Coloured Jones and Alexander polynomials unified through Lagrangian intersections in configuration spaces

Outline

Motivation

(I) Homological tools

(II) Topological model with immersed Lagrangians

(III) Topological model with embedded Lagrangians

Motivation

• Aim: Define topological models for $U_q(sl(2))$-quantum invariants

with nice homology classes

Topological model: graded intersection pairing of homology classes in coverings of configuration spaces

Th (Bigelow '00, Lawrence)
Noodles and Forks

skein rel

Jones polyn.

Explicit models

Th 1 (A. '20)
Unified topological model (immersed Lagrangians)
- weight space representations
- Martel's identification
- explicit classes

Th2 (A. '20)
Unified model over 3 variables (embedded Lagrangians)
- uses Th 1
- suitable for computations

Coloured Jones polyn.
$J_N(L,q) \in \mathbb{Z}[q^{\pm 1}]$

Coloured Alexander polyn.
$\Phi_N(L,\lambda) \in \mathbb{Z}[\xi^{\pm 1}, \xi^{\pm \lambda}]$

Existence type results

Th (A. '17)
- highest weight sp.
- Kohno's (Ito) identification over 2 variables
- discuss genericity questions

Th (A. '19)
- provides the homological meaning of the partial quantum trace

$1/\Delta_K$

Unified algebraically
Willetts '20

Representation theory

Intersections in configuration spaces

$\boxed{\text{Main result}}$

Fix $N \in \mathbb{N}$ - colour of the quantum invariant

Let L - oriented link ; $L = \widehat{\beta_n}$ for $\beta_n \in B_n$

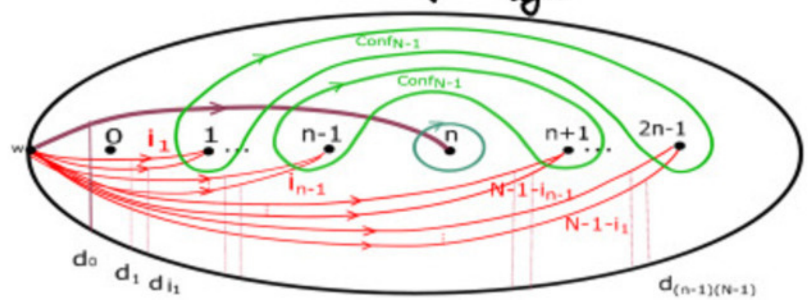
Construction : $\forall\, i_1,\dots,i_{n-1} \in \{0,\dots,N-1\} \rightsquigarrow$ Two Lagrangians

$$\mathcal{H}_*(\mathbb{Z}\oplus\mathbb{Z} \text{ covering}) \quad \left(\mathcal{F}_{i_1,\dots,i_{n-1}} \in \mathcal{H}^{-n}_{2n,(n-1)(N-1)+1} ; \mathcal{L}_{i_1,\dots,i_{n-1}} \in \mathcal{H}^{-n,\partial}_{2n,(n-1)(N-1)+1}\right)$$

|| not $\mathcal{H}_{2n,*}$

lifts

$\mathrm{Conf}_{(n-1)(N-1)+1}(\mathbb{D}_{2n})$



• **Def** (State sum of Lagrangian intersections)

$$\boxed{\Lambda_N(\beta_n) := u^{-w(\beta_n)}\, u^{n-1}\, x^{-n} \sum_{i_1,\dots,i_{n-1}=0}^{N-1} \left\langle (\beta_n \cup \mathbb{I}_n)\, \mathcal{F}_{i_1,\dots,i_{n-1}}, \mathcal{L}_{i_1,\dots,i_{n-1}} \right\rangle}$$

$\in \mathbb{Z}[x^{\pm1}, d^{\pm1}, u^{\pm1}]$

• $\boxed{\text{Th 2}}$ (A'20 Unified model through state sums of Lagrangian intersections)

The polynomial in 3 variables Λ_N recovers the N^{th} Coloured Jones and N^{th} Coloured Alexander polynomials for links:

$$\boxed{J_N(L,q) = \Lambda_N(\beta_n)\Big|_{\psi_{1,q,N}}}$$

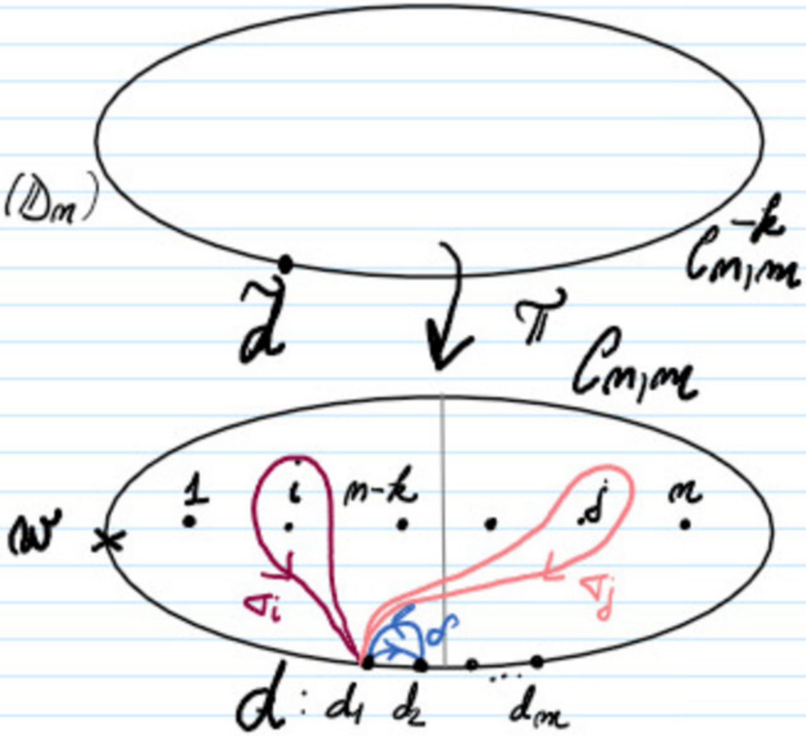
$$\boxed{\Phi_N(L,\lambda) = \Lambda_N(\beta_n)\Big|_{\psi_{1-N,\xi_N,\lambda}}}$$

