Topology Between The Waves V:
Open book decompositions and contact structures

March 10–14 2025

This document provides the list of talks for the 5th edition of Topology between the waves. Once you picked a talk, it is important that you speak with other speakers about your choices (notations, prototopycal examples, sources, etc.) to ensure consistency along the week. Do not hesitate to contact organizers if you have questions (titles of sections contains first name(s) of organizer(s) to help you direct your complains).

1. Open book decompositions: Delphine

In this section, following [Etn04, §2], we will study open book decompositions of closed oriented 3–manifolds and see three different proofs of existence.

1.1. **Definition and first properties.** Define open books and abstract open books. Give some (detailed) examples. State existence (the proof is postponed to the next talks). Define Murasugi sums and state Gabai's result on the connected sums of open books; sketch the proof. Define stabilizations. Explain how to obtain an open book with connected binding from any open book. Reference: [Etn04, §2].

1.2. **Proof of existence 1: branched covers.** The goal is to give the original Alexander's proof that every closed oriented 3–manifold has an open book decomposition. Define branched covers (see [Rol76, Chapter 10, §B] or [GS99, §6.3]). Show that every closed oriented 3–manifold is a branched cover of S^3, branched over some link [Ale20] (see also [Fei86]). Show that every link in S^3 is the closure of a braid [Ale23]. Conclude [Etn04, Theorem 2.1]. Illustrate with examples.

1.3. **Proof of existence 2: surgery presentations.** The goal is to give a second proof of the existence of open book decompositions, due to Rolfsen [Rol76] (see also the sketchy [Etn04, Page 6]). Define Heegaard diagrams, state existence (possibly give a brief proof) [Rol76, Chapter 9, §C]. Define Dehn surgeries and surgery presentations of 3–manifolds [Rol76, Chapter 9, §F,G]. Prove Lickorish–Wallace's theorem: any closed oriented 3–manifold admits a surgery presentation. Give Lickorish's proof [Lic62] (a good reference is [Rol76, Chapter 9, §I]), and precise the specific type of surgery presentation that can be obtained in this way [Rol76, Chapter 9, §I.8]. Deduce that every closed oriented 3–manifold admits an open book decomposition with planar pages [Rol76, Chapter 10, §K].

1.4. **Proof of existence 3: Lefschetz fibrations.** The goal is to give a third proof of the existence of open book decompositions, due to Harer [Har79] (see also the sketchy [Etn04, Pages 6-7]). Give the 4–dimensional version of Lickorish–Wallace's theorem studied in the previous talk [Etn04, Fact p.7] and explain how it follows from the statement in terms of surgery presentations of 3–manifolds. Define achiral Lefschetz fibrations and state Harer's result [Har79, Theorem 2.1], [Etn04, Page 6]. Show that these two results imply the existence of open book decompositions

[Har79, Corollary 2.2], [Etn04, Page 7]. Prove Harer's theorem [Har79, Chapter 1&2]. For a general treatment of Lefschetz fibration and the relation between critical points and 2–handles, see [GS99, Chapter 8], in particular §8.2.

2. Contact structures: Louis-Hadrien

The talks of this section aim to give the necessary material in contact topology/geometry before diving into the proof of the tight Giroux correspondence, as exposed in [LV1]. The lecture notes of Etnyre [Etn04] and Section 2 of the paper of Licata–Vértesi [LV1] should give an overview, while some other lectures notes of Etnyre [Etn01, Etn09, Mas13] and the book of [Gei08] should provide more details.

A pretalk with preliminaries of classical tools of differential geometry will very likely be given, so that if you think that some details about a basic notion is needed, but won't necessarily fit in your talk, it is a good idea to let the organizers know about it.

2.1. **Definition and detailed examples.** Define contact structures and contactomorphisms. Give detailled (3D) examples for these. Define the characteristic foliation of surfaces embedded in a contact 3-manifolds, give examples of those. Explain hyperbolic and elliptic points. State the theorem of extensions of diffeomorphisms preserving characteristic foliations and give ideas of the proof. Also define contact vector fields (coordinate with the speaker of talk 2.6). Basically follow [Etn01, Sections 2 and 3].

2.2. **Moser's trick consequences.** State and prove Gray's theorem [Gei08, Theorem 2.2.2] and then Darboux's theorem [Gei08, Theorem 2.6.2]. Note that there are many other possible sources for that. Explain [Etn01, Appendix to Section 3]. One should make sure to introduce Reeb vector fields.

2.3. **Overtwisted vs tight.** Define overtwisted disks and give examples (for instance from [Etn09, Section 2]). State and explain the theorem of Etnyre–Honda [Etn09, Theorem 2.31] or [Etn01, Theorem 4.3]. State and explain Eliashberg–Gromov result about symplectic filling [Etn01, Theorem 4.4]. State and explain unicity rel boundary of contact structure on B^3 [Etn01, Theorem 4.5] and solve [Etn01, Exercise 4.6]. Basically follow [Etn01, Section 4] with help of [Gei08, Section 4.5].

2.4. **Legendrian and transverse knots, existence of contact structures.** Sources are [Etn01, Section 5] and [Gei08, Chapter 3] Definition of Legendrian knots (links) in contact 3-manifolds. In $\mathbb{R}^3$, explain the front line projection (only has a cultural purpose for us, but is worth seeing). Prove that any curves in a contact manifold can be approximated by a Legendrian one. Mimic this for tranverse knots. Define Thurston–Bennequin invariant. Give examples (for instance for front line projections).

