Colored HOMFLY and quantum sl_N invariant

for the CIRM Workshop – Homological and quantum invariants

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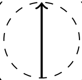
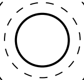
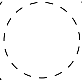
HOMFLY N-specialization

The framed HOMFLY N-specialization χ_N

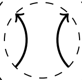
$q^{\frac{1}{N}}\chi_N\Big($  $\Big) - q^{-\frac{1}{N}}\chi_N\Big($  $\Big) = (q - q^{-1})\chi_N\Big($  $\Big)$

$\chi_N\Big($  $\Big) = q^{N-\frac{1}{N}}\chi_N\Big($  $\Big)$

and thus $\chi_N\Big($  $\Big) = [N]_q\chi_N\Big($  $\Big)$

The unframed HOMFLY N-specialization P_N

$$P_N(\mathcal{L}) = q^{(\frac{1}{N} - N) w(\mathcal{L})} \chi_N(\mathcal{L})$$

$q^N P_N\Big($  $\Big) - q^{-N} P_N\Big($  $\Big) = (q - q^{-1}) P_N\Big($  $\Big)$

The Hopf algebra $\mathcal{U}_q(sl_N)$ and its representations

Definition

$\mathcal{U}_q(sl_N)$ is the non-commutative $\mathbb{C}(q)$-algebra with :

generators : $K_i^{\pm 1},\ E_i,\ F_i, \quad 1 \leq i \leq N-1$

relations : $[-_i, -_j] = 0$ if $|i-j| \geq 2$, $[K_i, K_{i\pm 1}] = 0$,

$$K_i E_i = q^2 E_i K_i \quad , \quad K_i E_{i\pm 1} = q^{-1} E_{i\pm 1} K_i$$

$$K_i F_i = q^{-2} F_i K_i \quad , \quad K_i F_{i\pm 1} = q F_{i\pm 1} K_i$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

$$E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0$$

$$F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 = 0$$

Definition

$Rep_q(sl_N)$ = category of finite dimensional representations of $\mathcal{U}_q(sl_N)$

Definition

$\mathcal{U}_q(sl_N)$ has coproduct given by : $\Delta(K_i) = K_i \otimes K_i$,
$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$, $\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$
and counit $\varepsilon(K_i) = 1$, $\varepsilon(E_i) = \varepsilon(F_i) = 0.$

$\rightsquigarrow$ $Rep_q(sl_N)$ is monoidal with $\otimes_{\mathbb{C}(q)}$

Definition

$\mathcal{U}_q(sl_N)$ has antipode given by :
$S(K_i) = K_i^{-1}$, $S(E_i) = -E_i K_i^{-1}$, $S(F_i) = -K_i F_i$

$\rightsquigarrow$ $Rep_q(sl_N)$ is rigid

Proposition

$\mathcal{U}_q(sl_N)$ is (almost) quasitriangular and has a ribbon element

$\rightsquigarrow$ $Rep_q(sl_N)$ is braided and ribbon

Definition

Standard representation V basis $v_1, \dots, v_N$ and action :

$$K_i = \begin{pmatrix} 1 & 0 & \overset{i}{} & \cdots & & 0 \\ 0 & \ddots & & & & \\ {}_i & & q & & & \vdots \\ \vdots & & & q^{-1} & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}$$

$$E_i = \begin{pmatrix} 0 & & \overset{i+1}{} & \\ {}_i & \ddots & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad F_i = \begin{pmatrix} 0 & \overset{i}{} & & \\ & \ddots & & \\ {}_{i+1} & 1 & \ddots & \\ & & & 0 \end{pmatrix}$$

Note $L_i : \begin{cases} \{1, \dots, N-1\} & \to \quad \mathbb{Z} \\ j & \mapsto \quad \delta_{i,j} - \delta_{i-1,j} \end{cases}$

so $K_i \cdot v_j = q^{L_j(i)} v_j$.

Definition

$W \in Rep_q(sl_N)$, $\mu : \{0, \dots, N-1\} \to \mathbb{Z}$, $w \in W \setminus \{0\}$ is of weight μ if $K_i \cdot w = q^{\mu(i)} w$, $\forall i$.

w highest weight if $E_i \cdot w = 0$, $\forall i$

$\mathcal{U}_q(sl_N) \cdot w$ highest weight module

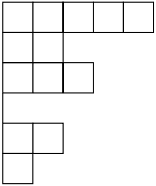
Examples

- $v_i \in V$ has weight L_i. Only v_1 is a highest weight vector. V is a highest weight module of highest weight L_1.
- $v_{i_1} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r}$ has weight $\sum_k L_{i_k}$

Highest weights of $V^{\otimes r}$?