specialisations of coefficients

I $\boxed{\text{Homological representations}}$

Fix $n, m, k \in \mathbb{N}$

$\mathbb{D}_n := D^2 \setminus \{1, \dots, n\} \rightsquigarrow C_{n,m} = \mathrm{Conf}_m(\mathbb{D}_n)$



- **Def** (Local system) Fix $0 \le k \le n$

$$\pi_1(C_{n,m}) \to \mathbb{Z}^n \oplus \mathbb{Z} \xrightarrow{\varphi^k} \mathbb{Z} \oplus \mathbb{Z}$$

$\langle \sigma_i \rangle \; \langle \delta \rangle \qquad \langle x \rangle \; \langle d \rangle$

$\sigma_i \mapsto \begin{cases} x, & 0 \le i \le n-k \\ -x, & i > n-k \end{cases} \qquad \delta \mapsto d$

$\rightsquigarrow C_{n,m}^{-k}$ covering sp

- Let $w \in \partial \mathbb{D}_n$; $\tilde{d} \in \pi^{-1}(d)$

- $\boxed{\text{Tools}}$: Homology of this covering sp.

① $\mathcal{H}_{n,m}^{-k} \subseteq H_m^{lf}\left(C_{n,m}^{-k}, \pi^{-1}(w); \mathbb{Z}\right)$

$B_n \circlearrowright$

Borel Moore w.r.t. punctures colissions

② $\mathcal{H}_{n,m}^{-k,\partial} \subseteq H_m^{cell}\left(C_{n,m}^{-k}, \partial; \mathbb{Z}\right)$

③ Prop : (A-Palmer) Intersection pairing :

$$\langle \, , \rangle : \mathcal{H}_{n,m}^{-k} \otimes \mathcal{H}_{n,m}^{-k,\partial} \to \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$$

Construction of homology classes

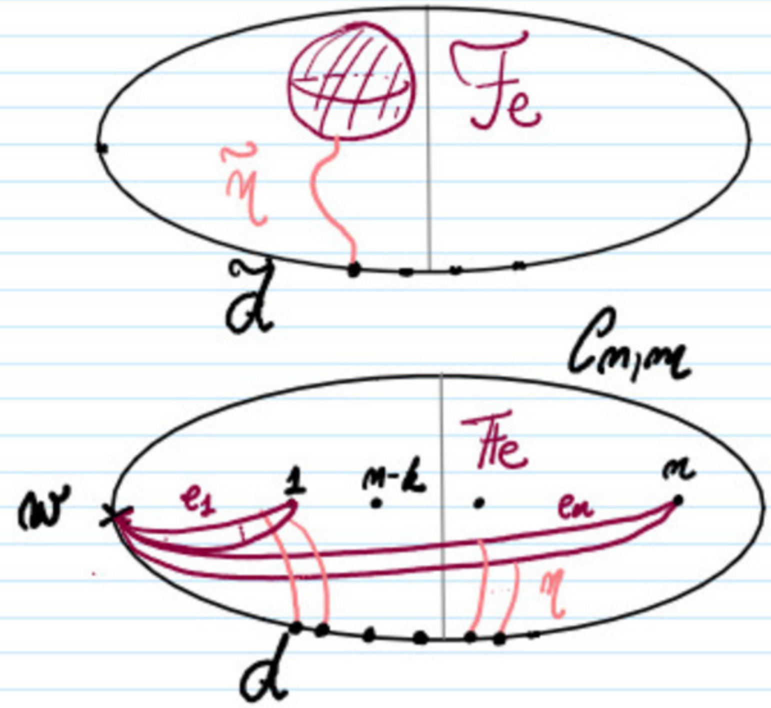
$E_{n,m} := \{ n\text{-partitions of } m \}$

$\ni \; e=(e_1,\dots,e_n) \rightsquigarrow \mathscr{F}_e \in \mathcal{H}^{-k}_{n,m}$

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$ lift through $\tilde{\eta}(1)$

$\left(\mathbb{T}_e \subseteq C_{n,m} \; ; \; \eta : d \to \mathbb{T}_e \right) \longrightarrow \tilde{\eta}$ - lift through $\tilde{d}$

given by the product of curves $\qquad$ path in the config.