2.5. **Contact Manipulation.** States and prove Martinet's theorem ([Etn01, Theorem 5.23] or [Gei08, Theorem 4.1.2]) Explain connected sum of contact manifolds, [Gei08, Pages 301] and other surgeries (only for contact 3-manifolds though). Following [Gei08]: first using symplectic cobordisms ans then restating everything without cobordism is probably a good plan (bear in mind that symplectic stuctures should have already appeared in talk 2.3, coordinate with the corresponding speaker).

2.6. **Convex surfaces.** This talk might be a bit longer than the other ones. It might be a good idea to have two speakers. Basically follow [LV1, Section 2.2] (with help of references therein and of [Etn01, Section 6], [Gei08, Section 4.6.2 and 4.8]) and [Mas13]. Recall definition of contact vector field (already given in talk 2.1), convex surface (with Legendrian boundary) and dividing curves. State genericity of convex surface [Etn01, Just after Exercise 6.1], if time allows explain a little bit where that come from. State and explain characterization using tubular neighborhood of the surface. Explain how to glue two convex surfaces along (a piece of) their (Legendrian) boundary: [LV1, Lemma 2.1]. Introduce the concept of convex isotopy. States and explain the statement (but not proof) 'All what matters is dividing curves' [LV1, Theorem 2.2]. Relate it to the 'Legendrian realization theorem' [Etn01, Theorem 6.3] or [Mas13, Lemma 19]. State Giroux's criterion [LV1, Theorem 2.5] and give idea of proof (see [Gei08, proof of Proposition 4.8.14]).

3. From open book decompositions to contact structures: Marco

The main reference for these two talks is [Etn04, Section 3].

3.1. **Contact structures supported by open book decompositions.** Define what it means for a contact structure to be supported by an open book decomposition of a 3-manifold. Prove that, if ξ is a contact structure on a 3-manifold M, and if (B, π) is an open book decomposition of M, then the following conditions are equivalent:

(1) ξ is supported by (B, π);
(2) ξ can be isotoped to be arbitrarily close to the tangent plane field $T\Sigma_\vartheta$ on any compact subset of a page Σ_ϑ of π, transverse to every page in a neighborhood of the binding B, and transverse to the binding B;
(3) there exists a Reeb vector field X for a contact structure isotopic to ξ that is positively tangent to B and positively transverse to every page Σ_ϑ.

Give examples of contact structures supported by open book decompositions. Warning: the proof of [Etn04, Lemma 3.5] contains 6 exercises.

3.2. **Existence, uniqueness up to isotopy, and invariance under positive stabilization.** Prove Thurston and Winkelnkemper's Theorem: for every abstract open book (Σ, φ), there exists a contact structure ξ_φ on M_φ. Warning: the proof of [Etn04, Lemma 3.13] contains 4 exercises. Prove that any two contact structures supported by the same abstract open book are isotopic. This defines a map from abstract open books to isotopy classes of contact structures. Prove that this map sends the Murasugi sum of two abstract open books (see [Etn04, Definition 2.16]) to the connected sum of the corresponding contact structures. Prove that this map is also invariant under positive stabilization of abstract open books (see [Etn04, Definition 2.19]). This establishes the existence of a map $\Psi : \mathcal{O}(M) \to \mathcal{C}(M)$, where $\mathcal{O}(M)$ is the set of abstract open books whose underlying 3-manifold is M, considered up to positive stabilization, and $\mathcal{C}(M)$ is the set of contact structures on M, considered up to isotopy.

4. From contact structures to open book decompositions: Jules et Quentin

The aim of the talks in this section is to prove that if two open book decompositions carry the same contact structure, they become isotopic after stabilizations (see [Etn04, §4]. We shall focus on the "tight case", following [LV1]. The plan of the proof is to show that the two open books determine two convex Heegaard splittings, and use stabilizations to make the two splittings isotopic. Then the isotopy discretization argument of Colin will decompose the isotopy in a sequence of moves that we can describe by stabilizations. You can refer to [Etn01, §6] for information about convex surfaces.

4.1. **Colin's Isotopy Discretization Theorem.** This is [LV1, Theorem 2.11] but the main reference should be [Hon02, Section 3] (notice that the published version differ from the arXiv version). Explain bypass attachments and prove the Colin's isotopy discretization theorem.

4.2. **Positive stabilizations of tight Heegaard splittings.** [LV1, Sec 3.2] Give the definition of a positive stabilization of a tight Heegaard splitting of a contact manifold [LV1, Definition 3.7], and explain [LV1, Lemma 3.8]: that such stabilizations preserve convexity and correspond to stabilizations of the corresponding open book. Explain [LV1, Remark 3.9]. State [LV1, Proposition 3.10] about bypass attachments and prove it (or sketch the proof if it feels too long).

4.3. **Proof of the Giroux correspondence, tight case.** The reference is [LV1, Section 6]. Explain the steps of the proof, recalling the related results proven previously. Mainly, you will need to explain [LV1, Theorem 5.10] and sketch a proof.

References

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