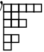
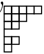
$\mu = \sum_{k=1}^{r} L_{i_k}$
$= \lambda_1 L_1 + \cdots + \lambda_N L_N$

$\sum_{k=1}^{N} \lambda_k = r$
$\lambda \vdash r$, $l(\lambda) \leq N$



$\lambda_1 = \#$ of 1 in the i_k's
$\lambda_2 = \#$ of 2 in the i_k's
$\vdots$
$\lambda_N = \#$ of N in the i_k's

$\forall \sigma \in \mathfrak{S}_r$, $v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(r)}}$ has weight μ.

$E_i \cdot v$$_{i+1} = \sum k' q^k v$$_{i} = 0?$

$\binom{\lambda_i + \lambda_{i+1}}{\lambda_{i+1}} \to \binom{\lambda_i + \lambda_{i+1}}{\lambda_{i+1} - 1} \rightsquigarrow$ solution v_λ if $\lambda_i \geq \lambda_{i+1}$

$\rightsquigarrow \lambda$ Young tableau

Definition : type 1 simple representations

$\lambda \vdash r$, $l(\lambda) \leq N$, Young tableau as above :
$V_\lambda = \mathcal{U}_q(sl_N) \cdot v_\lambda \subseteq V^{\otimes r}$ highest weight submodule of weight $\sum \lambda_k L_k$

Definition : 1-dimensional simple representations

$\vec{\varepsilon} \in \{1, -1\}^n, \ T_{\vec{\varepsilon}}(K_i) = \varepsilon_i, \ T_{\vec{\varepsilon}}(E_i) = T_{\vec{\varepsilon}}(F_i) = 0$

Theorem (Lusztig, Rosso)

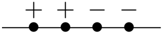
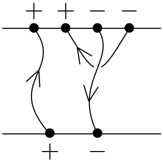
$Rep_q(sl_n)$ is semi-simple and its simples are the $T_{\vec{\varepsilon}} \otimes V_\lambda$.

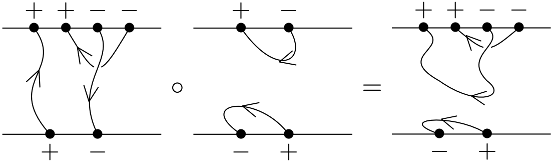
Corollary

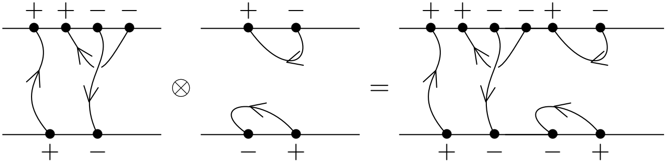
Submodules of $V^{\otimes r}$ give all type 1 modules

The Reshetikhin–Turaev functor on $Rep_q(sl_N)$

Definition : Tan^{fr}

objects :  morphisms : 





Theorem (Turaev)

$\exists ! RT_V : Tan^{fr} \to Rep_q(sl_N)$ monoidal such that $RT_V(+) = V$,

$RT_V(-) = V^*$, $RT_V($  $) = c_{V,V}$, $RT_V($  $) = ev_V$,

$RT_V($  $) = coev_V$ and $RT_V($  $) = \theta_V$.

Example : 

$RT_V($  $) = RT_V \left(\right.$  $\left. \right) = ev_V \circ c_{V,V^*} \circ (\theta_V \otimes Id_{V^*})$

Proposition (Sikora, Lin-Zheng)

The braiding $c_{V,V} =$  $: V \otimes V \to V \otimes V$ is given by :

$$c_{V,V}(v_i \otimes v_j) = q^{-\frac{1}{N}} \begin{cases} q v_i \otimes v_i & \text{if } i = j \\ v_j \otimes v_i & \text{if } i < j \\ v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & \text{if } i > j \end{cases}$$

The twist is $\theta_V =$  $= q^{N - \frac{1}{N}} Id_V : V \to V$.

