Des mousses et des Homs

Louis-Hadrien Robert

lrobert.perso.math.cnrs.fr/HDR/soutenance.pdf

Slides

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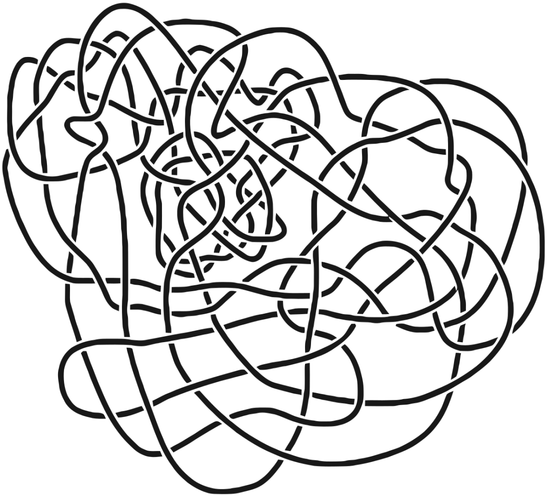
Thesis

12/12/24

Knot theory: a biased overview $K : \mathbb{S}^1 \hookrightarrow \mathbb{R}^3$



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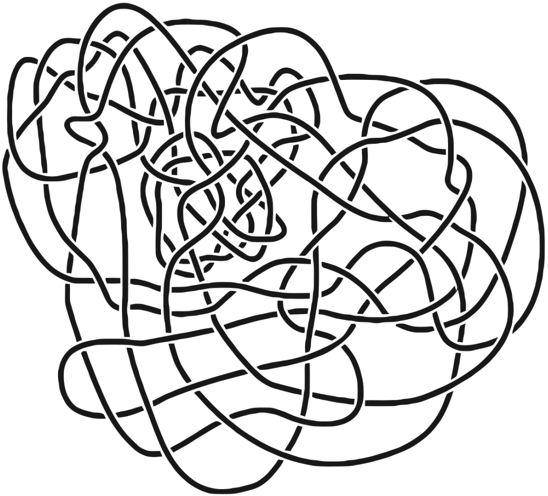

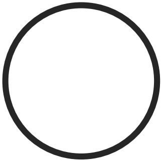

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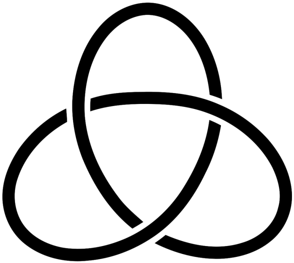

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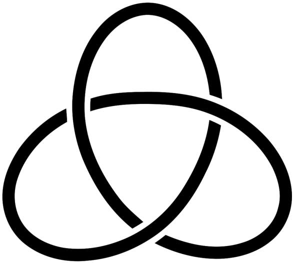

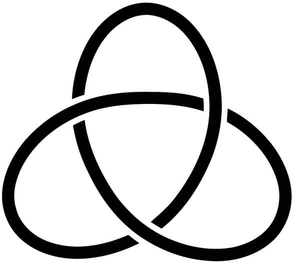

Knot theory: a biased overview

2000 Khovanov
1984 Jones
1927 Reidemeister
1923 Alexander
1894 Tait
1867 Kelvin
1833 Gauß

Carl Friedrich Gauß: the linking number

Von der *Geometria Situs*, die LEIBNITZ ahnte und in die nur einem Paar Geometern (EULER und VANDERMONDE) einen schwachen Blick zu thun vergönnt war, wissen und haben wir nach anderthalbhundert Jahren noch nicht viel mehr wie nichts.

Eine Hauptaufgabe aus dem *Grenzgebiet* der *Geometria Situs* und der *Geometria Magnitudinis* wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen.

Es seien die Coordinaten eines unbestimmten Punkts der ersten Linie x, y, z; der zweiten x', y', z' und

$$\iint \frac{(x'-x)(\mathrm{d}y\,\mathrm{d}z'-\mathrm{d}z\,\mathrm{d}y')+(y'-y)(\mathrm{d}z\,\mathrm{d}x'-\mathrm{d}x\,\mathrm{d}z')+(z-z')(\mathrm{d}x\,\mathrm{d}y'-\mathrm{d}y\,\mathrm{d}x')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}} = V$$

dann ist dies Integral durch beide Linien ausgedehnt

$$= 4\,m\pi$$

und m die Anzahl der Umschlingungen.