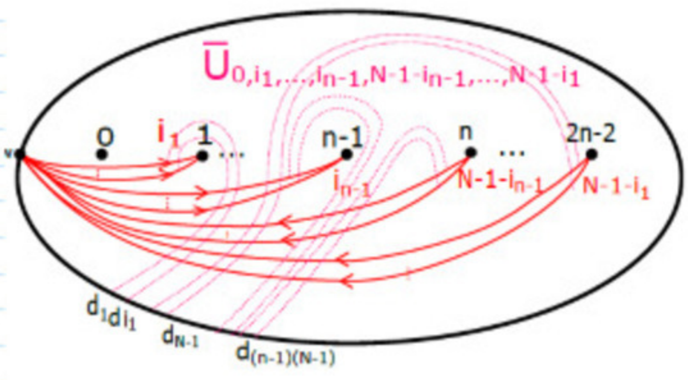


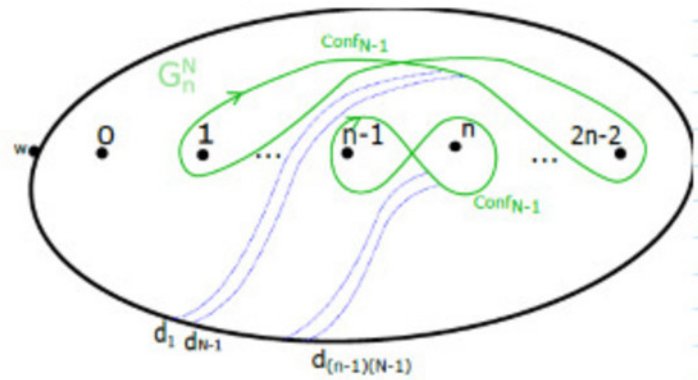
(II) Topological model with immersed Lagrangians

Context: L - oriented link ; $L = \hat{\beta}_n$, $\beta_n \in B_n$

Def (Main classes) $\underline{i} = (i_1,\dots,i_n)$: $i_k \in \{0,\dots,N-1\}$

$\tilde{\mathcal{U}}_{i_1,\dots,i_{n-1}} \in \mathcal{H}^{0}_{2n-1,(n-1)(N-1)} \qquad \mathcal{G}^N_n \in \mathcal{H}^{0,\partial}_{2n-1,(n-1)(N-1)}$





Def (Homology Classes)

$$\left(\mathcal{E}^N_n := \sum_{i_1,\dots,i_{n-1}=0}^{N-1} d^{\sum i_k} \cdot \tilde{\mathcal{U}}_{i_1,\dots,i_{n-1}} \; , \; \mathcal{G}^N_n \right)$$

- **Not** (Specialisation of coefficients) Let $c \in \mathbb{Z}$

$$\psi_{(c),q,\lambda} : \mathbb{Z}[u^{\pm1}, x^{\pm1}, d^{\pm1}] \to \mathbb{Z}[q^{\pm1}, q^{\pm\lambda}]$$

$$(u \rightsquigarrow q^{c\lambda})$$

$$\begin{cases} x \rightsquigarrow q^{2\lambda} \\ d \rightsquigarrow q^{-2} \end{cases}$$

- $\boxed{\text{Th 1}}$ (Topological model via immersed Lagrangians)

Let $\boxed{I_N(\beta_n) := \langle (\beta_n \cup \mathbb{1}_{n-1})\, \mathcal{E}_n^N, \mathcal{G}_n^N \rangle} \in \mathbb{Z}[x^{\pm1}, d^{\pm1}]$

Then, I_N recovers the N^{th} col. Jones and col. Alexander polyn.:

$$\boxed{J_N(L,q) = q^{(N-1)\,wr(\beta_n)}\, q^{(n-1)(N-1)} \cdot I_N(\beta_n)\big|_{\psi_{q,N-1}}} \qquad \psi_J := \psi_{q,N-1}$$

$$\boxed{\Phi_N(L,\lambda) = \xi_N^{-(N-1)\,wr(\beta_n)}\, \xi_N^{-(n-1)(N-1)} \cdot I_N(\beta_n)\big|_{\psi_{\xi_N,\lambda}}} \qquad \psi_\Phi := \psi_{\xi_N,\lambda}$$

Construction and idea of proof

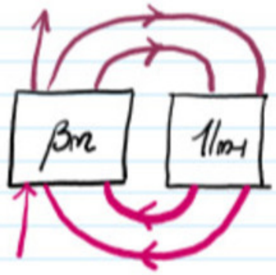
$\boxed{\text{Algebraic context}}$ $(U_q(sl(2)), R)$ over $\mathbb{Z}[q^{\pm1}, s^{\pm1}]$

Verma mod $\mathbb{Z}[q^{\pm1}, s^{\pm1}]$ $\qquad \hat{\mathbb{V}} := \langle v_0, v_1, \dots, v_{N-1}, v_N, \dots \rangle$ $\qquad \boxed{\text{Weight spaces}}$

$$\begin{tikzcd} & \hat{\mathbb{V}} & \\ & \mathbb{V}_N \arrow[u, phantom, "\cup"] \arrow[dl, "\psi_J"'] \arrow[dr, "\psi_\Phi"] & \\ V_N \arrow[d] & & V_\lambda \arrow[d] \\ J_N & & \Phi_N \end{tikzcd}
\qquad
\begin{tikzcd} \hat{V}_{n,m} \arrow[r, hook] & \hat{V}^{\otimes n} \\ \mathbb{V}_{n,m} \arrow[u, phantom, "\cup"] \arrow[r, hook] \arrow[d, "\psi_J"'] \arrow[dr, "\psi_\Phi"] & \mathbb{V}_N^{\otimes n} \arrow[u, phantom, "\cup"] \\ \mathbb{V}_{n,m}\big|_{\psi_J} & \mathbb{V}_{n,m}\big|_{\psi_\Phi} \end{tikzcd}$$

Step 1: Definition of J_N, Φ_N in this set-up

$L = \hat{\beta}_n$ link



Diagrammatically:

Coev: $1 \mapsto v_i \otimes v_{N-1-i}$

ev_J: $v_i \otimes v_{N-1-i} \mapsto s\,q^{-2i}$, 0 otherwise

ev_Φ: $v_i \otimes v_{N-1-i} \mapsto s^{1-N} q^{-2i}$, 0 otherwise

See both invariants from a construction over 2-variables

Extend ev_J and ev_Φ on all vectors from the Verma module with zero unless they are from V_N