Duality morphisms are :

$ev_V =$  $: \begin{cases} V^* \otimes V & \to & \mathbb{C}(q) \\ v^i \otimes v_j & \mapsto & \delta_{ij} \end{cases}$,

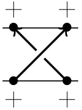
$coev_V =$  $: \begin{cases} \mathbb{C}(q) & \to & V \otimes V^* \\ 1 & \mapsto & \sum_i v_i \otimes v^i \end{cases}$ and for example

$ev_V^R =$  $: \begin{cases} V \otimes V^* & \to & \mathbb{C}(q) \\ v_i \otimes v^j & \mapsto & q^{2i-N-1} \delta_{ij} \end{cases}$

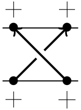
Proposition

$RT_V : Tan^{fr} \to Rep_q(sl_N)$ satisfies :

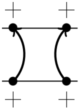
$$q^{\frac{1}{N}} RT_V \left(\right.$$



$$\left. \right) - q^{-\frac{1}{N}} RT_V \left(\right.$$

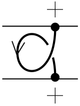


$$\left. \right) = (q - q^{-1}) RT_V \left(\right.$$

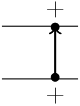


$$\left. \right)$$

and $RT_V \left(\right.$



$\left. \right) = q^{N - \frac{1}{N}} RT_V \left(\right.$



$\left. \right)$

Corollary

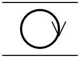
$RT(\mathcal{L}) = \chi_N(\mathcal{L}) \in End(\mathbb{C}(q)) \simeq \mathbb{C}(q)$

Proof

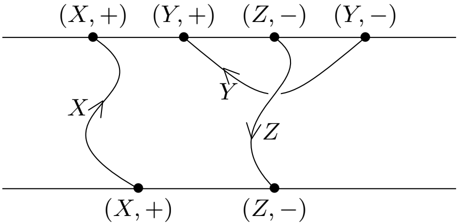
$$c_{V,V}^{-1}(v_i \otimes v_j) = q^{\frac{1}{N}} \begin{cases} q^{-1} v_i \otimes v_i & \text{if } i = j \\ v_j \otimes v_i - (q - q^{-1}) v_i \otimes v_j & \text{if } i < j \\ v_j \otimes v_i & \text{if } i > j \end{cases}$$

So $q^{\frac{1}{N}} c_{V,V}(v_i \otimes v_j) - q^{-\frac{1}{N}} c_{V,V}^{-1}(v_i \otimes v_j) = (q - q^{-1}) v_i \otimes v_j$.

Normalization

RT_V () $= RT_V($  $) \circ RT_V($  $) = \sum_{i=1}^{N} q^{2i-N-1} = [N]_q$

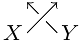
Définition : $Tan^{fr}_{Rep_q(sl_N)}$



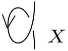
, $X, Y, Z \in Rep_q(sl_N)$

Théorème (Turaev)

$\exists! RT : Tan^{fr}_{Rep_q(sl_N)} \to Rep_q(sl_N)$ monoidal such that

$RT((X,+)) = X$, $RT((X,-)) = X^*$, $RT($  $) = c_{X,Y}$,

$RT($  $) = ev_X$, $RT($  $) = coev_X$ and

$RT($  $) = \theta_X$

The colored HOMFLY polynomial

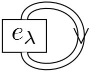
Definition

$\mathcal{L}_{\lambda^1,\dots,\lambda^s}$ colored link $\rightsquigarrow$ $\mathrm{Hom}_{Tan^{fr}_{Rep_q(sl_N)}}(\mathbb{C}(q),\mathbb{C}(q))$

$\chi^c_N(\mathcal{L}_{\lambda^1,\dots,\lambda^s}) := RT(\mathcal{L}_{V_{\lambda^1},\dots,V_{\lambda^s}})$

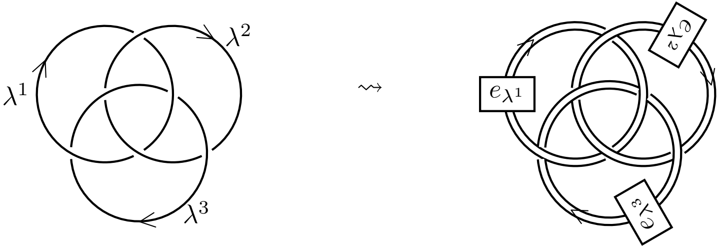
Proposition (Aiston)

$\exists e_\lambda \in Tan^{fr}$ such that $RT_V(e_\lambda) = p_\lambda : V^{\otimes r} \twoheadrightarrow V_\lambda \subseteq V^{\otimes r}$.

set $Q_\lambda = \hat{e_\lambda} =$ 

Proposition

$$\chi_N^c(\mathcal{L}_{\lambda^1,\dots,\lambda^s}) = \chi_N(\mathcal{L} \star (Q_{\lambda^1}, \dots, Q_{\lambda^s}))$$



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