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Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anders.

Johann Wolfgang von Goethe
[**Goe21**, Nr. 1279]

À me(s) nièce(s) et mon neveu.

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CHAPTER 1

Introductions

1. En français pour commencer

Ce mémoire vise à donner la formule d'évaluation des mousses et à donner un aperçu de ses conséquences directes ainsi que des constructions qu'elle suggère. La plupart des travaux présentés dans ce mémoire, sont en collaboration avec E. Wagner, pour deux d'entre eux la collaboration s'est étendue à Y. Qi et J. Sussan et pour l'un d'entre eux, elle s'est étendue à A. Beliakova et K. Putyra. Avant de parler des mousses il me faut expliquer dans quel contexte elles apparaissent et quel était le but original de la formule d'évaluation.

La topologie quantique est apparue dans les années 80 avec la découverte par Jones [**Jon85**] d'un invariant de nœuds d'une nature très différente de ceux qui étaient connus jusque-là (signature, polynôme d'Alexander, torsion, etc.), tous construits grâce à des techniques géométriques. Au départ ce polynôme est défini au moyen de méthodes issues des algèbres d'opérateurs mais fut rapidement reformulé comme un produit de la théorie des représentations du groupe quantique $U_q(\mathfrak{sl}_2)$. L'engouement suscité par cette découverte repose en partie sur l'aspect non-géométrique de son approche: la nature de l'information portée par le polynôme de Jones était alors des plus mystérieuses.

La topologie quantique s'est ensuite développée avec le formalisme de Reshetikhin–Turaev: ils ont donné un contexte très général à la définition de Jones permettant de multiples généralisations et ont extrait l'essence algébrique de ces invariants. Ceci a permis de définir des invariants de variétés de dimension 3, dit de Witten–Reshetikhin–Turaev ou WRT [**RT91**]. En même temps se développaient d'autres ramifications: étude des modules skein, formulation de la conjecture du volume, etc. Le comportement des invariants WRT a fait emerger l'espoir qu'ils soient en fait le reflet d'une théorie quadri-dimensionelle. Crane et Frenkel [**CF94**] ont formalisé cet espoir en suggérant un vaste et ambitieux programme de catégorification[1] des invariants quantiques. C'est cet espoir qui explique toute l'agitation autour de la catégorification en topologie en petites dimensions.

Le premier succès dans ce programme fut la définition d'une théorie homologique pour les nœuds par Khovanov [**Kho00**]. Les méthodes utilisées, notamment dans leurs reformulations par Bar-Natan [**BN02, BN05**] sont à la fois simples et complètement nouvelles. Le fait que cette théorie puisse parler des cobordismes entre les nœuds a immédiatement suscité l'enthousiasme de la communauté scien- tifique. Cette première étape fut rapidement suivie de quantité de travaux, à la fois pour l'étude des implications de cette découverte et pour la généraliser. On peut citer notamment la définition par Rasmussen [**Ras10**] d'un invariant dérivé de l'homologie de Khovanov qui donne une borne au genre lisse des nœuds levant ainsi partiellement

[1] Dans un autre contexte, l'homologie simpliciale (ou toute théorie homologique raisonnable) est une catégorification de la caractéristique d'Euler des espaces topologiques.

le voile sur la nature géométrique de l'information contenue par celle-ci et donc par le polynôme de Jones.

Les premières généralisations de l'homologie de Khovanov sont les homologies de Khovanov–Rozansky (ou homologies $\mathfrak{gl}_N$) qui catégorifient des invariants issus de la théorie des représentations de $U_q(\mathfrak{gl}_N)$. La définition originale de ces homologies repose sur les factorisations matricielles qui sont un outil algébrique cherchant à formaliser le fait que l'annulation du carré d'une différentielle dans un complexe $\mathbb{Z}_2$ gradué provient de l'annulation d'une somme de contributions locales. Autrement dit, localement le carré de la différentielle n'est pas nulle mais le devient globalement. Cet outil est très puissant mais beaucoup moins élémentaire que les techniques utilisées pour catégorifier le polynôme de Jones. Il faut néanmoins distinguer le cas $N = 3$ pour lequel Khovanov [**Kho04**] parvient à donner une définition totalement combinatoire. C'est pour cette approche qu'il introduit les mousses et c'est avec cette construction que j'ai travaillé pour ma thèse de doctorat.

Le fait que les homologies de Khovanov–Rozansky soient plus sophistiquées est naturel car l'algèbre de Hopf $U_q(\mathfrak{gl}_N)$ est plus compliquée que $U_q(\mathfrak{gl}_2)$. La nécessité d'avoir une définition plus combinatoire s'est néanmoins fait sentir. On peut au moins citer les travaux de Mackaay–Stošić–Vaz [**MSV11**] et de Queffelec–Rose [**QR16**] qui présentent des approches sans factorisations matricielles. Cela étant, dans un cas comme dans l'autre, il faut faire appel à de subtils artefacts pour arriver à la construction des homologies: la formule de Kapustin–Li pour le premier, la dualité de Howe à coefficients tordus et une certaines rigidité de la présentation dans l'autre. Notons néanmoins que ces deux approches font appel aux mousses. La formule d'évaluation de mousses que nous avons établie avec E. Wagner [**2**] est une réponse à cette quête de définition combinatoire de l'homologie de Khovanov–Rozansky. Mais il me faut l'admettre, cette quête était loin d'être nécessaire puisque la définition de Khovanov–Rozansky et ses reformulations étaient tout à fait satisfaisantes.

Ceci dit, à notre grande surprise, la formule d'évaluation des mousses a la bonne idées d'être équivariante. S'il m'est difficile d'expliquer ici ce que cela signifie pour lea lecteurice novice, disons qu'elle permet de travailler avec toutes les déformations des théories homologiques et ainsi d'en avoir une approche unifiée. Une conséquence importante de cette "unification" est la preuve par Ehrig–Tubbenhauer–Wedrich [**ETW18**] de la fonctorialité des homologies de Khovanov–Rozansky. La preuve de ce résultat déjà technique et diffcile, mais pas surprenant, aurait pu se passer de la formule d'évaluation des mousses, mais sa preuve en aurait été considérablement plus compliquée.

Une deuxième conséquence, que j'ai décidé d'éluder dans ce mémoire, est une reformulation combinatoire (avec M. Khovanov [**17, 19**]) de la théorie de jauge $SO(3)$ développée par Kronheimer–Mrowka [**KM16, KM19**] avec l'espoir d'obtenir une nouvelle preuve plus conceptuelle du théorème des quatre couleurs. Les objets que l'on considère alors sont des graphes non-orientés ce qui change certains aspects algébriques et combinatoire: on est par exemple forcé de travailler sur $\mathbb{Z}/2\mathbb{Z}$, ce qui simplifie considérablement les problèmes de signes. Ces études de graphes non-orientés se sont poursuivis avec J. Przytycki et M. Silvero en plus de M. Khovanov [**16**].

Une troisième conséquence est la définition d'une action de $\mathfrak{sl}_2$ sur les homologies de Khovanov–Rozansky [**8**]. Cette action vient en fait d'une action d'une algèbre de Lie plus grosse : la demie algèbre de Witt. Pour mettre en évidence cette action, on montre que certaines opérations sur les mousses commutent à l'évaluation. Ceci permet, presque gratuitement, d'obtenir le caractère bien défini de cette action. Sans l'évaluation des mousses il aurait fallut disposer d'une présentation complète de la catégorie des mousses et ceci nous échappe encore pour l'instant (voir cependant les travaux récents de Queffelec [**Que22**] pour les mousses $\mathfrak{gl}_2$). L'ajout de structures algébriques sur les homologies d'entrelacs raffine ces invariants mais surtout semble être une voie prometteuse pour la catégorification des invariants WRT.

Une quatrième conséquence, beaucoup plus inattendue, est la définition des homologies symétriques. Afin d'expliquer de quoi il retourne, il me faut faire un petit retour en arrière et revenir sur les invariants $\mathfrak{gl}_N$ de Reshetikhin–Turaev. Ceux-ci sont définis pour des entrelacs dont les composantes sont coloriés par des représentations de dimensions finies de $U_q(\mathfrak{gl}_N)$. Si l'on s'autorise à cabler les entrelacs, l'information qu'ils contiennent est complètement déterminée par les invariants obtenus en restreignant les coloriages aux puissances extérieures de la représentation vectorielle de $U_q(\mathfrak{gl}_N)$. Même dans le cas où toutes les composantes sont coloriées par la représentation vectorielle (cas dit non-colorié) les méthodes de calculs habituelles font en générales intervenir des puissances exterieures. Ainsi, les théories homologiques catégorifiant ces invariants se sont concentrées sur ces cas là: les composantes des entrelacs sont coloriées par des puissances exterieures de la représentation vectorielle.

Au niveau catégorifié, il n'y a aucune raison de penser que l'information contenue dans les homologies coloriés par des puissances extérieure contienne celle que l'on pourrait obtenir en catégorifiant les puissance symétriques. C'est pour cela qu'à la même période que d'autres [**Cau17, QRS18**], nous avons voulu catégorifier le calcul issu des puissances symétriques. Pour cela, des obstructions topologiques assez contraignantes nous empêche malheureusement de travailler avec des diagrammes d'entrelacs quelconques, nous devons nous restreindre aux fermetures de tresses. Dans ce nouveau contexte, nous donnons une évaluation des mousses qui repose fortement sur l'évaluation précédente et plus précisément sur sa composante équivariante. Ces homologies symétriques sont en effet nouvelles: si l'on considère le cas non-colorié, les groupes d'homologies sont différents des groupes d'homologies obtenus par les homologies de Khovanov–Rozansky. Le cas $N = 1$ est particulièrement criant : dans ce cas l'homologie de Khovanov–Rozanksy est triviale alors que l'homologie $\mathfrak{gl}_1$ symétrique est très loin de l'être. Il est en effet conjecturé que son rang est égalé au rang de l'homologie triplement graduée réduite.

Ces homologies symétriques sont encore très mal comprises et je compte poursuivre leurs études dans les années qui viennent. Ceci dit, nous savons déjà qu'elles sont munis d'action de $\mathfrak{sl}_2$ comme les homologie de Khovanov–Rozansky [**20**], mais j'ai décidé de ne pas en parler dans ce mémoire.

Elles ont de plus un produit dérivé assez surprenant: on arrive à construire à partir de l'homologie $\mathfrak{gl}_1$ symétrique une théorie homologique pour les nœuds qui catégorifie le polynôme d'Alexander [4]. Nous l'avons appelé homologie $\mathfrak{gl}_0$. Cette construction dépend très fortement d'un point base, c'est pour cela qu'elle fournit un invariant de nœuds et pas d'entrelacs. Nous avons réussi à montrer qu'il existe une suite spectrale de l'homologie $\mathfrak{gl}_0$ vers l'homologie de Floer pour les nœuds, la catégorification "classique" du polynôme d'Alexander. Comme nous savions aussi qu'il y avait une suite spectrale de l'homologie triplement graduée réduite vers l'homologie $\mathfrak{gl}_0$, ceci nous permet de montrer (d'une manière assez frustrante, pour l'instant) une conjecture de Dunfield–Gukov–Rasmusen [DGR06] stipulant l'existence d'une suite spectrale de l'homologie triplement graduée vers l'homologie de Floer pour les nœuds.

Organisation du mémoire. Comme suggéré dans cette introduction, ce mémoire se concentre sur les mousses (foams), qui sont des cobordismes entre des toiles (webs) et fait la place belle à la formule d'évaluation des mousses fermées, c'est-à-dire aux endomorphismes de la toile vide. En plus de cette introduction, le mémoire est divisé en trois chapitres:

- Le chapitre 2 intitulé "Webs" se concentre sur les objets planaires et unidimensionels que sont les toiles. Dans ce chapitre je détaille leur combinatoire, donne des définitions élémentaires pour leurs évaluations et je fais le lien avec la théorie des représentations. Afin d'anticiper sur la suite, je développe à la fois le point de vue extérieur et celui symétrique.
- Le chapitre 3 intitulé "Foams" donne la définition des mousses, de leurs coloriages et enfin de la formule d'évaluation. Je continue en esquissant la définition des homologies de Khovanov–Rozansky équivariantes. Finalement, je donne les formules permettant de définir une action de $\mathfrak{sl}_2$ sur les mousses et explique comment transférer cette structure aux homologies de Khovanov–Rozansky.
- Le chapitre 4 intitulé "Symmetric link homologies" commence par quelques prérequis combinatoire sur les graphes vinyls qui sont des toiles particulières. Je donne ensuite la formule d'évaluation des mousses vinyls (qui sont des mousses adaptées au contexte des graphes vinyls). Enfin j'explique comment construire les homologies symétriques à partir de cette évaluation. Je termine avec la définition de l'homologie $\mathfrak{gl}_0$ et les liens qu'elle entretient avec l'homologie triplement graduée réduite et l'homologie de Floer pour les nœuds.

Il y a quelques preuves dans ce mémoire qui correspondent à des (tout petits) résultats qui n'apparaissent nulle part ailleurs. Cela étant, lorsque cela m'a paru pertinent, j'ai donne des idées de preuve (pitch of the proof) ou encore des éléments d'histoire de la preuve (history of the proof) dans le cas de résultats formulés de manière compacte mais qui sont en fait l'aboutissement de nombreux travaux de diverses personnes.

2. Repeating in English

This thesis aims to provide and explain the foam evaluation formula and to give an overview of its direct consequences as well as the constructions it suggests. Most of the work presented in this thesis is in collaboration with E. Wagner; for two of them, the collaboration extended to Y. Qi and J. Sussan, and for one of them, it included A. Beliakova and K. Putyra. Before discussing foams, I need to explain the context in which they appear and the original purpose of the evaluation formula.

Quantum topology emerged in the 1980s with Jones' discovery [**Jon85**] of a knot invariant that was very different from those known until then (signature, Alexander polynomial, torsion, etc.), all constructed using geometric techniques. Initially, this polynomial was defined using methods from operator algebras but was quickly reformulated as a byproduct of the representation theory of the quantum group $U_q(\mathfrak{sl}_2)$. The excitement generated by this discovery partly stemmed from the non-geometric nature of the approach: the nature of the information carried by the Jones polynomial was then quite mysterious.

Quantum topology then developed with the Reshetikhin–Turaev formalism, providing a very general context for Jones' definition, allowing for multiple generalizations, and extracting the algebraic essence of these invariants. This made it possible to define invariants of 3-dimensional manifolds, known as Witten–Reshetikhin–Turaev (WRT) invariants [**RT91**]. At the same time, other branches developed: the study of skein modules, the formulation of the volume conjecture, etc. The behavior of WRT invariants raised hopes that they might actually reflect a four-dimensional theory. Crane and Frenkel [**CF94**] formalized this hope by suggesting an ambitious program of categorification[2] of quantum invariants. This hope explains the buzz around categorification in low dimensional topology.

The first success in this program was the definition of a homological theory for knots by Khovanov [**Kho00**]. The methods used, particularly in their reformulations by Bar-Natan [**BN02, BN05**], are both simple and completely novel. The fact that this theory could speak about cobordisms between knots immediately sparked excitement in the scientific community. This initial step was quickly followed by numerous studies, both to explore the implications of this discovery and to generalize it. Notably, Rasmussen's definition [**Ras10**] of an invariant derived from Khovanov homology provides a bound for the smooth genus of knots, thus partially revealing the geometric nature of the information contained within it, and thus within the Jones polynomial.

The first generalizations of Khovanov homology are the Khovanov–Rozansky homologies (or $\mathfrak{gl}_N$ homologies), which categorify invariants derived from the representation theory of $U_q(\mathfrak{gl}_N)$. The original definition of these homologies relies on matrix factorizations, an algebraic tool that aims to formalize the fact that the cancellation of the square of a differential in a $\mathbb{Z}_2$-graded complex comes from the cancellation of a sum of local contributions. In other words, locally, the square of the differential is not zero but vanishes globally. This tool is very powerful but much less elementary than the techniques used to categorify the Jones polynomial. However, one should distinguish the case $N = 3$, for which Khovanov [**Kho04**] managed to

[2] In another context, simplicial homology (or any reasonable homology theory) is a categorification of the Euler characteristic of topological spaces.

give a completely combinatorial definition. It is for this approach that he introduced foams, and it is on this construction that I worked for my doctoral thesis.

The fact that Khovanov–Rozansky homologies are more sophisticated is natural because the Hopf algebra $U_q(\mathfrak{gl}_N)$ is more complex than $U_q(\mathfrak{gl}_2)$. However, the need for a more combinatorial definition was felt. Notable examples include the work of Mackaay–Stošić–Vaz [**MSV11**] and Queffelec–Rose [**QR16**], which present approaches without matrix factorizations. However, in both cases, subtle artifacts are required to construct the homologies: the Kapustin–Li formula for the former, skew Howe duality and a certain rigidity of the presentation for the latter. Nevertheless, both approaches involve foams. The foam evaluation formula we established with E. Wagner [**2**] is a response to this quest for a combinatorial definition of Khovanov–Rozansky homology. But I must admit, this quest was not completely necessary since the Khovanov–Rozansky definition and its reformulations were quite satisfactory.

That being said, to our great surprise, the foam evaluation formula has the fortunate property of being equivariant. While it is difficult for me to explain what this means to a novice reader, let's say that it allows us to work with all the deformations of homological theories and thus provides us with a unified approach to them. An important consequence of this "unification" is the proof by Ehrig–Tubbenhauer–Wedrich [**ETW18**] of the functoriality of Khovanov–Rozansky homologies. The proof of this result, already technical and difficult, but not surprising, could have been done without the foam evaluation formula, but its proof would have been considerably more complicated.

A second consequence, which I have chosen to omit in this thesis, is a combinatorial reformulation (with M. Khovanov [**17, 19**]) of the $SO(3)$ gauge theory developed by Kronheimer–Mrowka [**KM16, KM19**] with the hope of obtaining a new, more conceptual proof of the four-color theorem. The objects considered then are unoriented graphs, which changes some algebraic and combinatorial aspects: for example, we are forced to work over $\mathbb{Z}/2\mathbb{Z}$, which greatly simplifies sign issues. These studies of unoriented graphs have continued with J. Przytycki and M. Silvero in addition to M. Khovanov [**16**].

A third consequence is the definition of an $\mathfrak{sl}_2$ action on Khovanov–Rozansky homologies [**8**]. This action actually comes from a larger Lie algebra action: the half-Witt algebra. To highlight this action, one shows that certain operations on foams commute with the evaluation. This allows us, almost for free, to establish the well-defined nature of this action. Without the foam evaluation, we would have needed a complete presentation of the foam category, which is still out of reach (see, however, recent work by Queffelec [**Que22**] for $\mathfrak{gl}_2$ foams). Adding algebraic structures to link homologies refines these invariants but, above all, seems to be a promising path for the categorification of WRT invariants.

A fourth, much more unexpected consequence is the definition of symmetric homologies. To explain this, I need to take a brief step back and revisit the Reshetikhin–Turaev $\mathfrak{gl}_N$ invariants. These are defined for links whose components are colored by finite-dimensional representations of $U_q(\mathfrak{gl}_N)$. If one allows for cabling of links, the information they contain is completely determined by the invariants obtained by restricting the colorings to exterior powers of the vector representation of $U_q(\mathfrak{gl}_N)$. Even in the case where all components are colored by the vector representation (the so-called uncolored case), usual calculation methods generally involve exterior powers. Thus, homological theories that categorify these invariants have focused on these cases: the components of the links are colored by exterior powers of the vector representation.