Step 2: We can use the weight spaces from the Verma module

$$\mathbb{Z}[q^{\pm1}, s^{\pm1}]$$

ev_J, ev_Φ: 0 unless $j_k = i_k$

$$\sum_{j} \alpha_j\, v_{j_0} \otimes v_{j_1} \otimes \cdots \otimes v_{j_{n-1}} \otimes v_{N-1-i_{n-1}} \otimes \cdots \otimes v_{N-1-i_1}$$

$$\uparrow \beta_n \cup 1_{n-1}$$

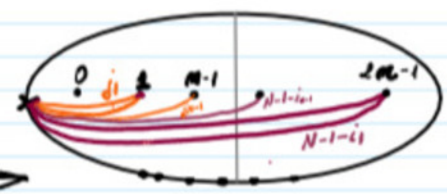
$$\sum_{i} v_0 \otimes v_{i_1} \otimes \cdots \otimes v_{i_{n-1}} \otimes v_{N-1-i_{n-1}} \otimes \cdots \otimes v_{N-1-i_1}$$

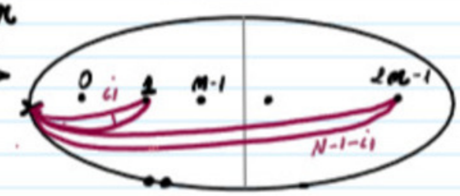
$$\uparrow$$

$$v_0$$

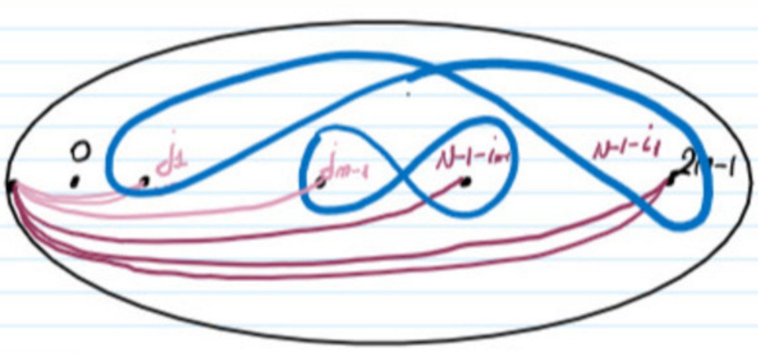
(all $\underline{i} = (i_1, \ldots, i_{n-1})$ with $i_k \in \{0, \ldots, N-1\}$)

Martel $\equiv$ A-Palmer $\equiv$





Step 3: We need a dual class, which intersects $\mathscr{F}_{j_1,\ldots,j_{n-1},N-1-i_{n-1},\ldots,N-1-i_1}$ non-zero iff $j_k = i_k$, $\forall k$



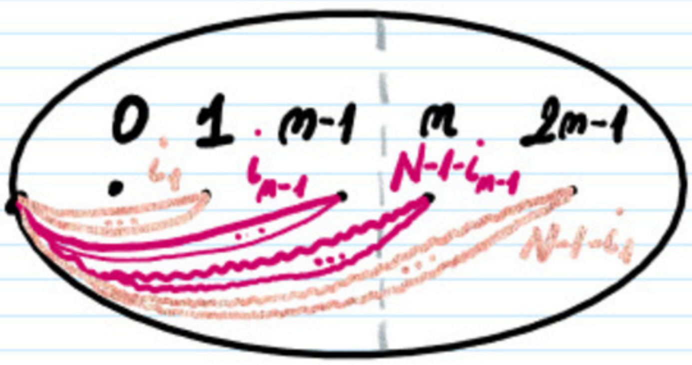
• **Corollary 1** (Recover Bigelow's model for the Jones polynomial)

Th 1 for $N=2$ recovers Bigelow's model:

$\mathcal{F}_m^2 \to$ forks $\quad \mathcal{G}_m^2 \to$ noodles

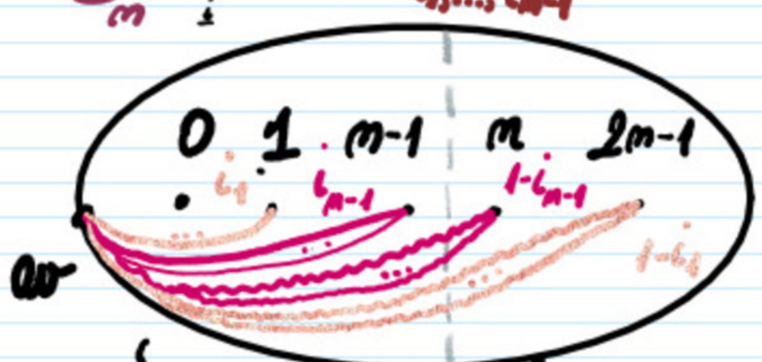
Proof For $i_1,\dots,i_{m-1} \in \{0,\dots,N-1\}$

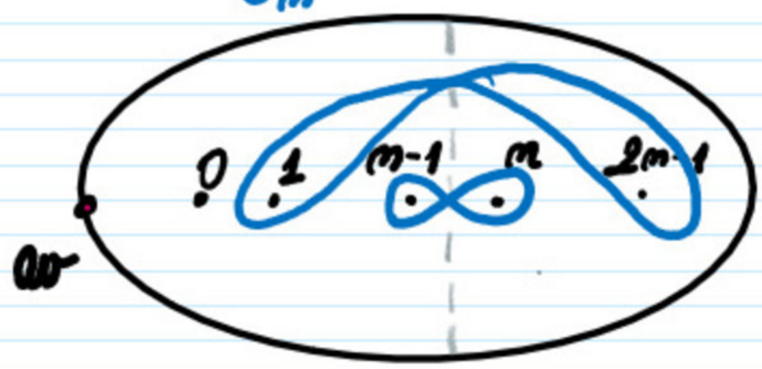
$\tilde{\mathcal{U}}_{i_1,\dots,i_{m-1}}$ $\qquad$ $\mathcal{G}_m^N$



