamental family of cells of $\hat{X}$.
for each cell a in X $\exists!$ lift. $\hat{e}_a \in \hat{e}$.
How to extract from this a Euler structure:
pick a base point in $\hat{X}$: $\hat{x}$;
for each cell $\hat{e}_a$ in $\hat{e}$
choose a path $\gamma_{\hat{e}_a}$ $\hat{x} \to$ to a point in $\hat{e}_a$.
project $\left(\sum (-1)^{\dim a} \gamma_{\hat{e}_a}\right)$ to $X \rightsquigarrow$ gives me a Euler structure.

b) Geometric Euler structures

M be mnfd. connected compact $\dim M \geq 3$. $\chi(M) = 0$.

Two non singular vector field on M are <u>homologous</u> if $\exists B^m$-ball inside M s.t.

$$u|_{M \setminus B} \underset{\substack{\text{among} \\ \text{non-singular} \\ \text{v. f.}}}{\overset{\text{homotopic}}{\simeq}} v|_{M \setminus B}$$

Easy to see: this ~~is~~ an equivalence relation.

The eq. classes are called geo. Euler classes and their set is denoted

$$\mathrm{Vect}(M).$$

H_1-action on $\mathrm{Vect}(M)$:

Let $[\gamma]$ in $H_1(M)$

γ simple closed curve.

Let $N(\gamma)$ be a tubular neighbd of $\gamma \subseteq M$.

$$N(\gamma) \simeq D^{m-1} \times S^1$$
$$(r, \alpha) \in D^{m-1} \times \mathbb{R}/_{2\pi\mathbb{Z}}$$

Let u be a non singular vect field.

Define u so that in $N(\gamma)$ u is given

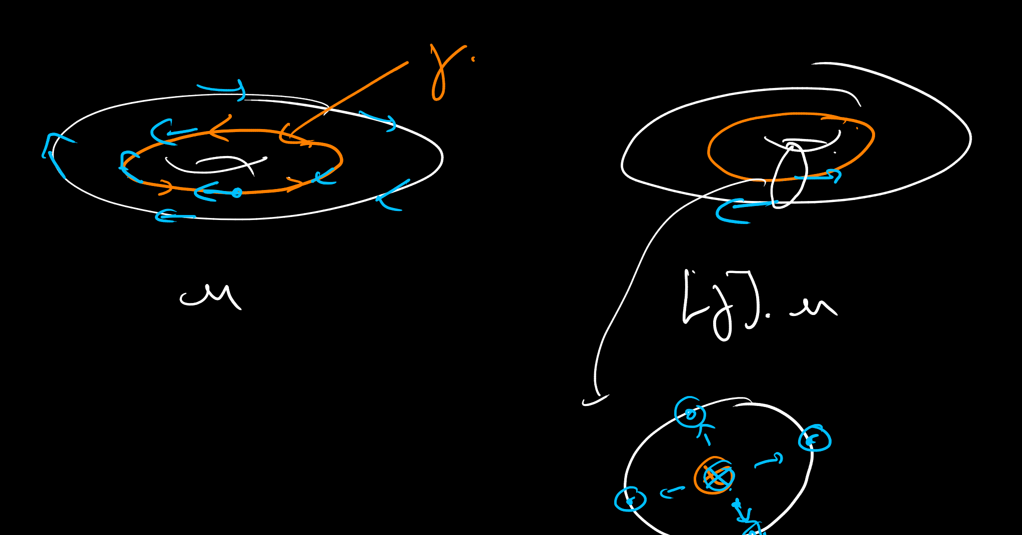
$$-\frac{\partial}{\partial \alpha}.$$

$$[\gamma].u|_{M \setminus N(\gamma)} = u.$$

In $N(\gamma)$

$$[\gamma].u = \cos(|r|\pi)\frac{\partial}{\partial \alpha} + \sin(|r|\pi)\frac{\partial}{\partial r}.$$

Picture in 3.d.



<u>Prop</u>: This defines a free and transitive action of $H_1(M)$ on $\mathrm{Vect}(M)$.

<u>Prop</u>: $\mathrm{Vect}(M) \simeq \mathrm{Eul}(M)$

<u>Sketch</u>: Construct a H_1-equivariant map from

$$\mathrm{Eul}(M) \longrightarrow \mathrm{Vect}(M)$$

construction not very difficult to construct.

<u>Prop</u>: $\mathrm{Vect}(M) \simeq \mathrm{Spin}^c(M)$.

<u>Sketch of Pf</u>:

As we have seen

From non-sing vect field $\rightsquigarrow$ spin^c structure.

claim: this map is H_1-equivariant $\square$

Euler class of spin^c str.

Let $\mathfrak{s} \in \mathrm{Spin}^c$

From previous bijection $\rightsquigarrow$ non sing vector field.

v.

$(v^\perp)$ is a rank 2 vector bundle, take a section s transverse to 0 section

$$S \cap M \in H_1(M) \simeq H^2(M, \partial M)$$

$\to$ This is (seen in $H^2(M, \partial M)$) is the Euler class of $\mathfrak{s}$.

Back to HFK.

$$\mathrm{HFK}(K) \overset{!}{\simeq} \mathrm{SFH}(Y(K))$$

$$\mathrm{SFH}(Y(K)) \simeq \bigoplus_{\mathfrak{s}} \mathrm{SFH}(Y(K), \mathfrak{s})$$

$$\mathrm{SFH}(Y(K), i) = \mathrm{SFH}(Y(K), \mathfrak{s})$$

$i \in \mathbb{Z}$. with $\mathfrak{s}$ s.t. $\langle c_1(\mathfrak{s}), [\Sigma] \rangle = i$. a Seifert