At the categorified level, there is no reason to think that the information contained in the homologies colored by exterior powers would include what one might obtain by categorifying the symmetric powers. For this reason, at the same time as others [**Cau17, QRS18**], we sought to categorify the calculations coming from symmetric powers. However, quite restrictive topological obstructions unfortunately prevent us from working with arbitrary link diagrams; we must restrict ourselves to braid closures. In this new context, we provide a foam evaluation that relies heavily on the previous evaluation, and more precisely on its equivariant component. These symmetric homologies are indeed new: in the uncolored case, the homology groups differ from those obtained by Khovanov–Rozansky homologies. The case $N = 1$ is particularly striking: in this case, Khovanov–Rozansky homology is trivial, while symmetric $\mathfrak{gl}_1$-homology is far from trivial. It is, in fact, conjectured that its rank equals the rank of the reduced triply graded homology.

These symmetric homologies are still poorly understood, and I plan to continue their study in the coming years. That said, we already know that they carry an $\mathfrak{sl}_2$ action, like the Khovanov–Rozansky homologies [**20**], but I have chosen not to discuss this in this thesis.

Moreover, they have a rather surprising derived product: from the symmetric $\mathfrak{gl}_1$-homology, one can construct a homological theory for knots that categorifies the Alexander polynomial [**4**]. We called it $\mathfrak{gl}_0$-homology. This construction strongly depends on a base point, which is why it provides an invariant for knots rather than links. We have shown that there exists a spectral sequence from $\mathfrak{gl}_0$-homology to Floer homology for knots, the "classical" categorification of the Alexander polynomial. Since we also knew that there was a spectral sequence from the reduced triply graded homology to $\mathfrak{gl}_0$ homology, this allows us to demonstrate (in a somewhat frustrating way, though) a conjecture by Dunfield–Gukov–Rasmussen [**DGR06**] stating the existence of a spectral sequence from the triply graded homology to Floer homology for knots.

Organization. As suggested in this introduction, this thesis focuses on foams, which are cobordisms between webs, and highlights the evaluation formula for closed foams, i.e., endomorphisms of the empty web. In addition to this introduction, the thesis is divided into three chapters:

- Chapter 2, titled "Webs" focuses on planar and one-dimensional objects known as webs. In this chapter, I detail their combinatorics, provide basic definitions for their evaluations, and link them to representation theory. To anticipate the next chapters, I develop both the exterior and symmetric perspectives.
- Chapter 3, titled "Foams" provides the definition of foams, their colorings, and finally, the evaluation formula. I continue by sketching the definition of equivariant Khovanov–Rozansky homologies. Finally, I present the formulas that define an $\mathfrak{sl}_2$ action on foams and explain how to transfer this structure to Khovanov–Rozansky homologies.

- Chapter 4, titled "Symmetric link homologies" begins with some combinatorial prerequisites on vinyl graphs, which are particular webs. I then provide the evaluation formula for vinyl foams (foams adapted to the context of vinyl graphs). Finally, I explain how to construct symmetric homologies from this evaluation. I conclude with the definition of $\mathfrak{gl}_0$ homology and the connections it has with reduced triply graded homology and Floer homology for knots.

There are a few proofs in this thesis that correspond to (very small) results that appear nowhere else. However, where I found it relevant, I provide proof sketches or elements of the history of the proof for results that are formulated concisely but are, in fact, the culmination of many studies by various people.

> La mathématique c'est l'art de donner le même nom à des choses différentes.
>
> Henri Poincaré [Poi08, p. 31]

3. Conventions

Following French notations, the symbol $\mathbb{N}$ denotes the set $\mathbb{Z}_{\geq 0}$ of non-negative integers and $\mathbb{N}^*$ denotes the set $\mathbb{Z}_{>0}$ of positive integers. The symbol q will be used in different fashions:

- Sometimes it will be an indeterminate and we will often consider Laurent polynomial in q with coefficient in $\mathbb{Z}$ or $\mathbb{N}$. For $n \in \mathbb{Z}$, set

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{1-n} + q^{3-n} + \cdots + q^{n-3} + q^{n-1} \in \mathbb{Z}[q, q^{-1}]. \tag{1}$$

  and for $k \in \mathbb{N}$, we set:

$$\begin{bmatrix} n \\ k \end{bmatrix} := \prod_{i=1}^{k} \frac{[n+1-k]}{[k]} \in \mathbb{Z}[q, q^{-1}]. \tag{2}$$

- In categories of graded modules it will denote the shift functor, so that if M is a graded module, $q^s M$ is the graded module with $(q^s M)_i = M_{i-s}$ for any integers s and i. Note however that we will often omit grading shifts. If $P(q) = \sum_{i \in \mathbb{Z}} a_i q^i$ is a Laurent polynomial in $\mathbb{N}[q, q^{-1}]$ and M is a graded module, then define:

$$P(q)M := \bigoplus_{i \in ZZ} q^i M^{a_i}. \tag{3}$$

  where M^{a_i} is the direct sum of a_i copies of M.

Diagrams (of webs and foams), should be read from bottom to top.

On the next page, there is first a bibliography listing my papers and preprints. Only results from references [1]–[8] are discussed in this thesis. The other bibliography starts on page 49.

My papers and preprints

[1] Louis-Hadrien Robert. A new way to evaluate MOY graphs, 2016. arXiv:1512.02370. 9, 12, 14, 16

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[3] Louis-Hadrien Robert and Emmanuel Wagner. Symmetric Khovanov–Rozansky link homologies. *J. Éc. polytech. Math.*, 7:573–651, 2020. arXiv:1801.02244, doi:10.5802/jep.124. 12, 40, 45

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CHAPTER 2

Web evaluations

1. Webs

This section provides the basics of webs which will be used throughout this thesis. It is based on results of [**1**, **5**] and [**3**, Appendix A].

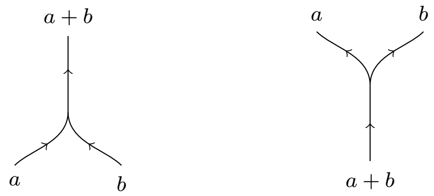
Definition 2.1.1. A *pre-web* is a finite uni-trivalent graph Γ (vertices have valency 1 or 3) endowed with an orientation and a *thickness* function $t\colon E(\Gamma) \to \mathbb{N}^*$ from the set of edges such that the flow is preserved at 3-valent vertices. This means that the sum of thicknesses of edges, counted with signs coming from orientation, is equal to zero. In particular 3-valent are of two different types: *split* and *merge* vertices. Pre-webs may include loops (as vertex-less edges) and may not be connected. At any 3-valent vertex, there are two *thin* edges and a single *thick* one. The set of (oriented) uni-valent vertices is the *boundary* of Γ and is denoted $\partial\Gamma$. If $\partial\Gamma = \emptyset$, the pre-web Γ is said to be *closed*.

Remark 2.1.2. It is sometimes convenient to allow the thickness of an edge to be equal to 0. In order to keep a split-and-merge dichotomy for 3-valent vertices, we decided not to allow such thickness in the definition. We might use thickness 0 edges, but this will only mean that the edges in question are meant not to exist.

Definition 2.1.3. Let Σ be an oriented smooth surface with a collared boundary. A *web in* Σ is a pre-web Γ endowed with a proper smooth[1] embedding of $(\Gamma, \partial\Gamma)$ in $(\Sigma, \partial\Sigma)$. When $\Sigma = \mathbb{R}^2$, the web is said to be *planar*.

Warning. The smoothness condition implies that a neighborhood of any 3-valent vertex is diffeomorphic to exactly one of the two following local models:

(4)



However, in figures we will often draw 3-valent vertices in a polygonal way. This is harmless since the orientations ensure that there is a unique way to smooth vertices.

Definition 2.1.4. Suppose that Σ is endowed with a nowhere vanishing vector field ξ (nowhere tangent to the boundary) and a Riemannian metric. A web Γ is *directed* if the unit tangent vector field of Γ is everywhere acute to ξ (with respect to the Riemannian metric).

The cases of directed webs we will consider are:

[1]By *smooth*, we mean that the restriction of the embedding to any oriented simple path is smooth.

(1) $\Sigma = [0,1]^2 \subset \mathbb{R}^2$ with $\xi = \frac{\partial}{\partial y}$ and the Riemmanian metric induced by the Euclidean scalar product on $\mathbb{R}^2$;
(2) $\Sigma = \{z \in \mathbb{C}$ such that $1 \leq |z| \leq 2\}$, $\xi = \frac{\partial}{\partial \theta}$ and the Riemmanian metric induced by the Euclidean scalar product.

Note that the annulus given in the second case is essentially obtained by taking the square of the first one and gluing top and bottom intervals of the boundary of $[0;1]^2$. One may also consider directed web in the annulus obtained by gluing the left and right intervals of the boundary of $[0;1]^2$, or the torus obtained by gluing both pairs of intervals. This would relate to affine braids and affine braid closures. While this is worth investigating, we will not consider these cases in this thesis.

1.1. Exterior colorings.

Definition 2.1.5. Let N be a non-negative integer, elements of the (canonically) ordered set $[\![N]\!] := \{1,\ldots,N\}$ are called *pigments*. Elements of its powerset $\mathcal{P}([\![N]\!])$ are called *colors*. A $\mathfrak{gl}_N$*-coloring* (or simply *coloring*, when the context is clear) is a map $c\colon E(\Gamma) \to \mathcal{P}([\![N]\!])$ such that:

- For all e in $E(V)$, $\#c(e) = t(e)$.
- At any vertex the color of the thick edge is the union of the colors of the two thin edges (which are necessarily disjoint).

A *coloring* of a web is a coloring of the underlying pre-web. The set of $\mathfrak{gl}_N$-colorings of the web (or pre-web) Γ is denoted $\mathrm{col}_N(\Gamma)$ (or sometimes, $\mathrm{col}(\Gamma)$, when the context is clear).

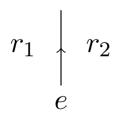
A *colored* web (resp. pre-web) is a web (resp. pre-web) endowed with a coloring.

Remark 2.1.6. For a map c satisfying the first condition, the last condition is equivalent to saying that at each 3-valent vertex v, the set $c(e_1)\Delta c(e_2)\Delta c(e_3)$ is empty where e_1, e_2 and e_3 are the three edges adjacent to v and Δ denotes the symmetric difference of sets.

If a pre-web has an edge with thickness strictly greater than N, then it does not admit any $\mathfrak{gl}_N$-coloring. Conversely one has:

Proposition 2.1.7. *Let Γ be a closed web in $\mathbb{S}^2$ (or any surface Σ diffeomorphic to a subsurface of $\mathbb{S}^2$) and denote $M := \max_{e\in E(\Gamma)} t(e)$, then Γ admits a $\mathfrak{gl}_M$-coloring.*

Sketch of proof. One may assume that $\Sigma = \mathbb{S}^2$. Consider the set of regions $R := \pi_0(\mathbb{S}^2 \setminus \Gamma)$. The flow condition on pre-webs implies that we can find a map $\ell\colon R \to \mathbb{Z}$ satisfying that if the boundary of the regions r_1 and r_2 share a common edge e of thickness a as follows:

(5) 

then $\ell(r_2) = \ell(r_1) + a$. In other words, the value of the map ℓ of two adjacent regions differ by the thickness of a common edge of their boundaries (and orientation gives the sign of this difference).

Define a map $\tilde{c}\colon E(\Gamma) \to \mathcal{P}(\mathbb{Z})$ using the following rule (given in the notation of (5)):

$$\tilde{c}(e) = [\ell(r_1);\ell(r_2)[\cap\mathbb{Z} \tag{6}$$

Post-composing $\tilde{c}$ with the map $\mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z}/M\mathbb{Z})$ and identifying the sets $\mathbb{Z}/M\mathbb{Z}$ with $[\![M]\!]$ gives a valid $\mathfrak{gl}_M$-coloring. □

Remark 2.1.8. The planarity assumption of the last is not necessary. However the proof is less constructive and requires an inductive argument on M.

Notation 2.1.9.

(1) Let $i \in [\![N]\!]$ be a pigment and (Γ, c) a closed colored web and, then the set of edges of Γ containing i in their colors is a disjoint union of oriented cycles denoted by $C_i(\Gamma, c)$.
(2) Let $i < j \in [\![N]\!]$ be two pigments and (Γ, c) a closed colored web, then the symmetric difference of $-C_i(\Gamma, c)$ and $C_j(\Gamma, c)$ is a disjoint union of oriented cycles denoted by $C_{ij}(\Gamma, c)$, it contains edges of Γ which contains j but not i and those which contains i but not j (with reverse orientations). It is convenient to define $C_{ji}(\Gamma, c) := C_{ij}(\Gamma, c)$.
(3) Let C be a collection of oriented simple closed curves in $\mathbb{R}^2$. The *rotational of* C is the number of simple closed curves in C which are positively oriented minus the number of simple closed curves in C which are negatively oriented. It is denoted $\rho(C)$.

Definition 2.1.10. Let Γ be a closed planar web, and c a $\mathfrak{gl}_N$-coloring of Γ. The $\mathfrak{gl}_N$-*degree* (or simply *degree*) of c is the integer defined by:

$$\deg_N(c) = \sum_{1 \le i < j \le N} \rho\left(C_{ij}(\Gamma, c)\right). \tag{7}$$

the *combinatorial* $\mathfrak{gl}_N$-*evaluation of* Γ (or simply $\mathfrak{gl}_N$-*evaluation* or *evaluation*) is the element of $\mathbb{N}[q, q^{-1}]$ defined by:

$$\langle \Gamma \rangle_N := \sum_{c \in \mathrm{col}_N(\Gamma)} q^{\deg_N(c)}. \tag{8}$$

For typographical reasons, we will often write $\langle \cdot \rangle$ instead of $\langle \cdot \rangle_N$.

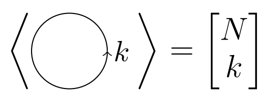
Note that if $N = 2$, the sum in (7) has only one term. The $\mathfrak{gl}_N$-degree is therefore a sum of $\mathfrak{gl}_2$-degrees of $\mathfrak{gl}_2$-colorings of a collections of sub-webs of Γ covering it (for a good notions of "sub-webs" and "covering", that we will not make explicit).

Lemma 2.1.11. *For any closed planar web* Γ*, the Laurent polynomial* $\langle \Gamma \rangle_N$ *is symmetric in* q *and* q^{-1}.

Sketch of the proof. The involution $\iota\colon [\![N]\!] \ni i \mapsto N - i + 1 \in [\![N]\!]$ induces an involution $\iota^\star$ on $\mathrm{col}_N(\Gamma)$ which is such that $\deg_N(\iota^\star(c)) = -\deg_N(c)$ for all $c \in \mathrm{col}_N(\Gamma)$. □

Proposition 2.1.12 ([1, Section 4])**.** *The* $\mathfrak{gl}_N$-*evaluation of webs satisfies the following local relations (and their mirror images):*

(9)



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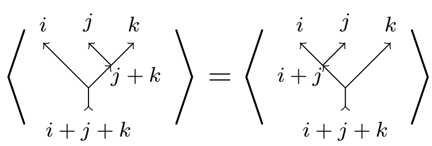
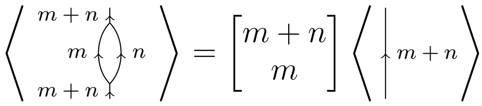
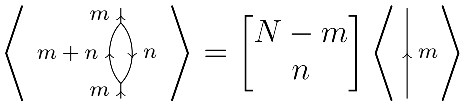
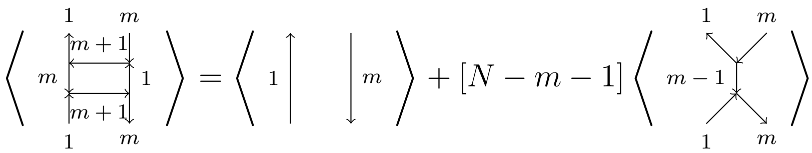
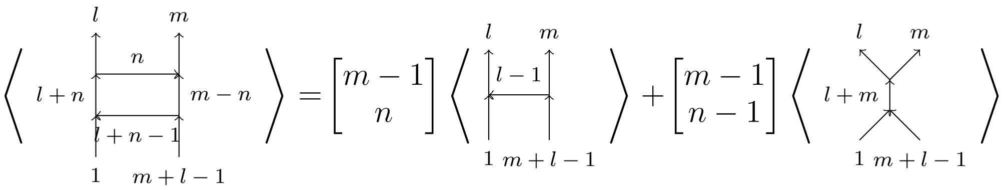
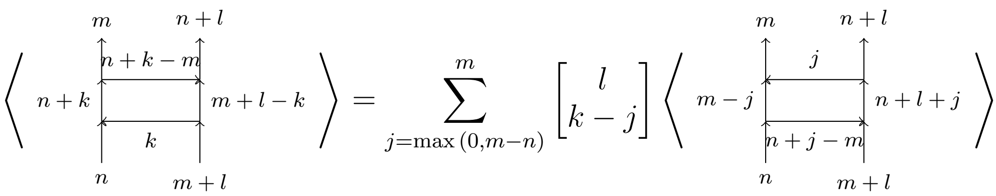


(11) 

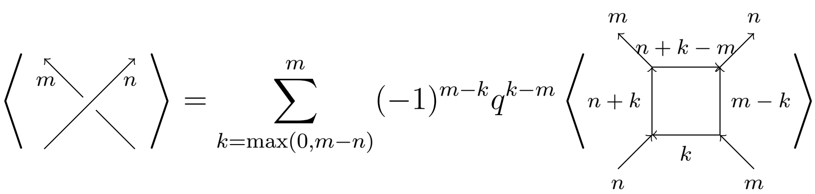
(12) 

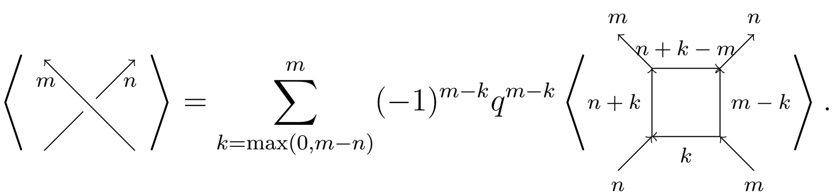
(13) 

(14) 

(15) 

Definition 2.1.13. One can extend $\langle\cdot\rangle_N$ to diagrams of planar knotted webs via the following formula:

(16) 

(17) 

In particular $\langle\cdot\rangle_N$ is computable on diagrams of oriented links colored by non-negative integers.

Proposition 2.1.14 ([MOY98])**.** *The quantity $\langle\cdot\rangle_N$ is an invariant of framed oriented links labeled by non-negative integers. If a framed oriented link L has a component labeled*[2] *by an integer $k > N$, then $\langle L\rangle_N = 0$.*

[2]In the literature, the labels of the components are *colors*, but should be thought of as thicknesses in our context.

The natural action of the symmetric group $\mathfrak{S}_N$ on $[\![N]\!]$ induces an $\mathfrak{S}_N$-action on $\mathrm{col}_N(\Gamma)$ for any pre-web Γ. Note that this action does not behave well with respect to the degree of coloring (in the case of a closed planar web). We now introduce more local moves on the set $\mathrm{col}_N(\Gamma)$. This is inspired by the work of A. Kempe on the Four Color Theorem [**Kem79**]. So far this notion does not play a major role in the study of webs. However, as we shall see, its 2-dimensional analogue is essential for foams.