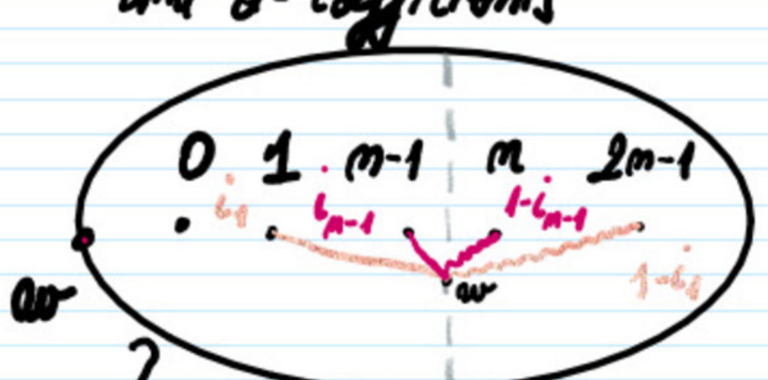
$\boxed{N=2}$ Jones polyn. $i_1,\dots,i_{m-1} \in \{0,\dots,N-1\}$

$\mathcal{E}_m^2 = \sum_i d^{\sum i_k} \tilde{\mathcal{U}}_{i_1,\dots,i_{m-1}}$ $\qquad$ $\mathcal{G}_m^2$

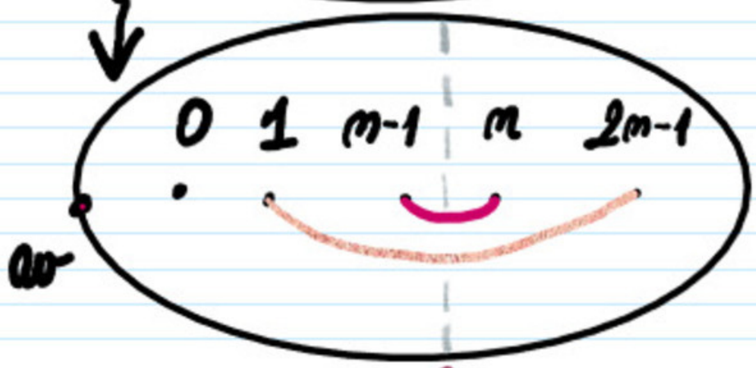


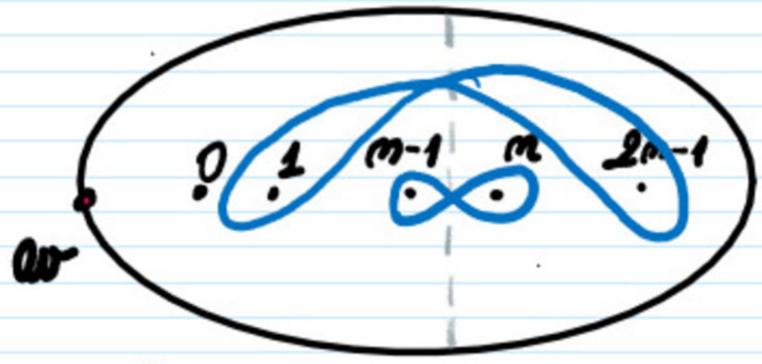


$\downarrow$ up to homotopy and d-coefficients



$\wr\downarrow$





$\mathcal{E}_m^2$ fork and $\mathcal{G}_m^2$ noodle

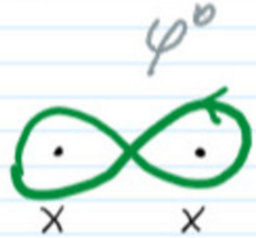
• **Corollary 2** (ADO invariants from $\mathbb{Z} \oplus \mathbb{Z}_N$-covering spaces)

The $N^{\underline{th}}$ coloured Alexander invariant is an intersection pairing in a $\mathbb{Z} \oplus \mathbb{Z}_N$-covering of a conf. sp. in the punctured disk.

Questions
1. Model with embedded Lagrangians
2. Simple lifts (paths η to the base point)
3. Geometrical meaning of the d-coeff.

III Model with embedded Lagrangians

Construction Ideas

① ② Replace 

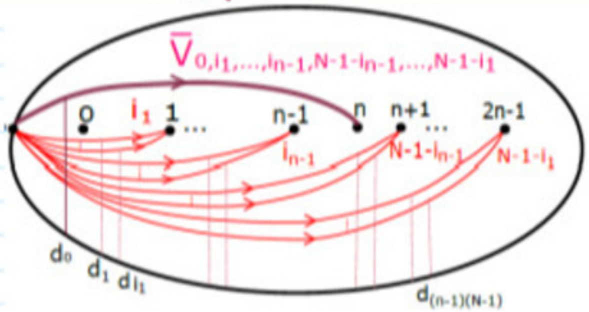
by 

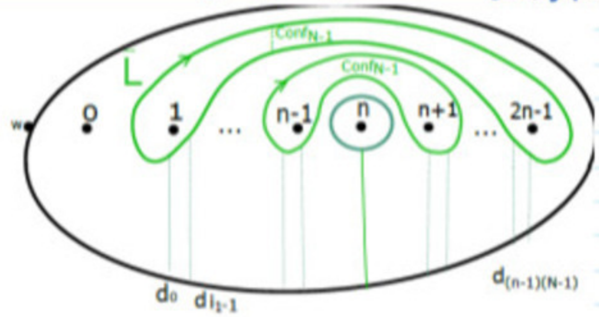
(change the local system)