Der Werth ist gegenseitig, d. i. er bleibt derselbe, wenn beide Linien gegen einander umgetauscht werden. 1833. Jan. 22.

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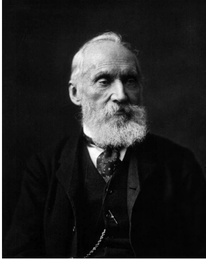
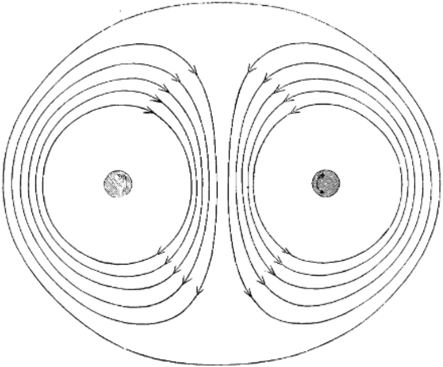
Lord Kelvin: the vortex model





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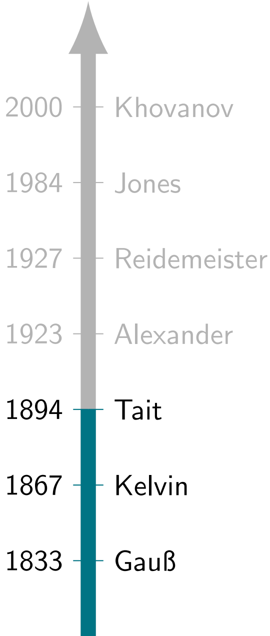
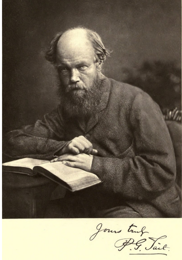
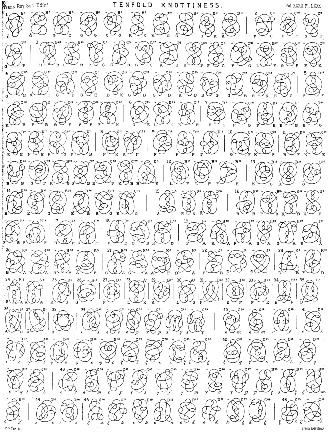


Peter Guthrie Tait: knot tables





Knot theory: a biased overview

- 2000 — Khovanov
- 1984 — Jones
- 1927 — Reidemeister
- 1923 — Alexander
- 1894 — Tait
- 1867 — Kelvin
- 1833 — Gauß

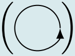
James W. Alexander II: the first knot polynomial

First definition using Dehn's knot group.

Normalization by Conway: $\Delta(K) \in \mathbb{Z}[t, t^{-1}]$.

Skein relations

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) = (t - t^{-1})\,\Delta\Big($  $\Big)$.

$\Delta\Big($  $\Big) = 1$

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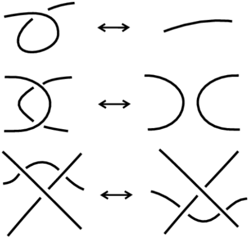
Kurt Reidemeister: playing with diagrams

La Trahison des Images, Magritte, 1928

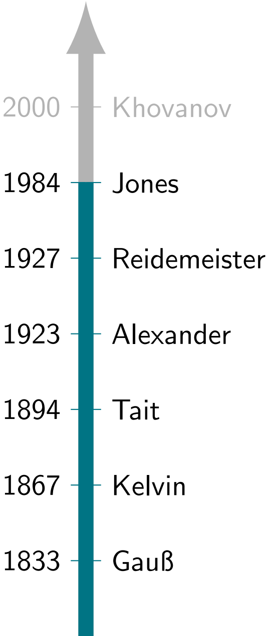


Reidemeister moves

Theorem (Reidemeister, 1927)

Two knot diagrams represent the same knot if and only if they can be related by Reidemeister moves.