Definition 2.1.15. Let Γ be a closed web, c a coloring of Γ and $i < j \in [\![N]\!]$ two pigments. Consider a connected component C of $C_{ij}(\Gamma, c)$ and define c' the $\mathfrak{gl}_N$-coloring of Γ which is identical to c on all edges not contained in C and which is equal the symmetric difference of $c(e)$ and $\{i, j\}$ for any edge e in C. The coloring c and c' are said to be related by an *ij-Kempe move along C*, or simply *Kempe-move.*

Two colorings of a given web Γ are *Kempe-equivalent* if they are related by a finite sequence of Kempe-moves. This is an equivalence relation.

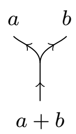
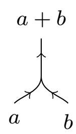
Question 2.1.16. *Are all $\mathfrak{gl}_N$-colorings of every planar web Kempe-equivalent?*

The statement holds for $N \leq 3$, see [**1**]. Orientability seems important since the 1-skeleton of the dodecahedron seen as an unoriented $\mathfrak{sl}_3$-web provides an example of a web for which there are different classes of Kempe-equivalences classes of $\mathfrak{sl}_3$-colorings, see [**19**].

2. Symcolorings

In this section, we define another evaluation of closed webs called symmetric evaluation. Heuristically it is obtained from the previous one by trading sets for multi-sets.

Definition 2.2.1. Let Γ be a web, for each vertex v of Γ define $W(v)$ to be the Laurent polynomial defined by the following formula:

(18) $W\left(\right.$  $\left.\right) := \begin{bmatrix} a+b \\ a \end{bmatrix} =: W\left(\right.$  $\left.\right),$

and $\langle\!\langle \Gamma \rangle\!\rangle_1$ by:

(19) $$\langle\!\langle \Gamma \rangle\!\rangle_1 := \prod_{\substack{v \in V(\Gamma) \\ v \text{ merge}}} = \prod_{\substack{v \in V(\Gamma) \\ v \text{ split}}} = \sqrt{\prod_{v \in V(\Gamma)}}.$$

The identities contained in (19) are not completly obvious.

Definition 2.2.2. Let X be a set. A *multi-subset of X* is an application $Y : X \to \mathbb{N}$. If $\sum_{x\in X} Y(x) < \infty$, the multi-subset Y is said to be *finite* and the sum is its *cardinal* (denoted by $\#Y$). If x is an element of X, the number $Y(x)$ is the *multiplicity of x in Y*. Let us fix two multi-subsets Y_1 and Y_2.

- The *disjoint union* of Y_1 and Y_2 (denoted $Y_1 \sqcup Y_2$) is the multi-subset $Y_1 + Y_2$.
- The *union* of Y_1 and Y_2 (denoted $Y_1 \cup Y_2$) is the multi-subset $\max(Y_1, Y_2)$.
- The *intersection* of Y_1 and Y_2 (denoted $Y_1 \cap Y_2$) is the multi-subset $\min(Y_1, Y_2)$.

Example 2.2.3. Let $X = \{a, b, c\}$, we consider the two multi-subsets of X given by:

$$\begin{array}{rccc} Y_1\colon & X & \to & \mathbb{N} \\ & a & \mapsto & 2 \\ & b & \mapsto & 0 \\ & c & \mapsto & 3 \end{array} \quad \text{and} \quad \begin{array}{rccc} Y_2\colon & X & \to & \mathbb{N} \\ & a & \mapsto & 1 \\ & b & \mapsto & 2 \\ & c & \mapsto & 0 \end{array}.$$

The multi-subset Y_1 can be represented by $\{a, a, c, c, c\}$ and Y_2 by $\{a, b, b\}$. The cardinal of Y_1 is equal to 5 and the cardinal of Y_2 is equal to 3. One has $Y_1 \sqcup Y_2 = \{a, a, a, b, b, c, c, c\}$, $Y_1 \cup Y_2 = \{a, a, b, b, c, c, c\}$ and $Y_1 \cap Y_2 = \{a\}$.

Of course, this model is meant to encode the notion of "set" whose elements may have multiplicities. A subset of X can be identified with its characteristic function and hence seen as a multi-subset of X.

The set of finite multi-subsets of X is denoted by $\mathcal{M}(X)$.

The symbol $[\![N]\!]$ still denotes the finite set of pigments $\{1, \dots, N\}$ endowed with its natural order.

Definition 2.2.4. Let Γ be a web. A $\mathfrak{gl}_N$*-symcoloring* (or simply *symcoloring*) of Γ is a map $c : E_\Gamma \to \mathcal{M}([\![N]\!])$, such that:

- For every edge e of Γ, $\#c(e) = t(e)$,
- For every vertex v of Γ, the multi-subset associated with the thick edge adjacent to v is equal to the disjoint union of the multi-subsets associated with the thin edges adjacent to v. This is the *flow condition* for symcolorings.

A *symcolored* web is a web endowed with a symcoloring. The set of $\mathfrak{gl}_N$-symcolorings of a web Γ is denoted by $\mathrm{scol}_N(\Gamma)$.

Notation 2.2.5. Let (Γ, c) be a symcolored web and i an element of $[\![N]\!]$. Denote by $G_i(\Gamma, c)$ the web[3] which as an embedded oriented graph is the same as Γ but whose thickness function is given by:

$$\begin{array}{rccc} t_{G_i(\Gamma,c)}\colon & E_\Gamma & \to & \mathcal{M}([\![N]\!]) \\ & e & \mapsto & t(e)(i). \end{array}$$

Let $i < j$ be two elements of $[\![N]\!]$. We denote by $G_{ij}(\Gamma, c)$ the web which as embedded graph is the same as Γ. The orientation of an edge e of $G_{ij}(\Gamma, c)$ is the same as the one of Γ if $m_j(c(e)) > m_i(c(e))$ and the reversed orientation otherwise. The thickness function of $G_{ij}(\Gamma, c)$ is given by:

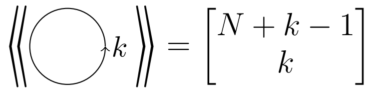
$$\begin{array}{rccc} t_{G_{ij}(\Gamma,c)}\colon & E_{G_{ij}(\Gamma,c)} & \to & \mathcal{M}([\![N]\!]) \\ & e & \mapsto & |c(e)(j) - c(e)(i)|. \end{array}$$

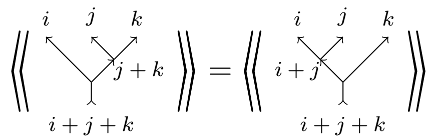
Definition 2.2.6. Let Γ be a web, the *combinatorial* $\mathfrak{gl}_N$*-symevaluation* (or simply $\mathfrak{gl}_N$*-symevaluation* or *symevaluation*) of Γ is the Laurent polynomial in q with positive coefficients defined by:

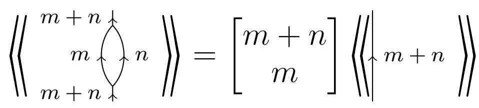
$$\langle\!\langle \Gamma \rangle\!\rangle_N = \sum_{c \in \mathrm{scol}_N(\Gamma)} \prod_{i \in [\![N]\!]} \langle\!\langle G_i(\Gamma, c) \rangle\!\rangle_1 \prod_{i<j \in [\![N]\!]} q^{\rho(G_{ij}(\Gamma,c))}. \tag{20}$$

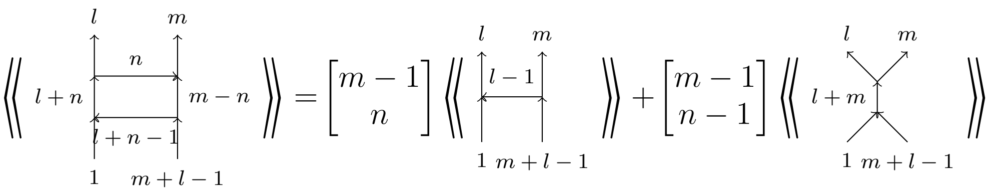
[3]In this construction, some edges may receive 0 as thickness. One should remove these edges (see Remark 2.1.2). The same applies for $G_i(\Gamma, c)$.

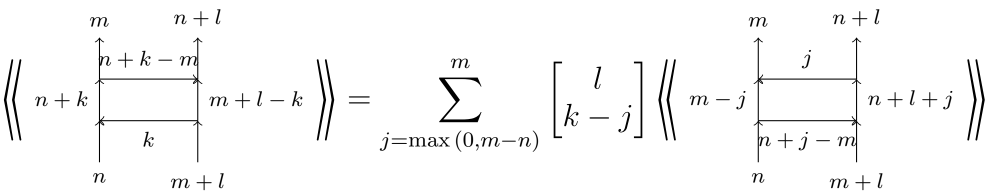
Proposition 2.2.7 ([5, Section 3])**.** *The symmetric $\mathfrak{gl}_N$-evaluation of webs satisfies the following local relations (and their mirror images):*

(21) 

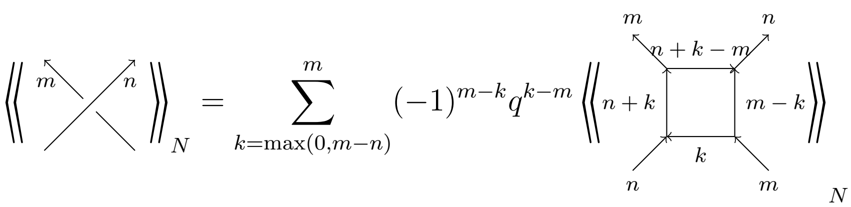
(22) 

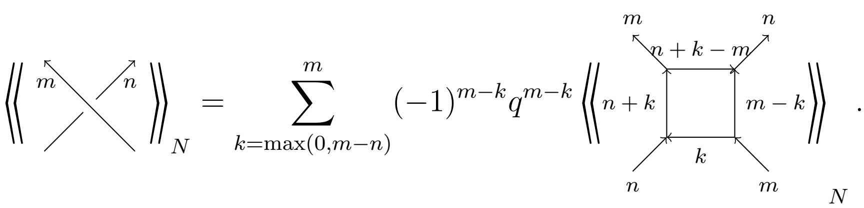
(23) 

(24) 

(25) 

Definition 2.2.8. One can extend $\langle\!\langle \cdot \rangle\!\rangle_N$ to diagrams of planar knotted webs via the following formula:

(26) 

(27) 

In particular $\langle\!\langle \cdot \rangle\!\rangle_N$ is computable on diagrams of oriented links labeled by non-negative integers.

Proposition 2.2.9 ([5])**.** *The quantity $\langle\!\langle \cdot \rangle\!\rangle_N$ computed on braid closures is an invariant of framed oriented links colored by non-negative integers.*

3. A glimpse of representation theory

Definition 2.1.10 and Definition 2.2.6 are purely combinatorial. They are inspired from —and coincide with— an evaluation coming from the representation theory of the quantum group $U_q(\mathfrak{gl}_N)$ interpreted through Reshetikhin–Turaev formalism. In this section, we will sketch this viewpoint, which provides a theoretical framework for polynomial quantum invariant. The advantage of the combinatorial approach is its proximity with the notions needed for the evaluation of foams. In order to differentiate these notion of evaluations of webs, we will denote the representation theoretic evaluations by $\langle\cdot\rangle_{\mathfrak{gl}_N}$ and $\langle\!\langle\cdot\rangle\!\rangle_{\mathfrak{gl}_N}$, while their combinatorial counterparts will be still denoted by $\langle\cdot\rangle_N$ and $\langle\!\langle\cdot\rangle\!\rangle_N$.

Definition 2.3.1. The quantum group $U_q(\mathfrak{gl}_N)$ is the unital associative algebra over $\mathbb{C}(q)$ generated by the elements $(E_i)_{1\leq i\leq N-1}$, $(F_i)_{1\leq i\leq N-1}$, $(L_i)_{1\leq i\leq N}$ and $(L_i^{-1})_{1\leq i\leq N}$, submitted to the following relations:

$$\begin{gathered}
L_iL_j = L_jL_i, \quad L_iL_i^{-1} = L_i^{-1}L_i = 1,\\
L_iF_i = q^{-1}F_iL_i, \quad L_{i+1}F_i = qF_iL_{i+1}, \quad L_iE_i = qE_iL_i, \quad L_{i+1}E_i = q^{-1}E_iL_{i+1},\\
L_jF_i = F_iL_j, \quad L_jE_i = E_iL_j \qquad \text{for } j\neq i, i+1,\\
E_iF_j - F_jE_i = \delta_{ij}\frac{L_iL_{i+1}^{-1} - L_i^{-1}L_{i+1}}{q-q^{-1}},\\
[2]F_iF_jF_i = F_i^2F_j + F_jF_i^2 \qquad \text{if } |i-j| = 1,\\
[2]E_iE_jE_i = E_i^2E_j + E_jE_i^2 \qquad \text{if } |i-j| = 1,\\
E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i \qquad \text{if } |i-j| > 1.
\end{gathered}$$

It is endowed with a structure of Hopf algebra by defining, the coproduct Δ, the co-unit ε and the antipode S by the following formula.

$$\begin{aligned}
&\Delta(L_i^{\pm1}) = L_i^{\pm1}\otimes L_i^{\pm1} && S(L_i^{\pm1}) = L_i^{\mp1} && \epsilon(L_i^{\pm1}) = 1\\
&\Delta(F_i) = F_i\otimes 1 + L_i^{-1}L_{i+1}\otimes F_i && S(F_i) = -L_iL_{i+1}^{-1}F_i && \epsilon(F_i) = 0\\
&\Delta(E_i) = E_i\otimes 1 + L_iL_{i+1}^{-1}\otimes E_i && S(E_i) = -E_iL_i^{-1}L_{i+1} && \epsilon(E_i) = 0
\end{aligned}$$

Proposition 2.3.2. *Define V_q to be an Nth dimensional $\mathbb{C}(q)$-vector space with basis $(b_i)_{i=1,\dots,N}$. The formulas (for $1\leq i,j\leq N$:*

$$\begin{aligned}
&L_ib_i = qb_i, && L_i^{-1}b_i = q^{-1}b_i,\\
&L_i^{\pm1}b_j = b_j && \text{if } i\neq j,\\
&E_ib_j = \delta_{j(i+1)}b_i && F_ib_j = \delta_{j(i+1)}b_j
\end{aligned}$$

endow V_q with a structure of $U_q(\mathfrak{gl}_N)$-modules. It is called the vector *(or* standard*) representation of $U_q(\mathfrak{gl}_N)$.*

Following [**ST19**], consider the tensor algebra $T^\bullet V_q$. This algebra is naturally graded and endowed with an action of $U_q(\mathfrak{gl}_N)$ which preserve the grading (i. e. for every integer a, T^aV_q is a $U_q(\mathfrak{gl}_N)$-submodule of $T^\bullet V_q$.). We consider two two-sided ideals E^2V_q and S^2V_q inside this algebra TV_q:

$$\begin{aligned}
&E^2V_q := \langle qb_i\otimes b_j - b_j\otimes b_i|\text{for } i<j\rangle \qquad \text{and}\\
&S^2V_q := \langle b_m\otimes b_m, b_i\otimes b_j + qb_j\otimes b_i|\text{for all } m \text{ and for } i<j\rangle.
\end{aligned}$$

Since these two ideals are homogeneous the quotients

$$\Lambda_q^\bullet V_q := T^\bullet V_q / SV_q \quad \text{and} \quad \mathrm{Sym}_q^\bullet V_q := T^\bullet V_q / EV_q$$

inherit a grading from $T^\bullet V_q$. For every integer a one can check that $\Lambda_q^a V_q$ and $\mathrm{Sym}_q^a V_q$ inherit also a $U_q(\mathfrak{gl}_N)$-action and that it is actually a simple module. The images of a pure tensor $x_1 \otimes \cdots \otimes x_a$ are denoted by

$$x_1 \wedge \cdots \wedge x_a \in \Lambda_q^\bullet V_q \quad \text{and} \quad x_1 \otimes \cdots \otimes x_a \in \mathrm{Sym}_q^a V_q \tag{28}$$

respectively.

We now define an analogous functor with $\mathrm{Sym}_q^\bullet V_q$ instead of $\Lambda_q^\bullet V_q$, The $\mathbb{C}(q)$ vector space $\Lambda_q^a V_q$ has dimension $\binom{N}{a}$ and is spanned by the vectors

$$(b_{i_1} \wedge b_{i_2} \wedge \cdots \wedge b_{i_a})_{1 \leq i_1 < i_2 < \cdots < i_a \leq N}. \tag{29}$$

If $1 \leq i_1 < i_2 < \cdots < i_a \leq N$ and $I = \{i_1, \ldots, i_a\}$, we write $b_I = b_{i_1} \wedge b_{i_2} \wedge \cdots \wedge b_{i_a}$.

If A and B are two subsets of a finite ordered set C, define:

$$|A < B| := \#\{(a, b) \in A \times B | a < b\}.$$

Proposition 2.3.3. *The following maps define morphisms of $U_q(\mathfrak{gl}_N)$-modules:*

$$\begin{array}{rrcl} \Lambda_{a,b}: & \Lambda_q^a V_q \otimes \Lambda_q^b V_q & \to & \Lambda_q^{a+b} V_q \\ & b_I \otimes b_J & \mapsto & \begin{cases} q^{-|J<I|} b_{I \sqcup J} & \text{if } I \cap J = \emptyset, \\ 0 & \text{otherwise,} \end{cases} \end{array} \tag{30}$$

$$\begin{array}{rrcl} Y_{a,b}: & \Lambda_q^{a+b} V_q & \to & \Lambda_q^a V_q \otimes \Lambda_q^b V_q \\ & b_K & \mapsto & \sum_{I \sqcup J = K} q^{|I<J|} b_I \otimes b_J, \end{array} \tag{31}$$

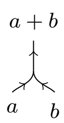
$$\begin{array}{rrcl} \overleftarrow{\cup}_a: & \mathbb{C}(q) & \to & \Lambda_q^a V_q \otimes (\Lambda_q^a V_q)^* \\ & 1 & \mapsto & \sum_{\# I = a} b_I \otimes b_I^*, \end{array} \tag{32}$$

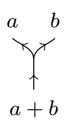
$$\begin{array}{rrcl} \overleftarrow{\cap}_a: & (\Lambda_q^a V_q)^* \otimes \Lambda_q^a V_q & \to & \mathbb{C}(q) \\ & f \otimes x & \mapsto & f(x) \end{array} \tag{33}$$

$$\begin{array}{rrcl} \overrightarrow{\cup}_a: & \mathbb{C}(q) & \to & (\Lambda_q^a V_q)^* \otimes \Lambda_q^a V_q \\ & 1 & \mapsto & \sum_{\# I = a} q^{-|I<\mathbb{P}| + |\mathbb{P}<I|} b_I^* \otimes b_I, \end{array} \tag{34}$$

$$\begin{array}{rrcl} \overrightarrow{\cap}_a: & \Lambda_q^a V_q \otimes (\Lambda_q^a V_q)^* & \to & \mathbb{C}(q) \\ & b_I \otimes b_J^* & \mapsto & q^{|I<(\mathbb{P}\setminus I)| - |(\mathbb{P}\setminus I)<I|} \delta_{IJ}, \end{array} \tag{35}$$

Using Reshetikhin–Turaev formalism, one can interpret planar webs (possibly with boundaries) as morphisms in $U_q(\mathfrak{gl}_N)$-mod using the following correspondence:

(36) 

$\rightsquigarrow \Lambda_{a,b}$ 

$\rightsquigarrow Y_{a,b}$

(37) 

$\rightsquigarrow \overrightarrow{\cup}_a$ 

$\rightsquigarrow \overleftarrow{\cup}_a$ 

$\rightsquigarrow \overrightarrow{\cap}_a$ 

$\rightsquigarrow \overleftarrow{\cap}_a$

To be more formal, this defines a functor $\langle\cdot\rangle_{\mathfrak{gl}_N}$ from the category Web to the category $U_q(\mathfrak{gl}_N)$-mod. In particular, every closed web Γ is mapped by this functor on an endomorphism of a one-dimensional representation of $U_q(\mathfrak{gl}_N)$, hence $\langle\Gamma\rangle_{\mathfrak{gl}_N} \in \mathbb{C}(q)$.