③ Add an extra puncture

• Def (Homology classes)

$\mathcal{F}_{i_1,\dots,i_{m-1}} \in \mathcal{H}^{-m}_{2m,(m-1)(N-1)+1}$ $\mathcal{L}_{i_1,\dots,i_{m-1}} \in \mathcal{H}^{-m,\partial}_{2m,(m-1)(N-1)+1}$





• $\boxed{\text{Th 2}}$ (A'20 Unified model through embedded Lagrangians)

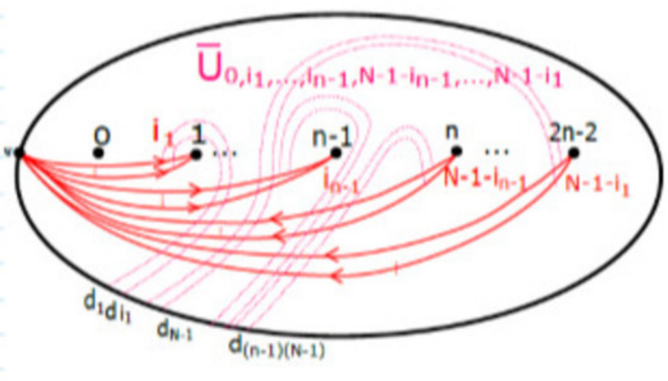
$$\boxed{\Lambda_N(\beta_n) := u^{-w(\beta_n)}\, u^{n-1}\, x^{-n} \sum_{i_1,\dots,i_{n-1}=0}^{N-1} \langle (\beta_n \cup \mathbb{1}_n)\, \mathcal{F}_{i_1,\dots,i_{n-1}}, \mathcal{L}_{i_1,\dots,i_{n-1}} \rangle}$$

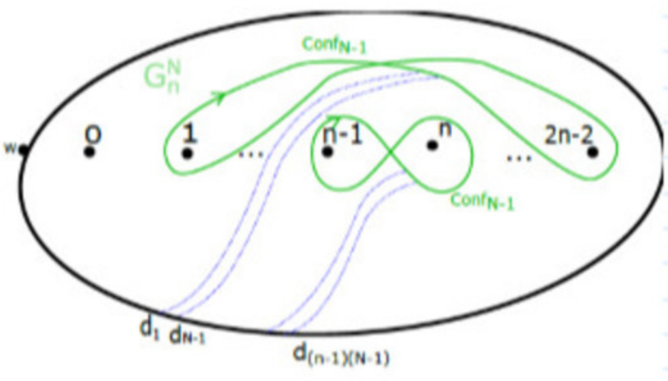
Then : $\begin{cases} \boxed{J_N(L,q) = \Lambda_N(\beta_n) \,/\, \psi_{1,q,N}} \\ \boxed{\Phi_N(L,\lambda) = \Lambda_N(\beta_n) \,/\, \psi_{1-N,q_N,\lambda}} \end{cases}$ specialisations of coefficients

• <u>Proof</u> We have two types of intersection pairings:

① Immersed classes $\longrightarrow$ $\boxed{\text{Th 1}}$

$\tilde{\mathcal{U}}_{i_1,\dots,i_{n-1}} \in \mathcal{H}^{0}_{2n-1,(n-1)(N-1)}$ $\quad \mathcal{G}^N_n \in \mathcal{H}^{0,\partial}_{2n-1,(n-1)(N-1)}$

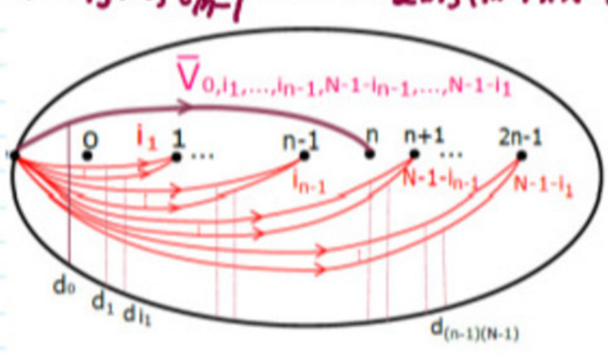


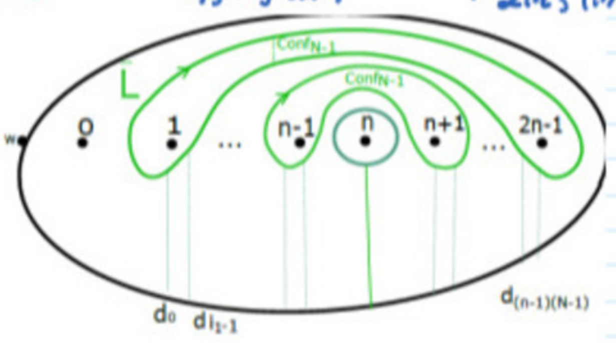


$\langle \, , \, \rangle_{①}$

② Embedded classes $\longrightarrow$ $\boxed{\text{Th 2}}$

$\mathcal{F}_{i_1,\dots,i_{n-1}} \in \mathcal{H}^{-n}_{2n,(n-1)(N-1)+1}$ $\quad \mathcal{L}_{i_1,\dots,i_{n-1}} \in \mathcal{H}^{-n,\partial}_{2n,(n-1)(N-1)+1}$