Knot theory: a biased overview



Vaughan Jones: quantum topology

Definition using Von Neuman algebras.
$V(K) \in \mathbb{Z}[q, q^{-1}]$.

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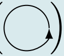
$q^{-2} V$ () $- q^2 V$ () $= (q - q^{-1})\, V$ ().

V () $= q + q^{-1}$

Reshetikhin–Turaev: reformulation and generalization using quantum groups and R-matrices.

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Mikhail Khovanov: categorification

Knot homology

Generalization: Khovanov–Rozansky

Link to topology: Jake Rasmussen

Toward smooth 4D-Poincaré conjecture?

Demystifying categorification

Usual slogan

Trade natural (relative) integers for (complexes of) vector spaces.

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A more accurate statement

Realize a ring as Grothendieck group of a category.

$$K_0(\mathcal{C}) = \left\langle [X], X \in \mathrm{ob}(\mathcal{C}) \;\middle|\; \begin{array}{l} [Y] = [X] + [Z], \text{when} \\ 0 \to X \to Y \to Z \to 0 \end{array} \right\rangle, \otimes \leftrightsquigarrow \cdot$$

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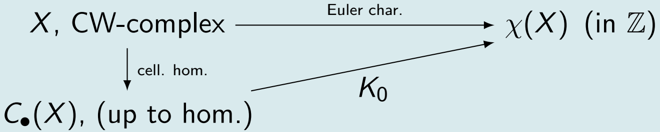
Example (Cellular homology)



Demystifying categorification: Khovanov homology

Categorification of the Jones polynomial

$$\begin{array}{ccc} K,\ \text{knot} & \xrightarrow{\text{Jones pol.}} & J(K)\ \ (\text{in } \mathbb{Z}[q,q^{-1}]) \\ \Big\downarrow{\scriptstyle\text{Khovanov hom.}} & \nearrow_{K_0} & \\ \mathrm{Kh}^{\bullet,\bullet}(K),\ (\text{up to hom.}) & & \end{array}$$

Here $\mathcal{C} = \mathrm{kom}(\mathbb{Z}\mathrm{-mod}_{\mathrm{gr}})_{/\mathrm{h}}$ and
$K_0(\mathcal{C}) \simeq \mathbb{Z}[q,q^{-1}]$ (via the graded Euler characteristic)

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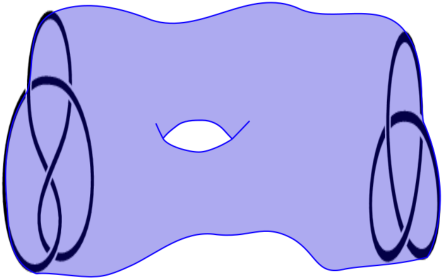
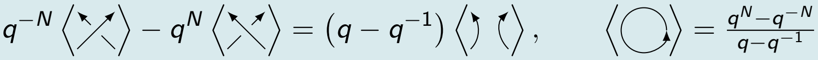


Toward the foam evaluation formula: $\mathfrak{gl}_N$-polynomial

Let $N \in \mathbb{N}^*$, $\langle \cdot \rangle = \langle \cdot \rangle_N$ is a polynomial link invariant.

Skein relation



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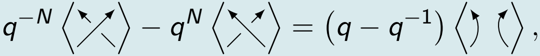
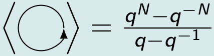
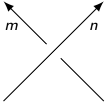
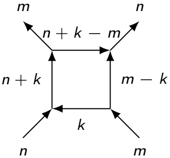






$\rightsquigarrow \sum_{k=\max(0,m-n)}^{m} (-1)^{m-k} q^{k-m}$



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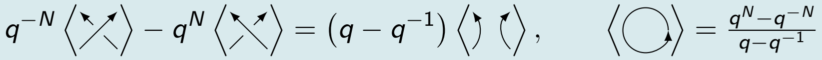

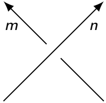

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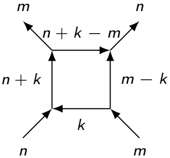