Proposition 2.3.4. *For any closed web* Γ*,* $\langle\Gamma\rangle_{\mathfrak{gl}_N} = \langle\Gamma\rangle_N$ *.*

Sketch of the proof. One checks that relations (25)–(21) holds also for the algebraic evaluation. This concludes, since these relations are enough to reduce any planar web to the empty web and that both evaluation agree on the empty web. □

The $\mathbb{C}(q)$ vector space $\mathrm{Sym}_q^a V_q$ has dimension $\binom{N+a-1}{a}$ and is spanned by the vectors

$$(b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_a})_{1\leq i_1 \leq i_2 \leq \cdots \leq i_a \leq N}.$$

If $1 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq N$ and $I = \{i_1, i_2, \dots, i_a\}$ is a multi-subset of $\mathbb{P}$, we write $b'_I = b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_a}$.

If A and B are two A and B are two multi-subsets of a finite ordered set X, define

$$|A < B| := \prod_{x<y\in X} A(x)B(y), \tag{38}$$

and

$$[A, B] = \prod_{x\in X} \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}. \tag{39}$$

Proposition 2.3.5. *The following maps define morphisms of* $U_q(\mathfrak{gl}_N)$*-modules:*

$$\begin{array}{rrcl} S\lambda_{a,b}: & \mathrm{Sym}_q^a V_q \otimes \mathrm{Sym}_q^b V_q & \to & \mathrm{Sym}_q^{a+b} V_q \\ & b'_I \otimes b'_J & \mapsto & q^{|J<I|} b'_{I\sqcup J} \end{array} \tag{40}$$

$$\begin{array}{rrcl} SY_{a,b}: & \mathrm{Sym}_q^{a+b} V_q & \to & \mathrm{Sym}_q^a V_q \otimes \mathrm{Sym}\Lambda_q^b V_q \\ & b'_K & \mapsto & \displaystyle\sum_{I\sqcup J=K} [I,J]_q q^{-|J<I|} b'_I \otimes b'_J \end{array} \tag{41}$$

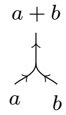
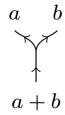
$$\begin{array}{rrcl} \overleftarrow{\Cup}_a: & \mathbb{C}(q) & \to & \mathrm{Sym}_q^a V_q \otimes (\mathrm{Sym}_q^a V_q)^* \\ & 1 & \mapsto & \displaystyle\sum_{\#I=a} q^{|I<J|} b'_I \otimes (b'_I)^* \end{array} \tag{42}$$

$$\begin{array}{rrcl} \overleftarrow{\Cap}_a: & (\mathrm{Sym}_q^a V_q)^* \otimes \mathrm{Sym}_q^a V_q & \to & \mathbb{C}(q) \\ & f \otimes x & \mapsto & f(x) \end{array} \tag{43}$$

$$\begin{array}{rrcl} \overrightarrow{\Cup}_a: & \mathbb{C}(q) & \to & (\mathrm{Sym}_q^a V_q)^* \otimes \mathrm{Sym}_q^a V_q \\ & 1 & \mapsto & \displaystyle\sum_{\#I=a} q^{-|I<\mathbb{P}|+|\mathbb{P}<I|} (b'_I)^* \otimes b'_I \end{array} \tag{44}$$

$$\begin{array}{rrcl} \overrightarrow{\Cap}_a: & \mathrm{Sym}_q^a V_q \otimes (\mathrm{Sym}_q^a V_q)^* & \to & \mathbb{C}(q) \\ & b'_I \otimes (b'_J)^* & \mapsto & q^{+|I<\mathbb{P}|-|\mathbb{P}<I|} \delta_{IJ} \end{array} \tag{45}$$

Using Reshetikhin–Turaev formalism, one can interpret planar webs (possibly with boundaries) as morphisms on $U_q(\mathfrak{gl}_N)$-mod using the following correspondence:

(46)  $\leftrightsquigarrow S\Lambda_{a,b}$  $\leftrightsquigarrow SY_{a,b}$

(47)  $\leftrightsquigarrow \overrightarrow{\Cup}_a$  $\leftrightsquigarrow \overleftarrow{\Cup}_a$  $\leftrightsquigarrow \overrightarrow{\Cap}_a$  $\leftrightsquigarrow \overleftarrow{\Cap}_a$

To be more formal, this defines a functor $\langle\!\langle \cdot \rangle\!\rangle_{\mathfrak{gl}_N}$ from the category Web to the category $U_q(\mathfrak{gl}_N)$-mod. In particular, every closed web Γ is mapped by this functor on an endomorphism of a one-dimensional representation of $U_q(\mathfrak{gl}_N)$, hence $\langle\!\langle \Gamma \rangle\!\rangle_{\mathfrak{gl}_N} \in \mathbb{C}(q)$.

Proposition 2.3.6. *For any closed directed web Γ in the annulus (see Item 2),* $\langle\!\langle \Gamma \rangle\!\rangle_{\mathfrak{gl}_N} = \langle\!\langle \Gamma \rangle\!\rangle_N$.

CHAPTER 3

Foam evaluation

In this chapter we explain the foam evaluation formula and inspect some of its consequences: new definition of Khovanov–Rozansky colored homologies, description of cohomology rings of partial flag varieties, and $\mathfrak{sl}_2$-symmetries of Khovanov–Rozansky homologies. The material in this chapter is extracted from [**2, 7, 8**]. For any integer a, Sym_a denotes the graded ring of symmetric polynomials in a variables with coefficients in $\mathbb{Z}$. Variables are always supposed to have degree 2. In particular, $\mathrm{rk}_q \mathrm{Sym}_a = \prod_{i=1}^{a} \left(1 - q^{2i}\right)^{-1}$.

1. Foams

Foams are 2-dimensional analogues of webs. As such many definitions concerning them will be very similar to what we have seen in the previous chapter.

Definition 3.1.1. Let M be an oriented smooth 3-manifold with a collared boundary. A *foam* $F \subset M$ is a finite collection of *facets*, that are compact oriented surfaces labeled with positive integers (their *thicknesses*) and glued together along their boundary points such that every point p of F has a closed neighborhood homeomorphic to one of the following:

- A disk, when p belongs to a unique facet,
- $Y \times [0, 1]$, where Y is the neighborhood of a merge or a split vertex in a web, when p belongs to three facets, or
- The cone over the 1-skeleton of a tetrahedron with p as the vertex of the cone (so that it belongs to six facets).

See Fig. 1 for a pictorial representation of these three cases. The set of points of the second type is a collection of curves called *bindings* and the points of the third type are called *singular vertices*. The *boundary* ∂F of F is the closure of the set of boundary points of facets that do not belong to a binding.

The thicknesses of facets and their orientations satisfy the same flow conditions at bindings as oriented edges in webs. Every binding is oriented so that:

- Its orientation agrees with the orientation of the boundary of the thick facet adjacent to it.
- Its orientation disagrees with the orientations of the boundary of the two thin facets adjacent to it.

It is understood that F coincides with $\partial F \times [0, \epsilon]$ on the collar of ∂M. We write $F^{(2)}$ for the collection of facets of F and $t(f)$ for the thickness of a facet f. A foam F is *decorated* if each facet $f \in F^{(2)}$ is assigned a symmetric polynomial $P_f \in \mathrm{Sym}_{t(f)}$. The *trivial decoration* on a facet f is the constant polynomial $1 \in \mathrm{Sym}_{t(f)}$. A foam F is *trivially decotared* if the decoration of every facet is trivial.

A foam is not necessarily connected and a facet can have empty boundary (in which case they are disjoint from the rest of the foam). Bindings are either closed intervals or circles. For a given foam F, its set of bindings is denoted by $F^{(1)}$ and its set of singular vertices is denoted by $F^{(0)}$.

Remark 3.1.2. Orientation of the manifold M and on the bindings induces for each binding a cyclic ordering of the set of facets adjacent to that binding.

One can also consider foams that are not embedded (or embedded in non-oriented 3-manifolds), most of what is discussed afterwards actally makes sense for those. However, one then need to endow each binding with a cyclic ordering of the set of facets which are adjacent to it with a compatility condition at singular vertices: namely that neighborhoods of vertices are embeddable in $\mathbb{R}^3$ (endowed with an orientation) such that the cyclic ordering on bindings is given by the right-hand rule (and the orientations of bindings).

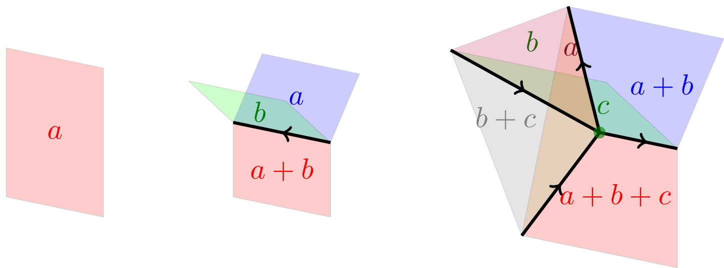


FIGURE 1. The three local models for a foam.

The boundary of a foam $F \subset M$ is a web in ∂M. In the case $M = \Sigma \times [0,1]$, a generic section $F_t := F \cap (\Sigma \times \{t\})$ is a web. The bottom and top webs F_0 and F_1 are called respectively the *input* and *output* of F. A *closed* foam is a foam with empty boundary.

Example 3.1.3. (1) A disjoint union of surfaces labelled by non-negative integers is a foam with no boundary, no binding and no singular point.
(2) If $\Gamma \subset \Sigma$ is a web, then $\Gamma \times [0,1]$ is naturally seen as a foam in $\Sigma \times [0,1]$ with input and output equal to Γ.
(3) The foam depicted in Fig. 2 describes a foam with six facets, four interval bindings and two singular points.

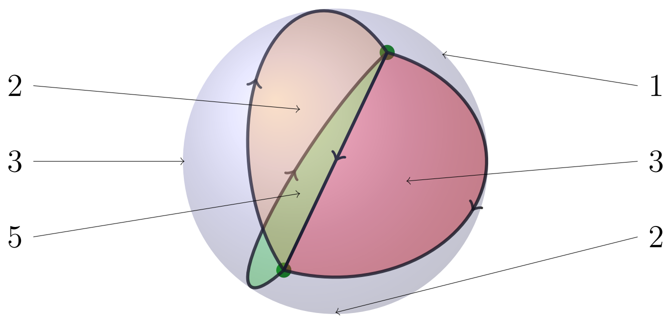


FIGURE 2. Example of a closed foam (without decoration).

Definition 3.1.4. Fix N a non-negative integer and consider a (non-decorated) foam F in a 3-manifold M, use the orientation of F and M to parallelize F:

- Replace each facets of thickness a by $a \times (N-a)$ parallel copies of that facet.
- Glue this surfaces together in the only embedded way at bindings. If the binding is surrounded by facets of thicknesses a, b and $a+b$, this needs gluing $ab + (a+b)(N-a-b))$ times along an interval (or a circle)
- Fill the remaining holes at singular vertices. At a singular vertex whose adjacent facets have thicknesses a, b, c, $a+b$, $b+c$ and $a+b+c$, this needs filling $ab+bc+ac+(a+b+c)(N-a-b-c)$ holes.

This gives rise to a surface S. The *N-degree* of F is the integer $\deg_N(F) := -\chi(S)$. If now F is decorated, define

$$\deg_N(F) := -\chi(S) + \sum_{f\in F^{(2)}} \deg(P_f)\,. \tag{48}$$

where $\deg(P_f)$ is the degree of P_f as an element of $\mathrm{Sym}_{t(f)}$, in particular it is even.

Example 3.1.5. (1) By definition, the N-degree is additive with respect to disjoint union.
(2) The N-degree of a surface S of thickness a is equal to $-a(N-a)\chi(S)$.
(3) If Γ is a web, then $\Gamma\times[0,1]$ (with trivial decoration on facets) has N-degree equal to 0.
(4) The foam of Example 3.1.3.(3) has N-degree equal to $-10N+34$.
(5) The N-degree is additive with respect to composition of foams (see Definition 3.1.6)

Definition 3.1.6. Let Σ be an oriented surface. Define the category Foam_Σ as follow:

- Objects of Foam_Σ are closed webs on Σ.
- The set of homomorphims $\hom_{\mathsf{Foam}_\Sigma}(\Gamma_0,\Gamma_1)$ is the set of foams in $\Sigma\times[0,1]$ with Γ_0 as input and Γ_1 as output up to ambient isotopy.
- Composition of morphisms is given by concatenation and resizing (and product of the decorations of facets which are identified). The collared boundary assumption ensures that the composition is well-defined. Associativity is guaranteed by the fact that we regard foams up to ambient isotopy.
- The identity of Γ is the foam $\Gamma\times[0,1]$ (trivially decorated).

Although defined for a general surface, we will mostly be interested in the case $\Sigma=\mathbb{R}^2$ and we set $\mathsf{Foam} := \mathsf{Foam}_{\mathbb{R}^2}$. In Chapter 4 we will consider a non-full subcategory of $\mathsf{Foam}_{\mathbb{S}^1\times[0,1]}$ where only directed webs and directed foams are considered. In both these cases, one has a monoidal structure given by disjoint union of webs and foams (side by side and concentrically respectively).

2. Colorings

The notion of coloring of foams is the obvious generalization of that for webs.

Definition 3.2.1. A *$\mathfrak{gl}_N$-coloring* (or *N-coloring* or *coloring*) of a foam F is a map $c\colon F^{(2)}\to\mathcal{P}(\llbracket N\rrbracket)$, such that:

- For each facet f, $\#c(f)=t(f)$.

- For each binding joining a facet f_1 with thickness a, a facet f_2 with thickness b, and a facet f_3 with thickness $a+b$, $c(f_1) \cup c(f_2) = c(f_3)$. This condition is called the *flow condition for colorings*.

A *colored foam* is a foam together with a coloring. The set of $\mathfrak{gl}_N$-colorings of F is denoted by $\mathrm{col}_N(F)$.

Remark 3.2.2. (1) Facets of thicknesses 0 are alway colored by the empty set.
(2) The symmetric group $\mathfrak{S}_N$ acts naturally on $\mathrm{col}_N(F)$.
(3) There are no extra condition at singular vertices.

A careful inspection of the behavior of colorings at bindings and singular points gives the following lemma:

Lemma 3.2.3. *(1) If (F, c) is a colored foam and i is an element of $[\![N]\!]$, the union (with the identification coming from the gluing procedure) of all the facets which contain the pigment i in their colors is a surface. It is called the* monochrome surface of (F, c) associated with i *and it is denoted by $F_i(c)$. The restriction we imposed on the orientations of facets ensures that $F_i(c)$ is oriented.*
(2) If (F, c) is a colored foam and i and j are two distinct elements of $[\![N]\!]$, the union (with the identification coming from the gluing procedure) of all the facets which contain i or j but not both in their colors is a surface. It is called the bichrome surface of (F, c) associated with i, j. *It is the symmetric difference of $F_i(c)$ and $F_j(c)$ and it is denoted by $F_{ij}(c)$. The restriction we imposed on the orientations of facets ensures that $F_{ij}(c)$ can be oriented by taking the orientation of the facets containing j and the reverse orientation on the facets containing i.*
(3) In the same situation, we may suppose $i < j$. We consider a binding joining the facets f_1, f_2 and f_3. Suppose that i is in $c(f_1)$, j is in $c(f_2)$ and $\{i, j\}$ is in $c(f_3)$. We say that the binding is positive with respect to (i, j) *if the cyclic order on the binding is (f_1, f_2, f_3) and* negative with respect to (i, j) *otherwise (see Fig. 3 for an illustration). The set $F_i(c) \cap F_j(c) \cap F_{ij}(c)$ is a collection of disjoint circles. Each of these circles is a union of bindings, for every circle the bindings are either all positive or all negative with respect to (i, j).*

FIGURE 3. Sign convention for bindings in $\mathbb{R}^3$ oriented by the right-hand rule when $i < j$.

The first two items of Lemma 3.2.3 are 2-dimensional analogues of the observations underlying the two first items of Notation 2.1.9. As such they contain the definitions of *monochrome* and *bichrome surfaces*. The last one yields the following definition:

Definition 3.2.4. Let (F,c) be a colored foam and $i<j$ be two pigments. A circle in $F_i(c)\cap F_j(c)\cap F_{ij}(c)$ is *positive* (resp. *negative*) *with respect to* (i,j) if it consists of positive (resp. negative) bindings. Denote by $\theta_{ij}^{+}(F,c)$ (resp. $\theta_{ij}^{-}(F,c)$) or simply $\theta_{ij}^{+}(c)$ (resp. $\theta_{ij}^{-}(c)$) the number of positive (resp. negative) circles with respect to (i,j) and set $\theta_{ij}(c)=\theta_{ij}^{+}(c)+\theta_{ij}^{-}(c)$.

In this 2-dimensional context one can define Kempe-moves analogously to Definition 2.1.15.

Definition 3.2.5. Let F be a closed foam, c a coloring of F and $i<j\in[\![N]\!]$ two pigments. Consider a connected component Σ of $F_{ij}(c)$ and define c' the $\mathfrak{gl}_N$-coloring of F which is identical to c on all facets not contained in Σ and which is equal to the symmetric difference of $c(f)$ and $\{i,j\}$ for any facet f in Σ. The coloring c and c' are said to be related by an *ij-Kempe move along* Σ, or simply *Kempe-move*.

3. Foam evaluation

The foam evaluation is a formula which associates with a closed decorated foam F in $\mathbb{R}^3$ (or $\mathbb{S}^3$) a symmetric polynomial in variables $X_1,\dots X_N$, i.e. an element of $\mathbb{Z}[X_1,\dots,X_N]^{\mathfrak{S}_N}$. It is a sum over all colorings c of F of rational functions associated with (F,c).

Let F be a foam and c be a coloring of F, define:

$$s(F,c)=\sum_{i=1}^{N} i\frac{\chi(F_i(c))}{2}+\theta_{ij}^{+}(F,c) \tag{49}$$

$$P(F,c)=\prod_{f\in F^{(2)}} P_f(X_{c(f)}) \tag{50}$$

$$Q(F,c)=\prod_{1\leq i<j\leq N} (X_j-X_i)^{\frac{\chi(F_{ij}(c))}{2}} \tag{51}$$

$$\tau_N\left(F,c\right)=(-1)^{s(F,c)}\frac{P(F,c)}{Q(F,c)} \tag{52}$$

In the definition of $P(F,c)$, recall that for each facet $f\in F^{(2)}F$, P_f is a symmetric polynomial in $t(f)$ variables, which can therefore be evaluated on any $t(f)$-element subset of a commutative ring, $X_{c(f)}$ refers to the set $\{X_i|i\in c(f)\}\subset\mathbb{Q}(X_1,\dots,X_N)$.