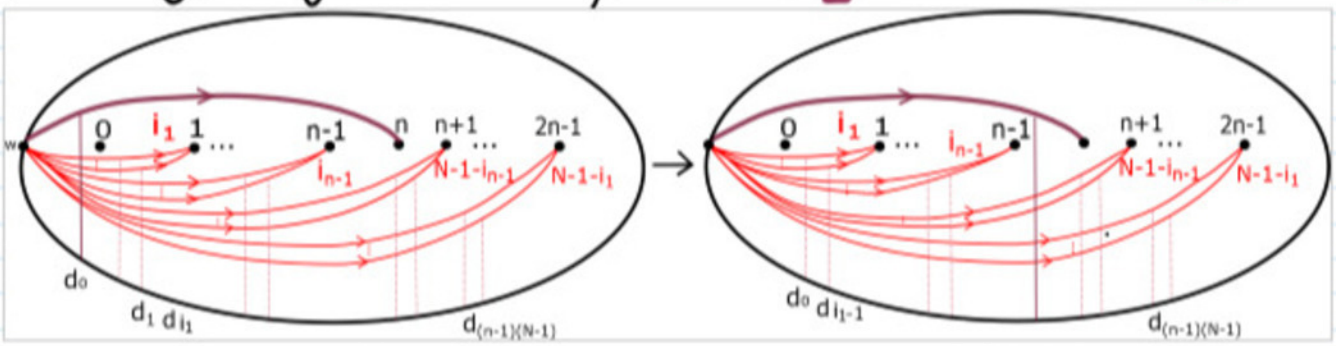
$\langle \, , \, \rangle_{②}$

• $\boxed{\text{Main Lemma}}$ (Relation between different Lagrangian intersections)

$\forall\ \underline{i} = (i_1,\dots,i_{n-1}) \in \{0,\dots,N-1\}^{n-1}$:

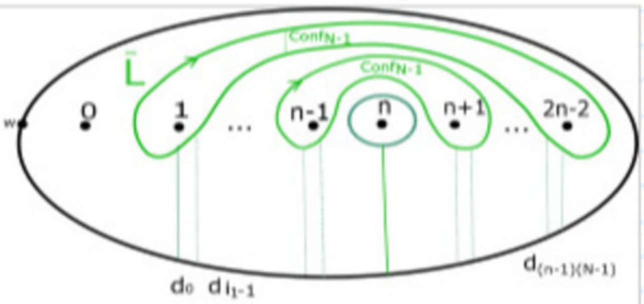
$$\boxed{\langle (\beta_n \cup \mathbb{1}_n)\, \mathcal{F}_{\underline{i}}, \mathcal{L}_{\underline{i}} \rangle_{②} = x^{n} \cdot d^{\Sigma i_k} \cdot \langle (\beta_n \cup \mathbb{1}_{n-1})\, \mathcal{U}_{\underline{i}}, \mathcal{G} \rangle_{①}}$$

(I) Change of the base point: $\mathcal{F}_{\underline{i}} = x^{n} \cdot d^{\Sigma i_k} \cdot \tilde{\mathcal{F}}_{\underline{i}}$



→ remove the middle point

$\langle\ ,\ \rangle_{2}$



$\langle\ ,\ \rangle_{3}$

We have

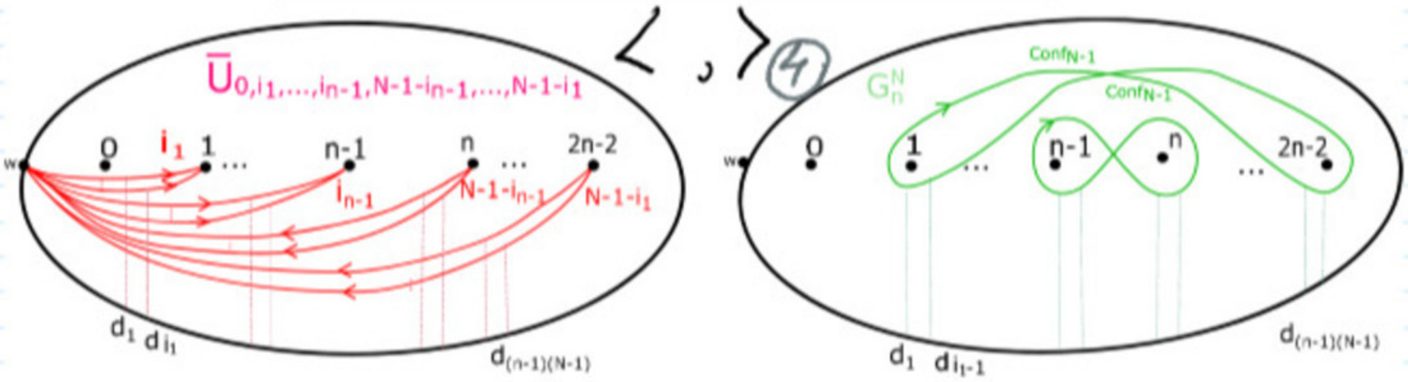
$$\boxed{\overset{B_n \cup \mathbb{1}_n}{\langle \circlearrowleft\ ,\ \rangle_{2}} \simeq \overset{B_n \cup \mathbb{1}_{n-1}}{\langle \circlearrowleft\ ,\ \rangle_{3}}}$$