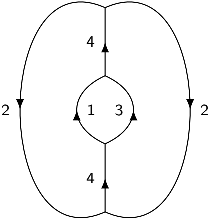

Web

$$\Gamma \mapsto \langle\Gamma\rangle \in \mathbb{N}[q^{\pm 1}]$$

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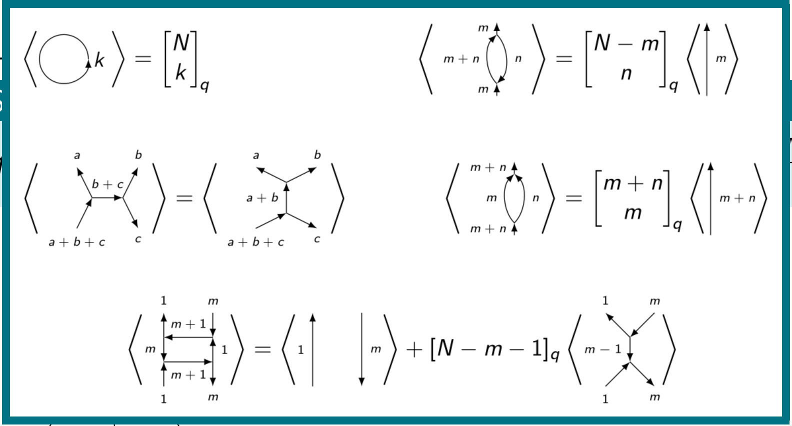

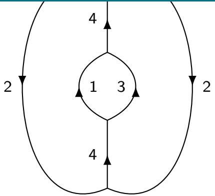

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Hypercube of resolutions $\quad + \quad$ "Singular 1+1-TQFT"

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$$\begin{aligned} \mathcal{F} : \quad \text{Web } \Gamma &\longmapsto \text{graded module} \\ \text{Foam } F &\longmapsto \text{graded module map} \end{aligned}$$

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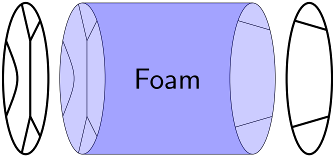

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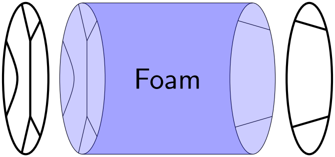

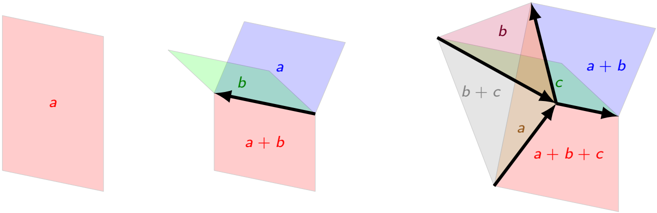

The foam evaluation formula

Definition (R-Wagner, [2])

$$\tau(F) = \sum_c (-1)^{\sum_i \frac{i\chi(F_i(c))}{2} + \sum_{i<j} \theta_{ij}^+(F,c)} \frac{\prod_f P_f(c(f))}{\prod_{i<j} (X_j - X_i)^{\frac{\chi(F_{ij}(c))}{2}}}$$

$$\in \mathbb{Z}[X_1, \ldots, X_N]^{S_N}$$

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Theorem (R–Wagner, [2])

The universal construction applied on τ gives rise to a singular 1+1-TQFT $\mathcal{F}_N$.

It categorifies the exterior MOY calculus.

Theorem (R-Wagner, [2])

The functor $\mathcal{F}_N$ and the hypercube of resolutions gives a definition of the (colored and equivariant) $\mathfrak{gl}_N$-homology.

The foam evaluation formula: some consequences

Theorem (Ehrig–Tubenhauer–Wedrich, 2018)

The $\mathfrak{gl}_N$-homology is fully functorial.

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Combinatorial counterpart to Kronheimer–Mrowka $SO(3)$-gauge theory.