The quantity $\tau_N\left(()\,F,c\right)\in\mathbb{Q}(X_1,\dots,X_N)$ is called the *colored evaluation* of (F,c). The symmetric group $\mathfrak{S}_N$ acts on $\mathbb{Q}(X_1,\dots,X_N)$ by permuting the variables. The quantity $s(F,c)$ is designed so that the following lemma holds:

Lemma 3.3.1. *Let σ be an element of $\mathfrak{S}_N$, then*

$$\tau_N\left(F,\sigma\cdot c\right)=\sigma\cdot\tau_N\left(F,c\right) \tag{53}$$

Let F be a closed foam, define the *evaluation of F* to be the quantity:

$$\tau_N\left(F\right)=\sum_{c\in \mathrm{col}_N(F)}\tau_N\left(F,c\right) \tag{54}$$

From Lemma 3.3.1 one immediately get that $\tau_N\left(F\right)$ is symmetric (remains unchanged when permuting variables). However its polynomiality is far from obvious.

Denote $\mathbb{Z}_N$ the graded ring $\mathbb{Z}[X_1, \dots, X_N]^{\mathfrak{S}_N}$. Recall that variables X_i are homogeneous of degree 2.

Proposition 3.3.2. *For any closed foam F in $\mathbb{R}^3$ (or $\mathbb{S}^3$), $\tau_N(F)$ is an element of $\mathbb{Z}_N$ of degree* $\deg_N(F)$.

Pitch of the proof. The only things to take care of are the factors $(X_j - X_i)^{\chi F_{ij}(c)/2}$ in $Q(F, c)$. Because of the $\mathfrak{S}_N$-symmetry given by Lemma 3.3.1, we may only focus on the factors $(X_2 - X_1)^{\chi F_{12}(c)/2}$. All connected oriented surfaces but the sphere have non-positive Euler characteristic. Hence only bichromatic spheres create non-polynomiality. A $(1, 2)$-Kempe-move along a sphere produces some antisymmetry in $X_1 \leftrightarrow X_2$ on the numerator so that it becomes divisible by $(X_2 - X_1)$. Hence grouping colorings properly shows that there is no $(X_2 - X_1)$ in the denominator and this proves the statement by symmetry. □

Example 3.3.3. If Γ is a web, then $F := \Gamma \times \mathbb{S}^1$ is a foam (or more precisely can be embedded in $\mathbb{R}^3$ to produce a foam). Let us decorate all facets with 1, the trivial decoration. For any coloring c of F, all monochromatic and bichromatic surfaces are disjoint unions of tori. Moreover, for every $1 \leq i < j \leq N$, $\theta^+_{ij}(c)$ is even. Therefore one has:

$$\tau_N(F) = (\langle \Gamma \rangle_N)_{|q \mapsto 1}\,. \tag{55}$$

Example 3.3.4. Let $1 \leq a \leq N$ and λ be a Young diagram with less than a rows and denote by s_λ the Schur polynomial in a variables associated with λ. consider F the foam which is a sphere of thickness a decorated by s_λ. The colorings of F are in one-to-one correspondence with a-element subsets $I \subset [\![N]\!]$. Let c_I be the coloring of F corresponding to I. One has:

$$\langle F, c \rangle_N = (-1)^{\left(\sum_{i\in I} i\right)} \frac{s_\lambda(X_I)}{\displaystyle\prod_{\substack{i<j \\ \#(i,j\cap I)=1}} (X_j - X_i)} \tag{56}$$

In order to compute $\langle F \rangle_N$, one can use the definition of the Schur polynomial $s_\lambda(X_I)$ as a quotient of a generalized Vandermonde $\Delta_\lambda(X_I)$ determinant by the standard Vandermonde determinant $\Delta(X_I)$.

$$\langle F \rangle_N = \sum_I (-1)^{\left(\sum_{i\in I} i\right)} \frac{\Delta_\lambda(X_I)\Delta(X_{[\![N]\!]\setminus I})}{\Delta(X_1, \dots X_N)}. \tag{57}$$

Using a the multi-line development formula for the determinant, one obtains that $\langle F \rangle_N = 0$ if λ does not contains the $a \times (N-a)$ rectangular Young diagram $\rho(a, N-a)$ and

$$\langle F \rangle_N = (-1)^{aN + a(a-1)/2} s_{\lambda \setminus \rho(a,N-a)}(X_{[\![N]\!]}) \in \mathbb{Z}_N \tag{58}$$

otherwise.

4. Universal construction

From the foam evaluation one can derive a functor from the category Foam to the category of $\mathbb{Z}_N$-modules using the so-called *universal construction* [BHMV95] that we detail in the following paragraphs. The monoidal structure on $\mathbb{Z}_N$-Mod is given by tensor product over the commutative ring $\mathbb{Z}_N$.

For Γ a web in $\mathbb{R}^2$, define

$$\widetilde{V}_N(\Gamma) := \bigoplus_{F \in \hom_{\mathsf{Foam}}(\emptyset, \Gamma)} q^{\deg_N(F)} \mathbb{Z}_N. \tag{59}$$

In other words, an element of $\widetilde{V}_N(\Gamma)$ is a formal linear combination of foams from the empty set to Γ. The $\mathbb{Z}_N$-module $\widetilde{V}_N(\Gamma)$ is free but is far from being finitely generated. It has a (homogeneous) basis given by elements of $\hom_{\mathsf{Foam}}(\emptyset, \Gamma)$

This construction immediately extends to a functor from Foam to $\mathbb{Z}_N$-Mod by defining for G in $\hom_{\mathsf{Foam}}(\Gamma_1, \Gamma_2)$, $\widetilde{V}_N(G)$ as the $\mathbb{Z}_N$-linear map mapping any foam F in $\hom_{\mathsf{Foam}}(\emptyset, \Gamma_1)$ on $G \circ F \in \hom_{\mathsf{Foam}}(\emptyset, \Gamma_2)$.

On each $\widetilde{V}_N(\Gamma)$, define a $\mathbb{Z}_N$-bilinear form $(\bullet;\bullet)_N$ by setting for any two foams F and G in $\hom_{\mathsf{Foam}}(\emptyset, \Gamma)$:

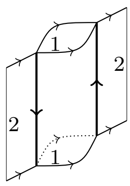
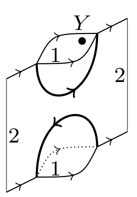
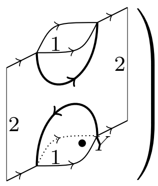
$$(F;G)_N := \tau_N\left(\overline{F} \circ G\right) \tag{60}$$

where $\overline{F}$ is the foam in $\hom_{\mathsf{Foam}}(\Gamma, \emptyset)$ given by taking the mirror image of $F \subset \mathbb{R}^2 \times [0,1]$ with respect to $\mathbb{R}^2 \times \{1/2\}$. Denote $K_N(\Gamma)$ the kernel (or radical) of this bilinear form and set $\mathcal{E}_N(\Gamma) = \widetilde{V}_N(\Gamma)/K_N(\Gamma)$. The $\mathbb{Z}_N$-module $\mathcal{E}_N(\Gamma)$ is called the *(exterior equivariant)* $\mathfrak{gl}_N$*-state space of* Γ.

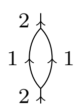
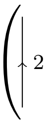
Just like $\widetilde{V}_N$, this immediately extends to a functor $\mathcal{E}_N$ from Foam to $\mathbb{Z}_N$-Mod. At this stage it is not clear that the modules $\mathcal{E}_N(\Gamma)$ are projective (and therefore free), finitely generated and that the functor $\mathcal{E}_N$ is monoidal. However this kind of construction always ensure lax-monoidality.

Theorem 3.4.1. *The functor $\mathcal{E}_N$ is monoidal and for any web Γ in $\mathbb{R}^2$, the $\mathbb{Z}_N$-module $\mathcal{E}_N(\Gamma)$ is free of graded rank equal to $\langle\Gamma\rangle_N$.*

Pitch of the proof. To prove Theorem 3.4.1, one proves that $\mathcal{E}_N$ satisfies a categorified version of Proposition 2.1.12: identities (9)–(15) are lifted to isomorphisms. To do this one decomposes identity morphisms as sums of two-by-two orthognal idempotents each of them being a composition of a projection and an injection. This boils down to showing that some local relations are satisfied by the foam evaluation formula. For instance one can show that:

(61) τ_N () $= \tau_N$ () $- \tau_N$ ()

where the relation is meant to hold locally and $\bullet Y$ means that the facets on which $\bullet$ stand are decorated by the one variable (symmetric) polynomial $Y \in \mathbb{Z}[Y]$, the other facets being trivially decorated. This identity, plus the fact that the two foams on the right-hand sides behave like orthogonal idempotents shows the following isomorphism:

(62) $\mathcal{E}_N$ () $\simeq \left(q + q^{-1}\right) \mathcal{E}_N$ ().

Wu [**Wu14**] showed that identities (21)–(25) are enough to reduce any web to the empty web. Since these relations are categorified (and that sum in the skein module corresponds to direct sum), this implies the projectivity statement as well as the rank statement (one also need to show that $\mathcal{E}_N(\emptyset) \simeq \mathbb{Z}_N$, which is rather simple). The first part of the statement is a consequence of the second one. □

Let S be a commutative ring and $\varphi\colon \mathbb{Z}_N \to S$. One can then endow S with a structure of $\mathbb{Z}_N$-module. Hence one can define an S-mod-valued functor $\mathcal{E}_N(\bullet;\varphi)$ by setting:

$$\mathcal{E}_N(\Gamma;\varphi) := S \otimes_{\mathbb{Z}_N} \mathcal{E}_N(\Gamma) \tag{63}$$

and similarly for morphisms.

Alternatively, one can define an S-valued foam evaluation formula by setting:

$$\tau_N(\bullet;\varphi) := \varphi \circ \tau_N(\bullet). \tag{64}$$

and reapply the universal construction. This provides another S-mod-valued functor. As an indirect consequence of Theorem 3.4.1, these two constructions are actually (canonically) isomorphic, so that we won't distinguished between the two. The construction of $\mathcal{E}_N(\cdot;\varphi)$ from $\mathcal{E}_N$ is usually referred to as *a base change.* If S is graded and φ respects that degree, then the functor is valued in the category of graded S-modules.

Usual base change in this context are given by the following ring morphisms:

$$\iota\colon \mathbb{Z}_N \hookrightarrow \mathbb{Z}[X_1,\dots,X_N] \tag{65}$$

$$\pi_{\mathbb{C}}\colon \mathbb{Z}_N \to \mathbb{C}, \quad E_i \mapsto 0 \tag{66}$$

$$\pi_{\underline{z}}\colon \mathbb{Z}_N \to \mathbb{C}, \quad X_i \mapsto z_i \tag{67}$$

where E_i is the ith elementary symmetric polynomial in variables $X_1,\dots,X_N$ and $\underline{z} = (z_1,\dots,z_N)$ is a tuple of complex numbers. The two first morphisms are graded while the last one is only filtered and therefore provides a functor valued in filtered $\mathbb{C}$-vector spaces.

5. A pinch of algebraic geometry

An easy consequence of Theorem 3.4.1 is that the graded rank of the $\mathfrak{gl}_N$-state space of the circle of thickness k is equal to $\begin{bmatrix} N \\ k \end{bmatrix}$. Incidentaly, this is also (up to a shift) the Poincaré polynomial of the cohomology ring of the Grasmannian variety $\mathrm{Gr}_{\mathbb{C}}(k,N) = \mathrm{Gr}(k,N)$. As we shall see we have more structural statement.

Lemma 3.5.1. *Let Γ be a web with a symmetry axis*[1] *D. Then the state space $\mathcal{E}_N(\Gamma)$ comes equipped with a Frobenius algebra structure.*

Let us denote by $\mathcal{E}_N(\Gamma, D)$ the $\mathfrak{gl}_N$-state space associated with Γ endowed with the Frobenius algebra structure and if $\varphi\colon \mathbb{Z}_N \to S$ is a morphism of commutative ring, denote $\mathcal{E}_N(\Gamma, D;\varphi)$ the same thing with the base change given by φ.

[1]The symmetry with respect to that symmetry axis should let Γ globally invariant but it changes all the orientations.

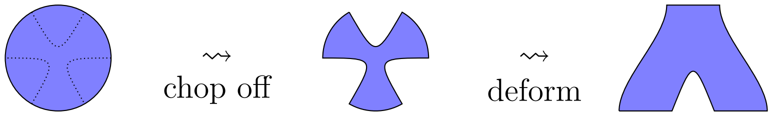


FIGURE 4. Construction of generalized pair of pants. Note that the regular pair of pants can be constructed in the same way.

Remark 3.5.2.
(1) Circles of various thicknesses provide examples of such webs. For those webs, Lemma 3.5.1 is related to the well-known facts that $(1+1)$-TQFTs are in one-to-one correspondence with commutative Frobenius algebra [**Koc04**].
(2) Different symmetry axis give (a priori) different Frobenius algebra structures. The Frobenius algebras arising may not be commutative.

PITCH OF THE PROOF. This follows from the existence of (analogues of) pair of pants, cups and caps for webs with a symmetry axis.

Let Γ be a web with a symmetry axis D. The line D partitions $\mathbb{R}^2$ into two half planes P_1 and P_2. Denote Γ_i the web (with boundary) $\Gamma \cap P_i$ (for $i \in \{1,2\}$). One can construct a foam $r(\Gamma)$ by rotating Γ_1 around D (in $\mathbb{R}^3$). The intersection of $r(\Gamma)$ with $\mathbb{R}^2 \times \mathbb{R}_{\leq 0}$ (resp. $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$) is a cap-like (resp. cup-like) foam for Γ.

The pairs of pants can be constructed as follow: start with $r(\Gamma)$, chop three cap-like (or cup-like) pieces off and finally deform the result to obtain a pair of pants. This procedure is schematized in Fig. 4.

Using a construction similar to arc-algebras, one gets a more structural way to understand this Frobenius algebra structure (see [**10**]). $\square$

Let B (resp. T) be the subgroup of $GL(N,\mathbb{C})$ of invertible upper triangular (resp. diagonal) matrices. The variety $\mathrm{Gr}(k,N)$ is naturally endowed with a right B (resp. T) action and we can consider its B-equivariant (resp. T-equivariant) cohomology ring. It is a module over $H^\bullet_B(\text{point}) \simeq \mathbb{Z}_N$ (resp. $H^\bullet_T(\text{point}) \simeq \mathbb{Z}[X_1,\dots,X_N]$). The reason we give these two versions is that the T-equivariant version is what is classically studied, while the B-equivariant version is what naturally arises in our construction.

Proposition 3.5.3. *The Frobenius algebra associated with the $\mathfrak{gl}_N$-state space of the circle of thickness k (denoted by $\mathbb{S}^1_k$) with a diameter D as symmetry axis is isomorphic to the (equivariant) cohomology of* $\mathrm{Gr}(k,N)$*, namely:*

$$\mathcal{E}_N\left(\mathbb{S}^1_k, D\right) \simeq H^\bullet_B(\mathrm{Gr}(k,N)), \tag{68}$$

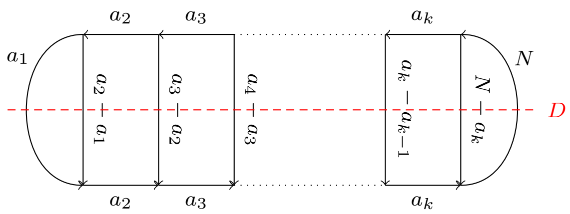
$$\mathcal{E}_N\left(\mathbb{S}^1_k, D; \iota\right) \simeq H^\bullet_T(\mathrm{Gr}(k,N)) \quad \textit{and} \tag{69}$$

$$\mathcal{E}_N\left(\mathbb{S}^1_k, D; \pi_{\mathbb{C}}\right) \simeq H^\bullet(\mathrm{Gr}(k,N)) \tag{70}$$

as Frobenius algebras over the rings $\mathbb{Z}_N$, $\mathbb{Z}[X_1,\dots,X_N]$ and $\mathbb{Z}$ respectively.

The proof of Proposition 3.5.3 not enlightening: it follows from the fact that the Frobenius algebra structures of the equivariant cohomology rings of Grassmannians are known (see for instance [**FIM14**]) and we can therefore write down explicit isomorphisms. We have a more general statement for partial flag variety: Let $\underline{a} := 0 < a_1 < \cdots < a_\ell < N$ be a strictly increasing sequence of positive integers and denote $\mathrm{Flag}_{\mathbb{C}}(\underline{a}) = \mathrm{Flag}(\underline{a})$ the complex partial flag variety indexed by $\underline{a}$. Define

$\theta(\underline{a})$, to be the web



and note that D is a symmetry axis for $\theta(\underline{a})$. Then Proposition 3.5.3 generalizes as follows:

Proposition 3.5.4. *The Frobenius algebra associated with the $\mathfrak{gl}_N$-state space of the web $\theta(\underline{a})$ with symmetry axis D is isomorphic to the (equivariant) cohomology of* $\mathrm{Flag}(\underline{a})$*, namely:*

(71) $$\mathcal{E}_N\left(\theta(\underline{a}), D\right) \simeq H^\bullet_B(\mathrm{Flag}(\underline{a}))$$

(72) $$\mathcal{E}_N\left(\theta(\underline{a}), D; \iota\right) \simeq H^\bullet_T(\mathrm{Flag}(\underline{a})) \quad \textit{and}$$

(73) $$\mathcal{E}_N\left(\theta(\underline{a}), D; \pi_{\mathbb{C}}\right) \simeq H^\bullet(\mathrm{Flag}(\underline{a}))$$

as Frobenius algebras over the rings $\mathbb{Z}_N$, $\mathbb{Z}[X_1, \dots, X_N]$ and $\mathbb{Z}$ respectively.

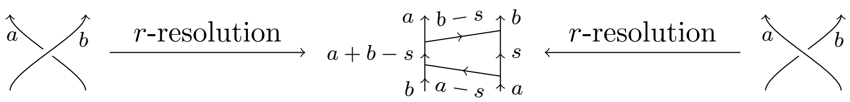
Proposition 3.5.3 and Proposition 3.5.4 in conjunction with the evaluation formula can be used to recover explicit formulas for the structural constants of these cohomology rings in terms of evaluation of decorated foams.

6. Exterior $\mathfrak{gl}_N$ link homology

The original motivation for the construction $\mathcal{E}_N$ is the definition of the equivariant $\mathfrak{gl}_N$-link homology. We sketch the construction in this section. All the material is inpired by [**KR08**, **Wu14**] and is a natural generalization of Khovanov original categorification of the Jones polynomial [**Kho00**].

Let $\overrightarrow{\mathbb{D}}$ be a colored oriented link diagram[2]: each component has a thickness which is a non-negative integer. It turns out that if a component has thickness strictly greater than N, then the whole construction is trivial and not interesting, but apart from that there is no reason to exclude these cases.