(II) Change the local system

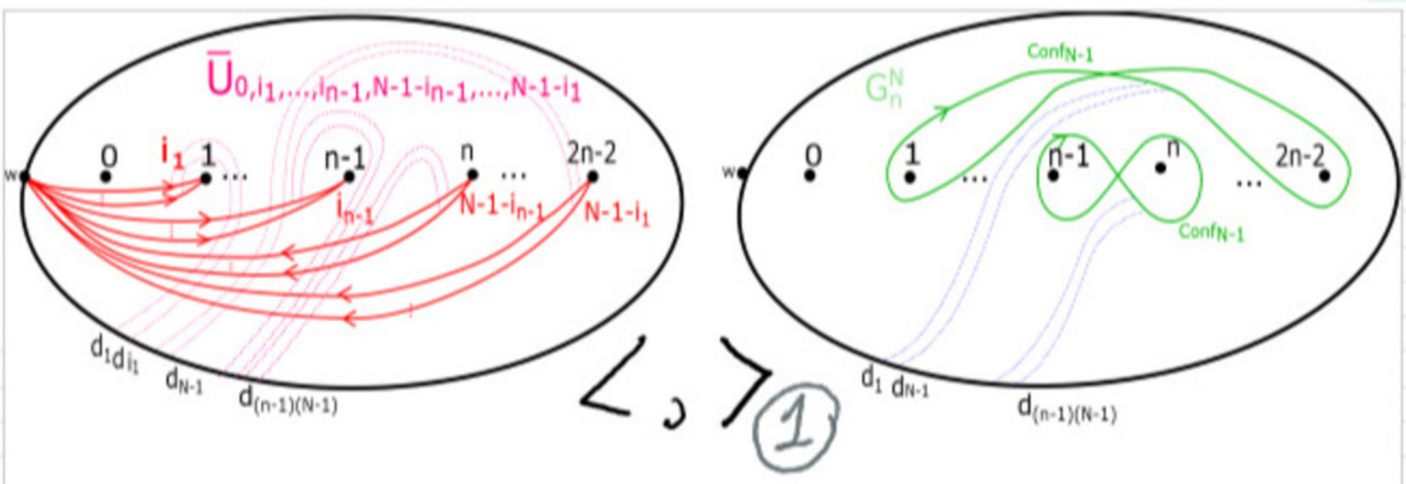
Then:

$$\boxed{\overset{B_n \cup \mathbb{1}_{n-1}}{\langle \circlearrowleft\ ,\ \rangle_{3}} = \overset{B_n \cup \mathbb{1}_{n-1}}{\langle \circlearrowleft\ ,\ \rangle_{4}}}$$

$\langle\ ,\ \rangle_{4}$



(III)

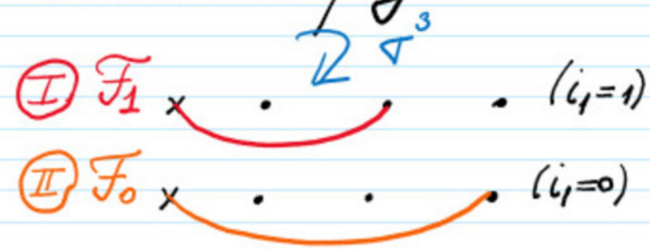


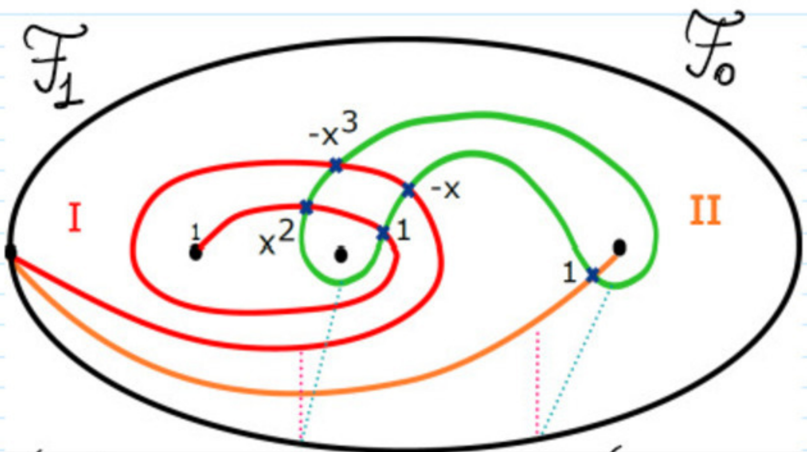
We give a geometrical argument using the properties of the pairing

$$\boxed{\overset{B_n \cup \mathbb{1}_{n-1}}{\langle \circlearrowleft\ ,\ \rangle_{4}} = \overset{B_n \cup \mathbb{1}_{n-1}}{\langle \circlearrowleft\ ,\ \rangle_{1}}}$$

• $\boxed{\text{Example}}$ $N=2$: Jones and Alexander polyn

T = trefoil knot : $\beta_2=\sigma^3$: $\boxed{n=2}$





$$\left(\sigma^3\mathscr{F}_1 \quad \mathscr{L}_1\right) \qquad \left(\sigma^3\mathscr{F}_0, \mathscr{L}_0\right)$$

$$= u^3\cdot u\cdot\left(\langle(\sigma^3\cup 1\!\!1_1)\, d\cdot\mathscr{F}_1, \mathscr{L}_1\rangle + \langle \sigma_1^3\mathscr{F}_0, \mathscr{L}_0\rangle\right)$$

$$\boxed{\Lambda_2(\sigma^3) = u^4\left(d(-x^3+x^2-x+1)+1\right)} \in \mathbb{Z}[u^{\pm1}, x^{\pm1}, d^{\pm1}]$$

$u=q^2$
$x=q^2$; $d=q^{-2}$

$u=q_2^{-\lambda}$
$x=q_2^{2\lambda}$; $d=q_2^{-2}=-1$

$$\boxed{J(T) = -q^8+q^2+q^6} \qquad \boxed{\Delta(T,x) = x-1+x^{-1}}$$