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Theorem (Qi–R–Sussan–Wagner, [7,8])

The $\mathfrak{gl}_N$-homology can be endowed with an $\mathfrak{sl}_2$-module structure.

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Project (Guérin–Roz)

Endow $\mathfrak{gl}_N$-homology with an action of the half-Witt algebra.

Symmetric homology

Can we play the same game with symmetric powers?

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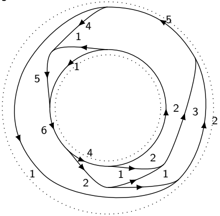
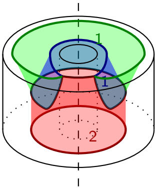
No, but. . . yes if we restrict to vinyl graphs and vinyl foams.





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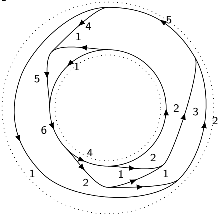

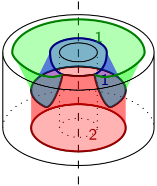

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- *The* **universal construction** *applied on σ gives rise to a singular 1+1-TQFT $\mathcal{G}_N$ categorifying the symmetric MOY calculus.*
- *The functor $\mathcal{G}_N$ and the* **hypercube of resolutions** *gives a definition of the symmetric $\mathfrak{gl}_N$-homology.*

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Conjecture (Marino, 2023)

The symmetric $\mathfrak{gl}_1$ homology has the same rank as the reduced triply graded homology.

$\mathfrak{gl}_0$-homology

By "localizing" the symmetric $\mathfrak{gl}_1$-homology, one obtains another homology theory, that we call $\mathfrak{gl}_0$-homology.

Theorem (R–Wagner, [4])

- *The $\mathfrak{gl}_0$-homology is a knot invariant, which categorifies the Alexander polynomial.*
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- *There is a spectral sequence from the reduced triply graded homology to the $\mathfrak{gl}_0$-homology.*

Theorem (Beliakhova–Putyra–R-Wagner, [6])

When working over $\mathbb{Q}$, there is a spectral sequence from the $\mathfrak{gl}_0$-homology to knot Floer homology.

$\mathfrak{gl}_0$-homology

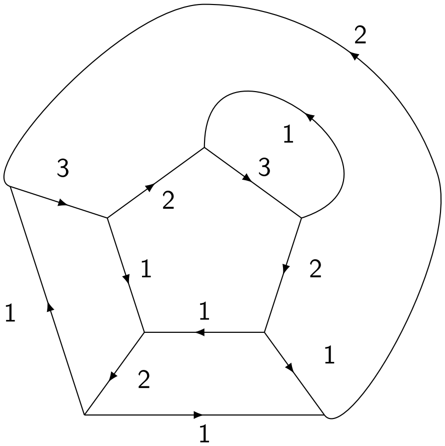
Corollary (Dunfield–Gukov–Rasmussen conjecture)

There is a spectral sequence from the reduced triply graded homology to knot Floer homology.

Corollary

Reduced triply graded and $\mathfrak{gl}_0$-homology detect: the unknot, the two trefoils, the figure-eight and the knot 5_1.

About the web evaluation $\langle \cdot \rangle$



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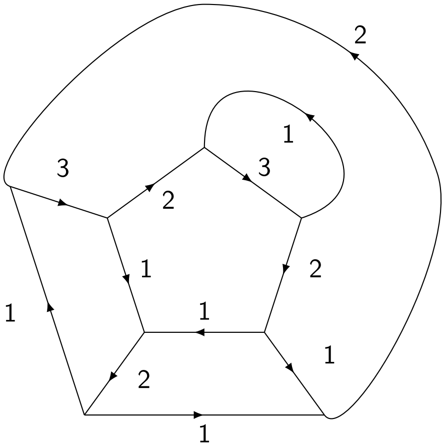

$[\![N]\!] = \{1, 2, 3, 4, \ldots, N\}$ (pigments).

Coloring:
$c : E(\Gamma) \longrightarrow \mathcal{P}([\![N]\!])$
with $|c(e)| = \mathrm{t}(e)$ and flow cond.