Denote $\mathsf{X} = \mathsf{X}\left(\overrightarrow{\mathbb{D}}\right)$ the set of crossings of $\overrightarrow{\mathbb{D}}$ and for x in X denote $\epsilon(x) \in \{-1, +1\}$ the sign of x. Let $R = R\left(\overrightarrow{\mathbb{D}}\right)$ be the set of map $r\colon \mathsf{X} \to \mathbb{Z}$ such that for each x in X, $0 \leq \epsilon(x) r(x) \leq \min(a, b)$ where a and b are the thicknesses of the strands involved in x. For $r \in R$, the r-resolution of $\overrightarrow{\mathbb{D}}$ is the web $\overrightarrow{\mathbb{D}}_r$ obtained by replacing each crossing x by a piece of web as prescribed here:

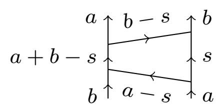
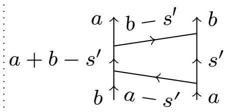
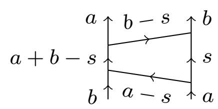
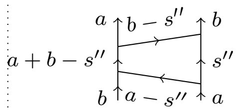
(74)


with $s = |r(x)|$. Consider the oriented graph $G\left(\overrightarrow{\mathbb{D}}\right)$ where vertices are the webs $(\overrightarrow{\mathbb{D}}_r)_{r \in R}$ and where an oriented edge $\overrightarrow{\mathbb{D}}_{r_1} \to \overrightarrow{\mathbb{D}}_{r_2}$ exists if and only if r_1 and r_2

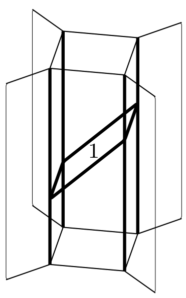
[2] The use of black board bold font is meant to emphasize that $\overrightarrow{\mathbb{D}}$ represent a framed link with the black board framing convention.

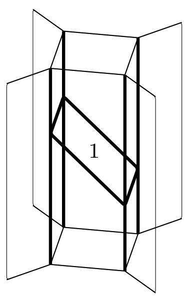
coincide on all elements of X except for a exactly one vertex, say x_0, for which $r_2(x_0) = r_1(x_0) + 1$. In that case, we say that the edge $\overrightarrow{\mathbb{D}}_{r_1} \to \overrightarrow{\mathbb{D}}_{r_2}$ is in the x_0-direction The graph $G\left(\overrightarrow{\mathbb{D}}\right)$ is a cartesian product of oriented Dynkin diagrams of type A. For each edge $e = r_1 \to r_2$ of $G\left(\overrightarrow{\mathbb{D}}\right)$, one associate a foam $F(e)$ from $\overrightarrow{\mathbb{D}}_{r_1}$ to $\overrightarrow{\mathbb{D}}_{r_2}$: recall that the webs $\overrightarrow{\mathbb{D}}_{r_1}$ and $\overrightarrow{\mathbb{D}}_{r_2}$ are identical except in a neighborhood of a vertex x_0 where they differ as follows (with $s = |r_1(x_0)|$, $s' = s+1$ and $s'' = s-1$):

(75)

	$\overrightarrow{\mathbb{D}}_{r_1}$	$\overrightarrow{\mathbb{D}}_{r_2}$
when $\epsilon(x_0) = +1$		
when $\epsilon(x_0) = -1$		

Away from a neighborhood of x_0, the foam $F(e)$ is given by $\overrightarrow{\mathbb{D}}_{r_1} \times [0,1]$ and its only non-trivial part is given by

(76) 

if $\epsilon(x_0) = +1$, and 

if $\epsilon(x_0) = -1$.

The graph $G\left(\overrightarrow{\mathbb{D}}\right)$ has webs as vertices and foams as edges. We can apply $\mathcal{E}_N$ to it and obtain a diagram in $\mathbb{Z}_N$-mod. The topological nature of the functor $\mathcal{E}_N$ immediately ensures that all squares commutes. Using the foam evaluation formula one can show that the composition of two composable maps in a given x_0-direction is 0. From this two facts, one obtains that this diagram can be flattened into a chain complex using a Koszul-like sign rule. Adjusting homological and q-grading approriatly, one obtains a bounded chain complex $C\left(\overrightarrow{\mathbb{D}}\right)$ of graded $\mathbb{Z}_N$-modules.

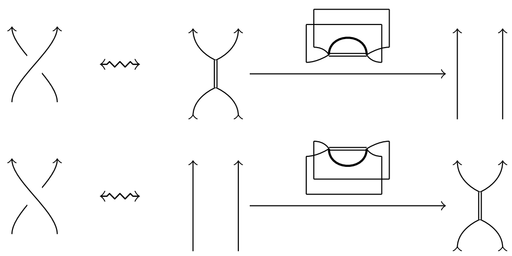
Theorem 3.6.1. *If two link diagrams $\overrightarrow{\mathbb{D}}_1$ and $\overrightarrow{\mathbb{D}}_2$ represent the same framed oriented colored link, then the chain complexes $C\left(\overrightarrow{\mathbb{D}}_1\right)$ and $C\left(\overrightarrow{\mathbb{D}}_2\right)$ are homotopy equivalent. The homotopy type of this complex is therefore a link invariant and so is its homology, we denote $\mathcal{KR}_N\left(\overrightarrow{\mathbb{L}}\right)$ the homology of the framed colored oriented link $\overrightarrow{\mathbb{L}}$.*

History of the proof. The first definition of these link homologies is due to Khovanov–Rozansky [**KR08**] using matrix factorization. Mackaay–Stošić–Vaz gave a foamy version of it using the Kapustin–Li formula [MSV11]. Another algebraico–foamy version was given by Queffelec–Rose [QR16]. The foundations for the proof of invariance (in the uncolored case) are given in [KR08] this was rephrased in [MSV11] in a language that applies mutatis mutandis here. It was reproved in [QR16] where they also explains how to cover the colored case from the uncolored one. □

Remark 3.6.2. We presented a framed version of the theory, however $\mathcal{KR}_N$ can be renormalized (by shifting both homological and q-gradings) to become an invariant of unframed colored links.

When every component of the link is colored with 1, this whole construction is simpler and in particular the complexes associated with crossings are of length 2 and are given by:

(77)



Due to a very clever argument of Bar-Natan, all variants of Khovanov-like homology are known to be projectively functorial: a cobordism between two links induces a well-defined (at least up to a sign) morphisms between their homologies. Moreover, some variations of Khovanov-like homology are known to be especially simple: this is the case for Lee's deformation of Khovanov homology for instance. In the $\mathfrak{gl}_N$ context, the homology theory obtained via the base change $\pi_{\underline{z}}$ for $z = (0, \dots, 0, 1)$, called the *fully deformed* $\mathfrak{gl}_N$*-homology*, has the same property of being (almost) trivial (see [RW16]).

To check if one can get rid of the sign ambiguity, it is enough to check the $\pi_{\underline{z}}$-base changed version of the homology. This fancy strategy was first used by Blanchet [Bla10] in the $\mathfrak{gl}_2$-case and really takes advantage of having an equivariant theory which can be specialized (i.e. base-changed) to the fully-deformed version. Using this strategy, Ehrig–Tubbenhauer–Wedrich proved full-functoriality of Khovanov–Rozansky homology.

Theorem 3.6.3 ([ETW18])**.** *The* $\mathbb{Z}_N$*-equivariant* $\mathfrak{gl}_N$*-link homology is functorial and so are all their versions obtained by base-change.*

7. Symmetries

In this section, we describe an $\mathfrak{sl}_2$-action on $\mathfrak{gl}_N$-state spaces and explain how $\mathfrak{gl}_N$-homology can be adapted to be endowed with such an action. This is detailed in [7, 8]. We can in fact show that a larger Lie algebra (a half of the Witt algebra) acts on $\mathfrak{gl}_N$-state spaces, however this richer action is not yet[3] proven to be transferable to $\mathfrak{gl}_N$-homology. We need to invert 2 in this section. Let $\Bbbk$ be a commutative ring where 2 is invertible and $\Bbbk_N$ the ring $\Bbbk[X_1, \dots, X_N]^{\mathfrak{S}_N}$.

[3]This is a project of my PhD student A. Guérin together with F. Roz, a PhD student at Columbia University.

For us $\mathfrak{sl}_2$ is the $\mathbb{Z}$-Lie algebra generated by $\mathbf{e}, \mathbf{h}$ and $\mathbf{f}$ subject to the following relations:

$$[\mathbf{e}, \mathbf{f}] = \mathbf{h}, \quad [\mathbf{h}, \mathbf{e}] = 2\mathbf{e}, \quad [\mathbf{h}, \mathbf{f}] = -2\mathbf{f}. \tag{78}$$

It can be graded by declaring that $\deg \mathbf{e} = -2, \deg \mathbf{h} = 0$ and $\deg \mathbf{f} = 2$. The Lie algebra $\mathfrak{sl}_2$ acts on the polynomial ring $\mathbb{Z}[Y]$ via the following operators:

$$\mathbf{e} \leftrightsquigarrow -\partial_Y, \qquad \mathbf{h} \leftrightsquigarrow -2Y\partial_Y, \qquad \text{and} \qquad \mathbf{f} \leftrightsquigarrow Y^2\partial_Y. \tag{79}$$

Since $\mathfrak{sl}_2$ is a Lie algebra its category of modules is monoidal and it also acts on $\mathbb{Z}[Y_1, \dots, Y_k] \simeq \mathbb{Z}[Y]^{\otimes k}$ via

$$\mathbf{e} \leftrightsquigarrow \sum_{i=1}^{k} \partial_{Y_i}, \qquad \mathbf{h} \leftrightsquigarrow -2\sum_{i=1}^{k} Y_i\partial_{Y_i}, \qquad \text{and} \qquad \mathbf{f} \leftrightsquigarrow -\sum_{i=1}^{k} Y_i^2\partial_{Y_i}. \tag{80}$$

This action restrict to the ring of symmetric polynomials. If variables in these polynomial rings have degree 2, this action respects the grading.

Note that any foam in $\mathbb{R}^3$ can be isotoped so that it appears as a composition of basic foams, where *basic foams* are traces of isotopies and foams which are a product of a web with an interval expect in a ball where is it is given by one of the models given in Fig. 5. A foam which is a composition of basic foam, is said to be *in good position.*

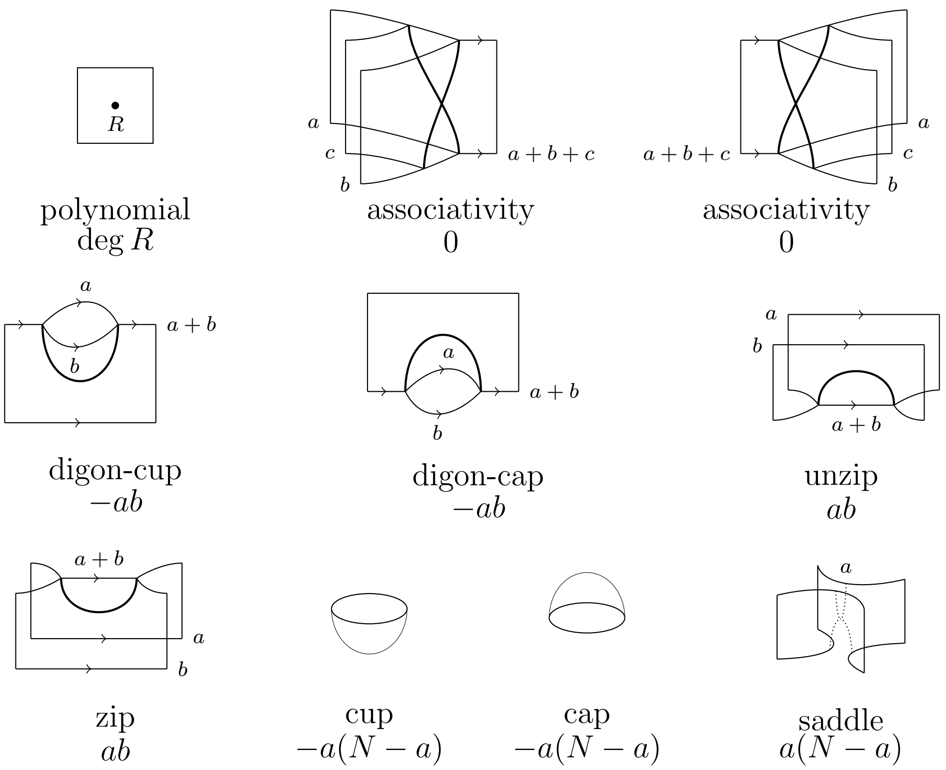
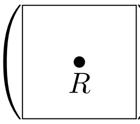
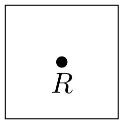


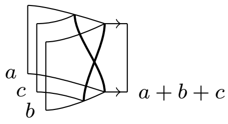
FIGURE 5. The contribution to the degree of a basic foam is given below the name of each of the local models.

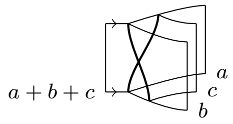
We fix two parameters $\mathsf{t}_1, \mathsf{t}_2$ in $\Bbbk$ and denote $\bar{\mathsf{t}} = (1 - \mathsf{t})$ for $\mathsf{t} \in \Bbbk$. Let Γ_1, Γ_2 be two webs and denote $\widetilde{W}_N(\Gamma_1, \Gamma_2)$ the free $\Bbbk_N$-module generated by foams in $\hom_{\mathsf{Foam}}(\Gamma_1, \Gamma_2)$ in good position. We will define three operators $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{h}$ on

$\widetilde{W}_N(\Gamma_1, \Gamma_2)$. They follow the Leibniz rule with respect to composition of foams. On traces of isotopies they are all equal to 0. The operator $\mathbf{e}$ acts via $-\sum_i \frac{\partial}{\partial x_i}$ on polynomials and by 0 on any other basic foam. The operators $\mathbf{h}$ and $\mathbf{f}$ are defined as follows, where on a facet of thickness a, $\spadesuit_i$ denotes the ith power sum polynomial in a variables, namely $\sum_{k=1}^{a} x_k^i$:

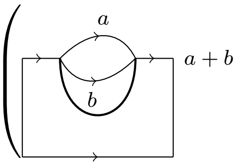
(81) $\mathbf{h}\Big($ 

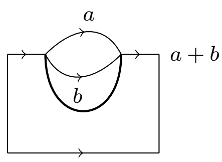
$\Big) = -\deg R \cdot$ 

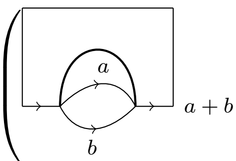
(82) $\mathbf{h}\Big($ 

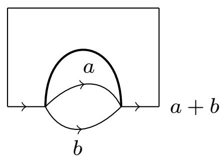
$\Big) = \mathbf{h}\Big($ 

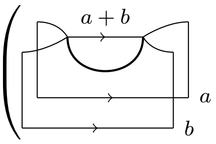
$\Big) = 0$

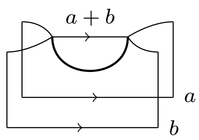
(83) $\mathbf{h}\Big($ 

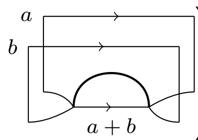
$\Big) = ab(\mathsf{t}_1 + \mathsf{t}_2) \cdot$ 

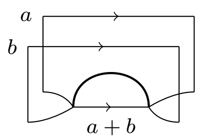
(84) $\mathbf{h}\Big($ 

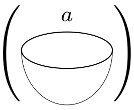
$\Big) = ab(\overline{\mathsf{t}_1} + \overline{\mathsf{t}_2}) \cdot$ 

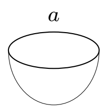
(85) $\mathbf{h}\Big($ 

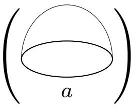
$\Big) = -ab(\overline{\mathsf{t}_1} + \overline{\mathsf{t}_2}) \cdot$ 

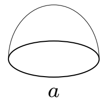
(86) $\mathbf{h}\Big($ 

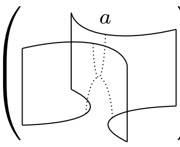
$\Big) = -ab(\mathsf{t}_1 + \mathsf{t}_2) \cdot$ 

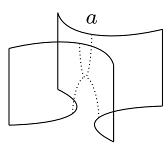
(87) $\mathbf{h}\Big($ 

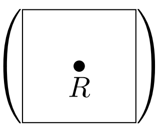
$\Big) = a(N - a) \cdot$ 

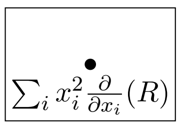
(88) $\mathbf{h}\Big($ 

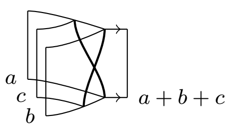
$\Big) = a(N - a) \cdot$ 

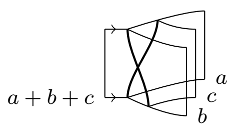
(89) $\mathbf{h}\Big($ 

$\Big) = -a(N - a) \cdot$ 

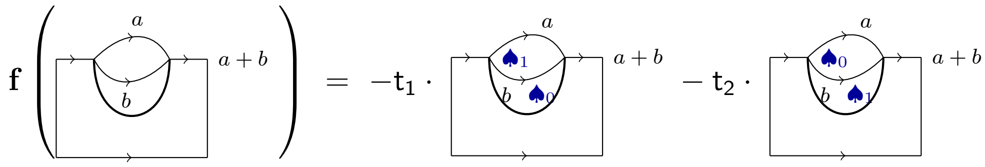
(90) $\mathbf{f}\Big($ 

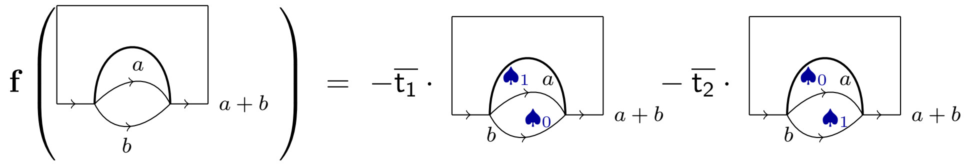
$\Big) =$ 

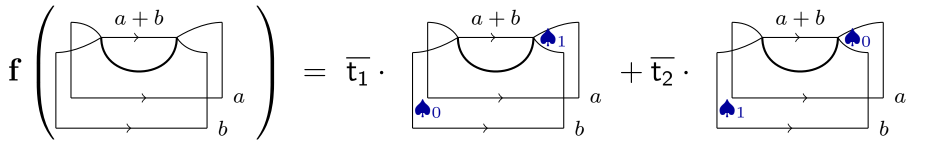
(91) $\mathbf{f}\Big($ 

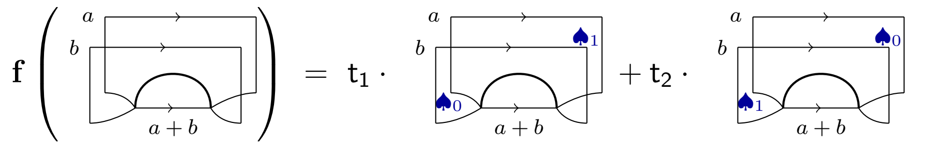
$\Big) = \mathbf{f}\Big($ 

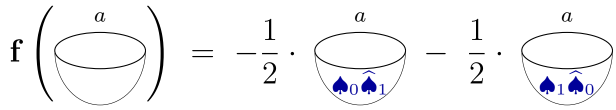
$\Big) = 0$

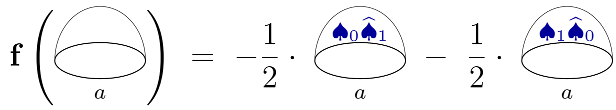
(92) 

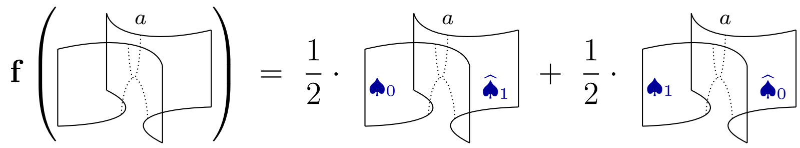
(93) 

(94) 

(95) 

(96) 

(97) 

(98) 

The action of $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{h}$ were designed so that the following proposition holds:

Proposition 3.7.1. *Let F be a closed foam (in $\mathbb{R}^3$) in good position, then for all $g \in \{\mathbf{e}, \mathbf{f}, \mathbf{h}\}$,*

$$\tau_N(g \cdot F) = g \cdot \tau_N(F).$$

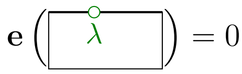
From this, one obtains:

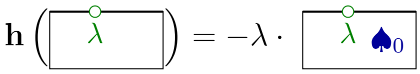
Corollary 3.7.2. *The operators $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{h}$ induce an $\mathfrak{sl}_2$-action on the $\mathfrak{gl}_N$-state space $\mathcal{E}_N(\Gamma; \Bbbk_N)$ for any web Γ.*

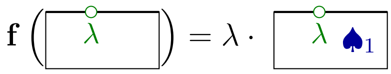
Since $\mathfrak{sl}_2$ acts on foams by the Leibniz rule, a necessary condition for a foam F to induce an $\mathfrak{sl}_2$-module map between $\mathfrak{gl}_N$-state spaces is the action of $\mathfrak{sl}_2$ on F to be trivial. This is often not the case (otherwise our $\mathfrak{sl}_2$-action would not be very interesting). However we can fix that at the cost of twisting the $\mathfrak{sl}_2$-action on state spaces. In order to encode the twist we need to add extra information on webs. This is why we introduce green-dotted webs.

Definition 3.7.3. A *green-dotted* web is a web Γ endowed with a finite collection D of *green dots*, that are marked points with multiplicities (in $\Bbbk$) located in the interior of edges of Γ. These green dots are depicted on webs by the following symbol: $\circ$. If a given edge carries several green dots, they may be replaced by one green dot on this edge with the sum of all multiplicities.

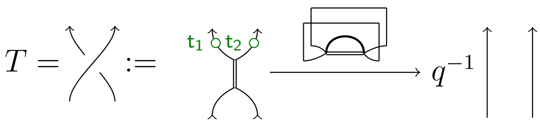
If a web Γ has green dot, the $\mathfrak{sl}_2$ action on a state space associated with Γ is modified as follows (keeping in mind that the Leibniz rule still applies):

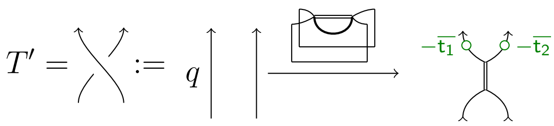
(99)


(100)


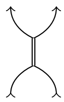
(101)


We now explain how to modify the construction of exterior $\mathfrak{gl}_N$-homology in the uncolored[4] case.

(102)


(103)


where in both complexes we assume (as in [20]) that the term



sits in cohomological degree 0. In these diagrams $\mathcal{E}_N$ has been omitted to maintain readability.

The green dots in the diagrams defining T and T' ensure that the morphism is $\mathfrak{sl}_2$-equivariant. Moreover the proof of invariance can be made $\mathfrak{sl}_2$-compatible.

For a link L, define $\mathcal{KR}_N^{\mathfrak{sl}_2}(L;R) := \mathcal{KR}_{N;\mathtt{t}_1,\mathtt{t}_2}^{\mathfrak{sl}_2}(L;R)$ to be the Khovanov–Rozansky $\mathfrak{gl}_N$-homology of L with coefficients in a ring R, equipped with the action of the Hopf algebra $\mathcal{U}(\mathfrak{sl}_2)$. When the coefficient ring R is clear from context we will also write $\mathcal{KR}_N^{\mathfrak{sl}_2}(L)$ for simplicity.

Theorem 3.7.4. *The homology $\mathcal{KR}_{N;\mathtt{t}_1;\mathtt{t}_2}^{\mathfrak{sl}_2}(L;\mathbb{r}_N)$ is an invariant of framed oriented links.*

When cobordisms have non-trivial topology then $\mathfrak{sl}_2$ acts on them non-trivially preventing them to induce $\mathfrak{sl}_2$-equivariant maps on $\mathfrak{gl}_N$-homology, however if one restricts to link concordance then one obtains $\mathfrak{sl}_2$-equivariance.

Let Links_{Con} be the category whose objects are framed links and whose morphisms are link concordances. Recall that a link concordance is a link cobordism which is an embedding of cylinders inside $\mathbb{R}^3 \times [0,1]$ with boundary components the given links (each cylinder having boundary components in both $\mathbb{R}^3 \times \{0\}$ and $\mathbb{R}^3 \times \{1\}$). The Euler characteristic of a link concordance is always zero.

Let $\mathcal{C}^{\mathfrak{sl}_2}(\mathbb{k}_N)$ be the homotopy category of finite complexes of $\mathbb{k}_N$-modules endowed with an extra $\mathfrak{sl}_2$-action but where we don't require the homotopies to intertwine the $\mathfrak{sl}_2$-action (it is called the $\mathfrak{sl}_2$-relative homotopy category, see [8]).

[4]The colored case is a future project for my PhD student A. Guérin.

Corollary 3.7.5. *The functor $\mathcal{KR}_N$ from restricts to a functor*

$$\mathcal{KR}_N \colon \mathsf{Links}_{Con} \longrightarrow \mathcal{C}^{\mathfrak{sl}_2}(\mathbb{k}_N).$$

A concordance C is *ribbon* if the projection to the $[0,1]$ factor restricts to a Morse function on C with only index 0 and 1 critical points. Using a result of Zemke on maps induced by ribbon concordances [**Zem19**], we obtain:

Corollary 3.7.6. *If C is a ribbon concordance from L_0 to L_1, then $\mathcal{KR}_N^{\mathfrak{sl}_2}(L_0; \mathbb{r}_N)$ is a direct summand of $\mathcal{KR}_N^{\mathfrak{sl}_2}(L_1; \mathbb{r}_N)$ as $\mathfrak{sl}_2$-module.*

For the remaining of this chapter, we assume the link L is a knot and we restrict to the case $N = 2$ and set $\mathbb{r} = \mathbb{Q}$ sor that $\mathbb{r}_N = \mathbb{r}_2 = \mathbb{Q}[E_1, E_2]$, with $E_1 = X_1 + X_2$ and $E_2 = X_1 X_2$.

Fixing a base point p on L, we endow the homology $\mathcal{KR}_2^{\mathfrak{sl}_2}(L; R)$ with the action of the homology of the unknot

$$A = \mathbb{Q}[E_1, E_2][x]/(x^2 - E_1 x + E_2)$$

by placing an unknot near the base point of L and then merging the unknot to the link at the base point. The homology of L then acquires an action of $\mathbb{Q}[x]$.

For a knot L, denote by $\mathcal{KR}_N^{\mathfrak{sl}_2,\mathrm{tor}}(L; \mathbb{r}_2)$ the torsion elements of its homology with respect to the $\mathbb{Q}[x]$-action. Let $\mathcal{KR}_N^{\mathfrak{sl}_2,\mathrm{free}}(L; R)$ be the free part with respect to this action.

Proposition 3.7.7. *The space $\mathcal{KR}_2^{\mathfrak{sl}_2,\mathrm{tor}}(L; R)$ is an $\mathfrak{sl}_2$-subrepresentation.*

This immediately implies the following result.

Corollary 3.7.8. *The subspace $\mathcal{KR}_2^{\mathfrak{sl}_2,\mathrm{free}}(L; R)$ is an $\mathfrak{sl}_2$-quotient representation of the entire homology.*

We are now able to record the relationship between the Rasmussen invariant of a link and $\mathfrak{sl}_2$-structure on its equivariant Khovanov homology.

Corollary 3.7.9. *Let L be a knot and let $\mu(L)$ be the highest weight of $\mathcal{KR}_2^{\mathfrak{sl}_2,\mathrm{free}}(L; R)$. If $\mathsf{t}_1 + \mathsf{t}_2 = 1$, then $s(L) = \mu(L) - 1$.*

CHAPTER 4

Symmetric link homology

The first aim of this chapter is to describe a categorification of the symmetric MOY calculus at least in the case of directed webs in an annulus. Then we explain how to derive from that a link homology theory categorifying colored $\mathfrak{gl}_N$-link invariant associated with symmetric power of the vector representation of $U_q(\mathfrak{gl}_N)$. This is based on [**3, 4, 6**].

Note that the first symmetric power is just like the first exterior power: both are the vector representation. However in the case of links colored by this representation, the homology described is this chapter is substantially different from the one described in Section 6 of Chapter 3.

1. Annular calculus

Let $\mathbb{A}$ be the annulus $\{(x, y) \in \mathbb{R}^2 | 1 \leq \sqrt{x^2 + y^2} \leq 2\}$ endowed with the vector field $x\partial_y - y\partial_x$. A *vinyl graph* is a closed directed web in $\mathbb{A}$. The *index* of a vinyl graph is the sum of the thicknesses of the edges which intersect a ray r (the flow condition ensures that it is well-defined). An example of a vinyl graph is given in Fig. 1. A vinyl graph Γ is *thin* if the thicknesses of every edge of Γ is either 1 or 2.

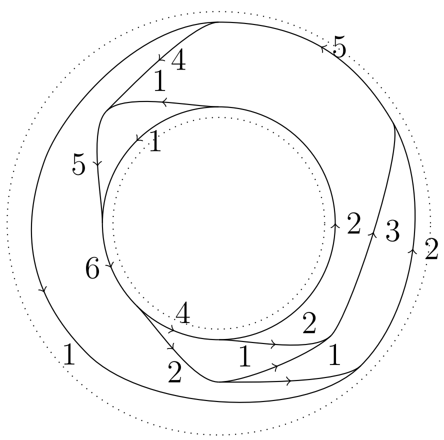


FIGURE 1. Example of a vinyl graph of index 7.

A peculiar class of examples of vinyl graph is given by concentric disjoint union of circles of various thicknesses drawn in the annulus. If $\underline{k} = (k_1, \ldots, k_\ell)$ is an ℓ-tuple of positive integer, $\mathbb{S}_{\underline{k}}$ denotes such a collection of circles (with k_1 the thickness of the innermost circle and k_ℓ that of the outermost circle). We identify a positive integer k with the 1-tuple (k), so that $\mathbb{S}_k$ denotes a single circle of thickness k.

A *braid-like web* is a directed web in $[0, 1] \times [0, 1]$ endowed with the vector field ∂_y, with boundary included in $[0, 1] \times \{0, 1\}$. If the thicknesses and the location of the boundary points in $\{0\}$ and $\{1\}$ agree, one can close up a braid-like web to obtain a vinyl graph. This procedure is described in Fig. 2.

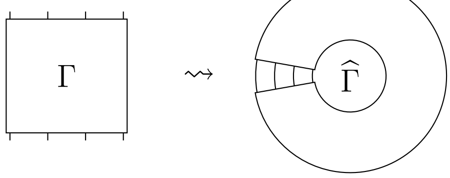
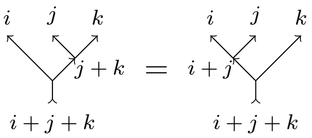
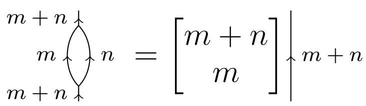
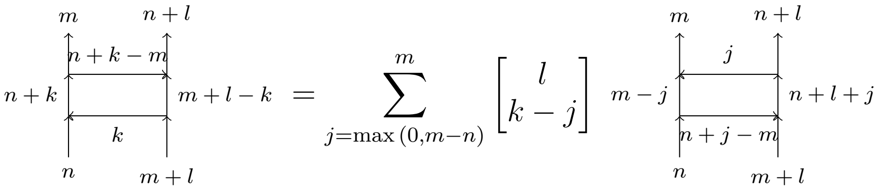


FIGURE 2. Closing a braid-like web into a vinyl graph.

Consider the following set of relations between $\mathbb{Z}[q, q^{-1}]$-linear combinations of vinyl graphs (and their mirror images):

(104)


(105)


(106)


For any positive integer k, one can consider the skein module over $\mathbb{Z}[q, q^{-1}]$ generated by vinyl graphs of index k modulo relations (104)–(106), we denote it by $\mathrm{skein}_q(\mathbb{A}, k)$.

Proposition 4.1.1 ([**QR16**]). *The* $\mathbb{Z}[q, q^{-1}]$*-module* $\mathrm{skein}_q(\mathbb{A}, k)$ *is free generated by* $\mathbb{S}_{\underline{k}}$ *for* $\underline{k} = (k_1, \dots k_\ell)$ *with* $k_1 \leq \dots \leq k_\ell$ *and* $\sum_{i=1}^{\ell} k_i = k$.

The fact that the family $(\mathbb{S}_{\underline{k}})_{\underline{k}}$ is free is not difficult to prove. The proof of this results is algorithmic: Queffelec and Rose explain how one can inductively use relations (104)–(106) (and their mirror images) to reduce any vinyl graph until one reaches collection of circles. It is then easy to show that if $\underline{k}$ and $\underline{k'}$ are equal up to reordering, then $\mathbb{S}_{\underline{k}}$ and $\mathbb{S}_{\underline{k'}}$ are equal in $\mathrm{skein}_q(\mathbb{A}, k)$. Based on computational evidence, they conjecture the following:

Conjecture 4.1.2. *Let* Γ *be a vinyl graph of index* k *let* $\lambda_{\underline{k}}$ *be the elements of* $\mathbb{Z}[q, q^{-1}]$ *so that*

$$\Gamma = \sum \lambda_{\underline{k}} \mathbb{S}_{\underline{k}} \tag{107}$$

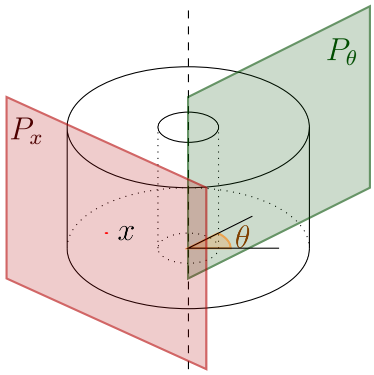
in $\mathrm{skein}_q(\mathbb{A}, k)$*, then* $\lambda_{\underline{k}} \in \mathbb{N}[q, q^{-1}]$.

It turns out that this conjecture is implied by to the so-called Stanley–Stembridge conjecture [**SS93**] which states that the chromatic function of unit interval graphs can be positively and "unimodularly" written in terms of E-polynomials. Part of this conjecture have just been proven [**Hik24**].

For some of the coefficients, one could actually show (before [Hik24]) that they belong to $\mathbb{N}[q, q^{-1}]$ for all vinyl graphs. This is the case for $\lambda_{(k)}$.

2. Symmetric evaluation

We will consider foam in the thickened annulus $\mathbb{A} \times [0,1]$. If $x := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is an element of $\mathbb{A} \times [0,1]$, we denote by t_x the vector $\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$, by v the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and by P_x the plane spanned by t_x and v. If θ is an element of $[0, 2\pi[$, P_θ is the half-plane $\left\{ \begin{pmatrix} \rho \cos\theta \\ \rho \sin\theta \\ t \end{pmatrix} \middle| (\rho, t) \in \mathbb{R}_+ \times \mathbb{R} \right\}$. Planes parallel to $\mathbb{R} \times \mathbb{R} \times \{0\}$ are called *horizontal*.



Definition 4.2.1. Let k be a non-negative integer and Γ_0 and Γ_1 two vinyl graphs. A *vinyl foam from* Γ_0 *to* Γ_1 is a foam F with boundary embedded in $\mathbb{A} \times [0,1]$ such that:

- $F \cap (\mathbb{A} \times \{0\}) = -\Gamma_0$ and $F \cap (\mathbb{A} \times \{1\}) = \Gamma_1$.
- For every point x of F, the normal line to F at x is *not* contained in P_x.

See Fig. 3 for an example.

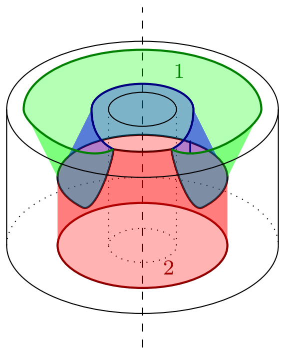


FIGURE 3. An example of a vinyl foam

Remark 4.2.2. If F is a vinyl foam from Γ_0 to Γ_1 then necessarily, these two vinyl graphs have the same index.

Vinyl graphs of index k and vinyl foam between them (up to ambient isotopy) form a category that we denote Vinyl_k. The disjoint union of theses categories (for $k \in \mathbb{N}$) is also a category (denoted Vinyl_∞) and can be endowed with a monoidal structure given by concentric disjoint union.

By essence, for $k \geq 1$, there is no closed foam in the category Vinyl_k, so that for having a foam evaluation one needs to have a replacement for closed foam, this role is played by vinyl foams from $\mathbb{S}_k$ to $\mathbb{S}_k$. If F is such a vinyl foam, denotes $\widehat{F}$, the (non vinyl) closed foam obtained by capping and cupping it with two disks of thickness k. The evaluation of F will be based on the exterior $\mathfrak{gl}_k$-evaluation of $\widehat{F}$. Before we can give the formula of we need to define a some maps between in polynomial rings. We actually need to work over $\mathbb{Q}$, and denote $\mathbb{Q}_N$ the ring $\mathbb{Q}[T_1, \dots, T_N]^{\mathfrak{S}_N}$.

Notation 4.2.3. The set of Young diagrams with at most a rows and at most b columns is denoted by $T(a,b)$ and the set of Young diagrams with at most a rows is denoted by $T(a,\infty)$. The rectangular Young diagram with a rows and b columns is denoted by $\rho(a,b)$.

Denote the graded algebra $\mathbb{Q}_N[x_1, \dots, x_k]^{\mathfrak{S}_k}$ by A_k and by $J_{N,k}$ the ideal of the algebra $\mathbb{Q}_N[x_1, \dots, x_k]$ generated by

$$\left\{ \prod_{i=1}^{N} (x_j - T_i) \middle| j = 1, \dots, k \right\}. \tag{108}$$

Note that elements of this set of generators are indeed symmetric in the $T_\bullet$. Denote by $M_{N,k}$ the $\mathbb{Q}_N$-algebra

$$A_k \,/ (J_{N,k} \cap A_k) \ ,$$

seen as a $\mathbb{Q}_N$-module. The indeterminates $x_\bullet$ have degree 2, just like the indeterminates $T_\bullet$ appearing in the definition of $\mathbb{Q}_N$. Note that this ideal is graded so that $M_{N,k}$ is graded.