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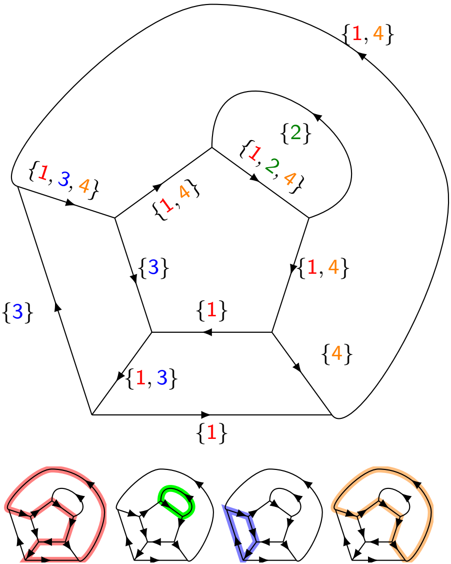

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$\Gamma_{ij}(c) = \Gamma_i(c) \Delta \Gamma_j(c)$ ($i < j$)

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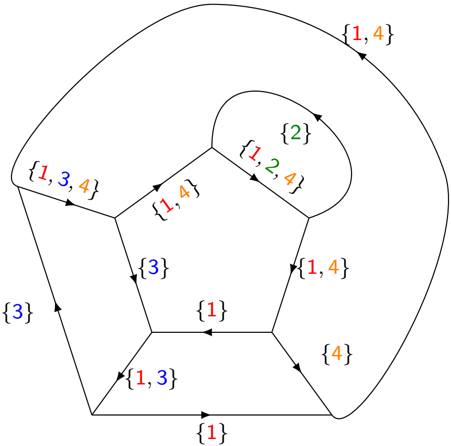

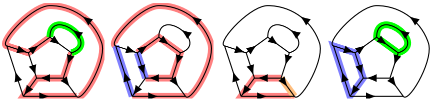

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$\rho = \# \circlearrowleft - \# \circlearrowright$

$\deg(c) = \sum_{i<j} \rho(\Gamma_{ij}(c))$

Definition (R, [1])

$\langle \Gamma \rangle = \sum_c q^{\deg(c)}$

Claim

- *For a given N and given Γ, the degree of all colorings have the same parity.*
- $\langle \Gamma \rangle$ *is symmetric in* $q \leftrightarrow q^{-1}$.

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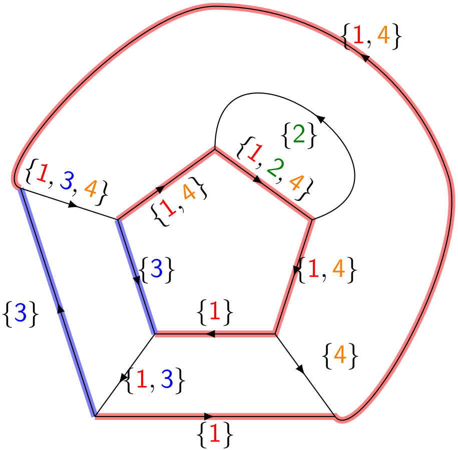
Conjecture

For any N and any Γ, the polynomial $\langle\Gamma\rangle$ is unimodal (its coefficients increase then decrease).

Problem

- *Find an efficient way to compute the degree of* $\langle\Gamma\rangle$.
- *Find a condition for* $\langle\Gamma\rangle$ *to be monic.*
- *Given a colored web (Γ, c), find a colored foam (F, C), with $\partial F = \Gamma$ and $\partial C = c$.*

Kempe moves



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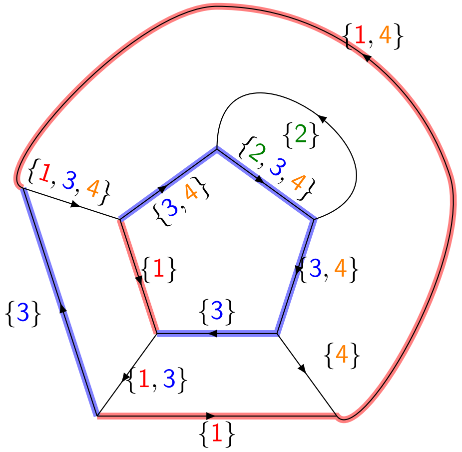