If $\lambda = (\lambda_1, \dots, \lambda_k)$ is a Young diagram with at most k rows, define $\mathbf{x}^\lambda := \prod_{i=1}^k x_i^{\lambda_i}$. Denote by $m_\lambda(x_1, \dots, x_k)$ the symmetric polynomial $\sum_{\lambda'} \mathbf{x}^{\lambda'}$, where λ' runs over all *distinct* permutations of λ. Denote by $\tilde{m}_\lambda(x_1, \dots, x_k)$ the symmetric polynomial $\sum_{\lambda'} \mathbf{x}^{\lambda'}$, where λ' runs over *all* permutations of λ. The family $(m_\lambda)_{\lambda \in T(k,\infty)}$ is a $\mathbb{Z}$-basis of the ring of symmetric polynomials in k variables with coefficients in $\mathbb{Z}$ (see [Mac15]). The family $(\tilde{m}_\lambda)_{\lambda \in T(k,\infty)}$ is a $\mathbb{Q}$-basis of the ring of symmetric polynomials in k variables with coefficients in $\mathbb{Q}$.

Lemma 4.2.4. *The $\mathbb{Q}_N$-module $M_{N,k}$ is free and has a basis given by images in $M_{N,k}$ of*

$$(m_\lambda(x_1, \dots, x_k))_{\lambda \in T(k, N-1)}$$

seen as element of A_k.

Denote $\epsilon_{N,k}$ the following morphism of $\mathbb{Q}_N$-modules:

$$\begin{array}{rccl} \epsilon_{N,k}: & M_{N,k} & \to & \mathbb{Q}_N \\ & m_\lambda & \mapsto & \begin{cases} 1 & \text{if } \lambda = \rho(k, N-1), \\ 0 & \text{if } \lambda \neq \rho(k, N-1), \end{cases} \end{array}$$

Proposition 4.2.5. *The $\mathbb{Q}_N$-linear map $\epsilon_{N,k}$ endows the $\mathbb{Q}_N$-algebra $M_{N,k}$ with a structure of symmetric algebra. In particular $M_{N,k}$ is a commutative Frobenius algebra.*

Denote $\Upsilon_{N,k}$ the composition of the projection from A_k to $M_{N,k}$ with $\epsilon_{N,k}$.

Definition 4.2.6. The *equivariant $\mathfrak{gl}_N$-symmetric evaluation* of a vinyl foam F from $\mathbb{S}_k$ to $\mathbb{S}_k$ is given by:

$$\upsilon_N(F) := \Upsilon_{N,k}\left(\tau_k\left(\widehat{F}\right)\right) \in \mathbb{Q}_N,$$

where $\tau_k\left(\widehat{F}\right)$ denotes the (exterior) $\mathfrak{gl}_k$-evaluation of the closed foam $\widehat{F}$.

Note that the definition of $\upsilon_N(F)$ depends on the index k of F, we intentionally removed the dependency on k in the notations of $\upsilon_N(F)$ because eventually, we will use this evaluation for all positive integer k.

We are now almost ready to apply the universal construction again. Let Γ be a vinyl graph in $\mathbb{A}$ of index k. Define

$$\widetilde{W}_N(\Gamma) := \bigoplus_{F \in \hom_{\mathsf{Vinyl}_k}(\mathbb{S}_k, \Gamma)} q^{\deg_N(F) - k(N-1)} \mathbb{Q}_N. \tag{109}$$

As in Section 4, this construction immediately extends to a functor from Vinyl_k to $\mathbb{Q}_N$-Mod

On each $\widetilde{W}_N(\Gamma)$, define a $\mathbb{Q}_N$-bilinear form $((\bullet;\bullet))_N$ by setting for any two foams F and G in $\hom_{\mathsf{Vinyl}_k}(\mathbb{S}_k, \Gamma)$:

$$((F;G))_N := \upsilon_N\left(\overline{F} \circ G\right) \tag{110}$$

where $\overline{F}$ is the foam in $\hom_{\mathsf{Vinyl}_k}(\Gamma, \mathbb{S}_k)$ given by taking the mirror image of $F \subset \mathbb{A} \times [0,1]$ with respect to $\mathbb{A} \times \{1/2\}$. Denote $L_N(\Gamma)$ the kernel (or radical) of this bilinear form and set $\mathcal{S}_N(\Gamma) = \widetilde{W}_N(\Gamma)/L_N(\Gamma)$. The $\mathbb{Q}_N$-module $\mathcal{S}_N(\Gamma)$ is called the *symmetric (equivariant) $\mathfrak{gl}_N$-state space of* Γ. As for the exterior case, this extends to a functor $\mathcal{S}_N$ from Vinyl_∞ to $\mathbb{Q}_N$-Mod.

Theorem 4.2.7. *The functor $\mathcal{S}_N$ is monoidal and for any vinyl graph Γ, the $\mathbb{Q}_N$-module $\mathcal{S}_N(\Gamma)$ is free of graded rank equal to $\langle\!\langle \Gamma \rangle\!\rangle_N$.*

The vinyl constraint we put on foams simplify considerably the topology of the construction. In order to express this simplicity we need the following definition.

Definition 4.2.8. Let Γ be a vinyl graph of index k. A vinyl foam F from $\mathbb{S}_k$ to Γ is *tree-like*, if for any $\theta \in [0, 2\pi]$, the intersection of F, with the half-plane P_θ is a tree and all non-trivial decorations of F are located on facets which intersects Γ.

It is actually quite easy to construct a tree-like foam from $\mathbb{S}_k$ to Γ, moreover except from their decorations, there are all equal in $\mathcal{S}_N(\Gamma)$, as the following proposition states, so that their topologies do not really matter as far as $\mathcal{S}_N$ is concerned.

Proposition 4.2.9. *Let Γ be a vinyl graph of index k, if two tree-like foams F and F' agree on their decorations (which can be compared since they are located on facets touching Γ), then they are equal in $\mathcal{S}_N(\Gamma)$.*

Finally, tree-like foams are enough to understand the $\mathcal{S}_N(\Gamma)$:

Proposition 4.2.10. *For any vinyl graph Γ, the symmetric $\mathfrak{gl}_N$ state space $\mathcal{S}_N(\Gamma)$ is spanned by tree-like webs.*

From Proposition 4.2.9 and Proposition 4.2.10 one can deduce a formulation of the functor $\mathcal{S}_N$ in a foam-free language. This is detailed in [4] and [**Mar23a**]. Even if one can get rid of the topology of foams, finding basis of the symmetric $\mathfrak{gl}_N$-state spaces is in general quite difficult (and this causes a problem for computing symmetric link homology that we will define in the next section). My student L. Marino solves this problem in the case $N = 1$ for thin vinyl graphs [Mar23a].

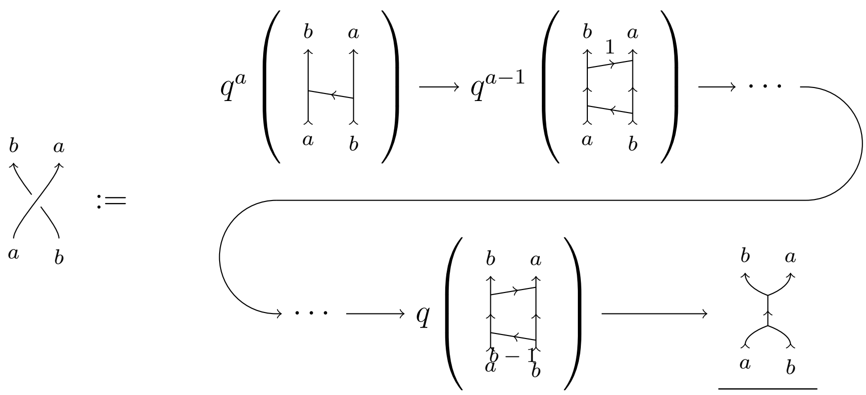
As in the exterior case, if R is a ring and $\varphi\colon \mathbb{Q}_N \to R$ is a ring morphisms, then one can perform a base change and get state spaces which are free R-module. In the case where $R = \mathbb{Q}$ and φ maps the variables $T_1, \dots, T_N$ to 0, this base change is graded and we speak about *non-equivariant symmetric* $\mathfrak{gl}_N$*-state spaces.*

3. Symmetric link homology

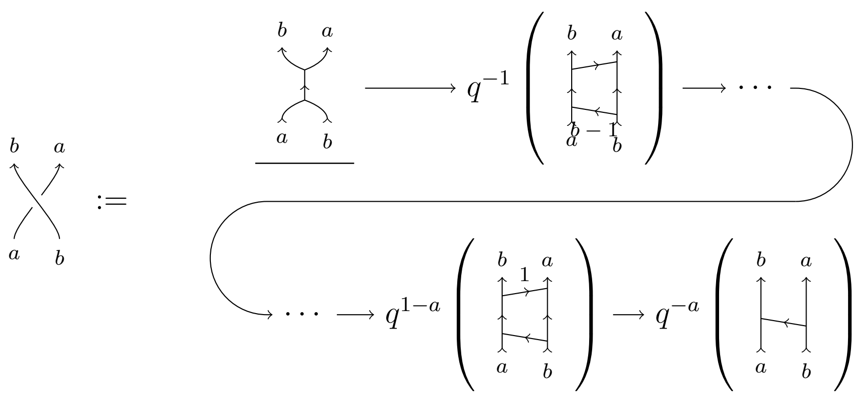
The construction of symmetric link homology appeared independently in three different papers [Cau17, QRS18, 3]. In this document, we follow [3], it is the only one which is equivariant. In order to construct a link homology out of $\mathcal{S}_N$, one uses the same strategy as for exterior homology: namely we replace crossings by complexes of vinyl graphs. The diagrams used to defined these complexes are extremly close from that for exterior homology. There are two notable difference though:

- Both homological and q-grading are different (and positive and negative crossings are swapped).
- Since we work with vinyl graphs, the only link diagrams we consider are braid closures.

Consider $\overrightarrow{\mathbb{D}}$ a colored link diagram given as closure of a (colored) braid diagram. As for the exterior case, we construct an hyper-rectangle of vinyl graphs and vinyl foams using the following local rules (for $a \leq b$):

(111)


and

(112)


If $a > b$, the complexes are the same, only the last (or first) diagrams need to be changed: the rung point right instead of left.

Applying the functor $\mathcal{S}_N$ to the hyperrectangle and flattening it produces a finitely generated chain complex of $\mathbb{Q}_N$-modules $\mathrm{CS}_N(\mathbb{B})$.

Theorem 4.3.1. *If $\mathbb{B}_1$ and $\mathbb{B}_2$ are two colored braid closure diagram representing the same colored framed link L, then the homology of $\mathrm{CS}_N(\mathbb{B}_1)$ and $\mathrm{CS}_N(\mathbb{B}_2)$ are isomorphic and is denoted $\mathrm{HS}_N(L)$. Their (common) Euler characteristic is equal to $\langle\!\langle L \rangle\!\rangle_N$.*

Pitch of the proof. The statement on the Euler characteristic follows from the very definition of CS_N, once one has noticed that the definitions of the complexes associated with crossings are lifts of the definitions of $\langle\!\langle \bullet \rangle\!\rangle_N$ on crossings (26) and (27).

On the one hand, the proof of invariance for braid relations is relatively standard and that for the first Markov move (trace-like property) is trivial. On the other hand, the proof of invariance by stabilization is quite convoluted. So far we do not have an elementary proof. The proof relies on an algebraic description of the functor $\mathcal{S}_N$: one puts a differential d_{-N} inspired from the work of [**Cau17**] on the Hochschild homology on (singular) Soergel bimodules associated with vinyl graph[1] It turns out that the homology with respect to this differential is concentrated in Hochshild degree 0 and is isomorphic to the symmetric $\mathfrak{gl}_N$ state space. One uses invariance of the triply graded homology (constructed using Soergel bimodules) and the compatibility of the invariance with the differential d_{-N} to deduce invariance of $\mathrm{HS}_N(L)$. □

As a byproduct of this convoluted proof, we obtain

Proposition 4.3.2. *There is a spectral sequence from the (unreduced) triply graded homology to the symmetric $\mathfrak{gl}_N$ link homology.*

The case $N = 1$ is peculiar, indeed in that case, the polynomial invariant is always trivial. However, the symmetric link $\mathfrak{gl}_1$-homology is far from being trivial. L. Marino wrote a program [**Mar23b**] to compute it. Based on the results of her program on small knots, she conjectures that it has the same rank than the reduced triply graded homology (in the uncolored case) [**Mar23a**].

4. $\mathfrak{gl}_0$-homology

Revall that a vinyl graph is *thin* if the thicknesses of its edges are included in $\{1, 2\}$. The resolutions of uncolored braid closure diagram are all thin.

Definition 4.4.1. A *pointed* thin vinyl graph, is a thin vinyl graph Γ together with a base point $\star$ located in the interior of an edge of thickness 1, which itself is innermost.

If $(\Gamma, \star)$ is a pointed vinyl graph of index k, one can show that the coefficient $\lambda_{(k)}$ of $\mathbb{S}_k$ in its expansion in collection of circles (see Proposition 4.1.1) is divisible by $[k]$. The Laurent polynomial $\lambda_{(k)}/[k]$ has non-negative coefficients and is denoted $\langle\!\langle \Gamma \rangle\!\rangle'_0$.

[1] Soergel bimodules are associated with braid-like webs, but their Hochschild homology only depends on their closure.

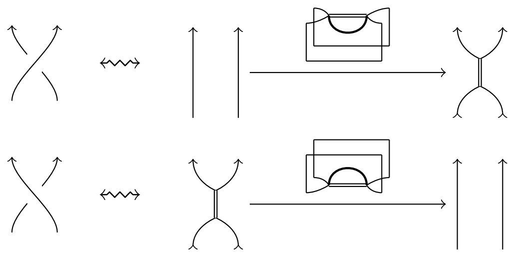
If $(\Gamma, \star)$ is a pointed thin vinyl graph of index k, denote $G_{\Gamma,\star}$ the vinyl foam from Γ to Γ, which is $\Gamma \times [0,1]$ with trivial decoration everywhere but on the facet corresponding to $\star$ where the decoration is the one-variable (symmetric) polynomial x^{k-1}. This foam induces an endormorphism of the non-equivariant symmetric $\mathfrak{gl}_1$-state space that we denote $g_{\Gamma,\star}$.

The $\mathfrak{gl}_0$-*state space* associated with $(\Gamma, \star)$ is the graded q-vector space $q^{1-k}\mathrm{Im}(g_{\Gamma,\star})$. It is denoted $\mathcal{S}_0'(\Gamma, \star)$. Because $\mathcal{S}_1(\Gamma)$ is defined via universal construction, $\mathcal{S}_0'(\Gamma, \star)$ can also be seen as a (q-shifted) quotient of $\mathcal{S}_1(\Gamma)$.

Proposition 4.4.2. *For any pointed thin vinyl graph $(\Gamma, \star)$, $\mathcal{S}_0'(\Gamma, \star)$ is graded vector space of dimension $\langle\!\langle \Gamma \rangle\!\rangle_0'$.*

Let $(\mathbb{B}, \star)$ be a (uncolored) braid diagram closure with a base point on an innermost edge. We can apply the same hypercube as for the symmetric homology (it is a cube and not a rectangle in this case, since all crossing corresponds to complexes of length 2). Namely replace crossings using the following complexes:

(113)



Applying the functor $\mathcal{S}_0'$ to the hypercube and flattening it produces a finite dimensional chain complex of graded $\mathbb{Q}$-vector space $\mathrm{C}_{\mathfrak{gl}_0}(\mathbb{B}, \star)$.

Theorem 4.4.3. *If $(\mathbb{B}_1, \star_1)$ and $(\mathbb{B}_2, \star_2)$ are two pointed (uncolored) braid closure diagram with the basepoint on a innermost edge such that $\mathbb{B}_1$ and $\mathbb{B}_2$ represent the same knot K, then the homology of $\mathrm{C}_{\mathfrak{gl}_0}(\mathbb{B}_1, \star_1)$ and $\mathrm{C}_{\mathfrak{gl}_0}(\mathbb{B}_2, \star_2)$ are isomorphic and is denoted $\mathrm{H}_{\mathfrak{gl}_0}(K)$. Their (common) Euler characteristic is equal to the Alexander polynomial of K.*

Pitch of the proof. As usual, invariance under braid relation is standard. Invariance under the first Markov move is almost trivial. Unlike for the symmetric homology, invariance under stabilization is actually rather easy because for any resolution which is not connected the corresponding $\mathfrak{gl}_0$-state space is equal to $\{0\}$. The real challenge is the invariance under change of base point, which follow from a quite delicate description of $\mathfrak{gl}_0$-state spaces. □

Although relating the $\mathfrak{gl}_0$-state spaces with Soergel bimodules is not needed for the proof of invariance, using the same techniques as for the symmetric homology, we obtain:

Proposition 4.4.4. *There is a spectral sequence from the reduced triply graded homology to the $\mathfrak{gl}_0$-homology.*

There is actually a more surprising spectral sequence which relate the $\mathfrak{gl}_0$-homology to the "classical" categorification of the Alexander polynomial.

Theorem 4.4.5. *There is a spectral sequence from the $\mathfrak{gl}_0$-knot homology to the knot Floer homology $\widehat{\mathrm{HFK}}$.*

Pitch of the proof. The spectral sequence is actually a spectral sequence of coefficients and is therefore less conceptual than one could have hoped. This relies on a reformulation of Heegaard–Floer homology in terms of a twisted hypercube of resolution developed by Gilmore [**Gil16**] and Manolescu [**Man14**] where the base ring is $\mathbb{Z}_2[t^{-1}, t]]$ (where t is a twisting parameter). One first challenge is to promote their results in characteristic zero, which boils down to dealing with signs and therefore orientations of moduli spaces. Once this is done, one can work over the ring $\mathbb{Z}[t^{-1}, t]]$. The second challenge is to get rid of the deformation parameter t (i.e. setting $t = 1$). However, the fact that $1 - t$ is not a zero divisor is important in Gilmore–Manolescu's approach. This two constraints seem pretty much incompatible. For overcoming this difficulty, we use a pseudo-completion technique: one can promote any $\mathbb{Z}[t^{-1}, t]$-module M to a $\mathbb{Z}[t^{-1}, t]]$-module M' by extending the scalars. There is then a natural map $\phi\colon M \to M'$ and this map may not be injective. One can therefore mod out by the kernel of ϕ and get $\mathbb{Z}[t^{-1}, t]$-module $\widetilde{M}$. One can set $t = 1$ in $\widetilde{M}$. Applying this technique to the twisted complex of Gilmore–Manolescu (whose homology is isomorphic to knot Floer homology), one can relate it to the hypercube providing the $\mathfrak{gl}_0$-homology. Indeed one can prove that knot Floer homology is equal to the last page of a Bockstein-like spectral sequence with $\mathfrak{gl}_0$-homology on the second page. □

From Proposition 4.4.4 and Theorem 4.4.5 and since spectral sequences can be composed, one obtains the following corollary which established Dunfield–Gukov–Rasmussen conjecture [**DGR06**], in an admittedly deceptive way since the spectral sequence obtained is seems pretty much artificial.

Corollary 4.4.6. *There is a spectral sequence from the reduced triply grade homology (for knots) homology to the knot Floer homology* $\widehat{\mathrm{HFK}}$.

However, we can transfer some detection result known for knot Floer homology to $\mathfrak{gl}_0$-homology and reduced triply graded homology:

Corollary 4.4.7. *The reduced triply graded homology and the* $\mathfrak{gl}_0$*-homology detect the unknot, the two trefoils, the figure-eight knot, and the cinquefoil.*

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