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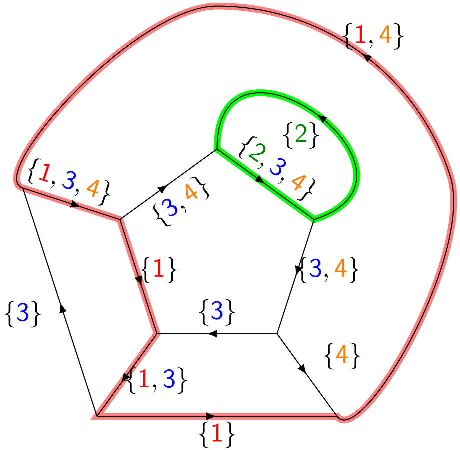
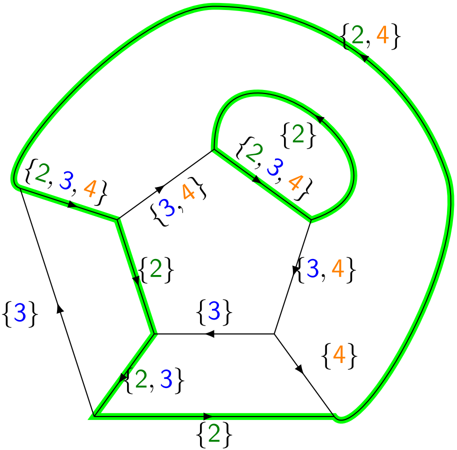
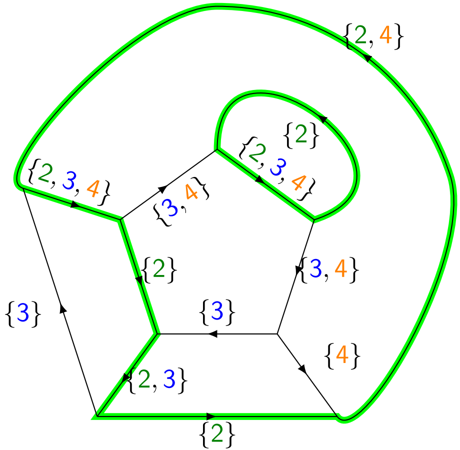


Kempe moves



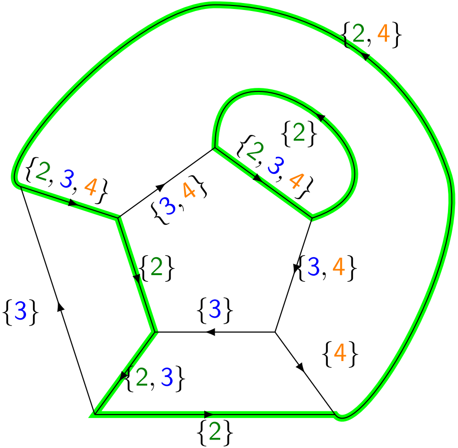
Kempe moves



Definition

Two colorings are Kempe equivalent if one can transform one into the other using Kempe moves.

Kempe moves



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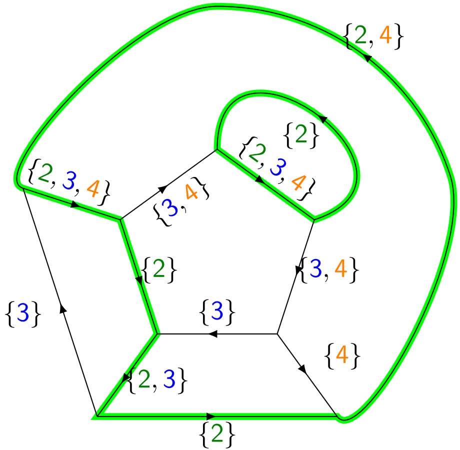
Are all colorings of a given web Kempe equivalent?

Kempe moves



Definition

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Question

Are all colorings of a given web Kempe equivalent?

Yes if $N \leq 3$.

Thank you!