Homotopy (and other) braids

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0. Introduction

Classical braid group can be defined as the fundamental group of configuration space or as the mapping class group of a disc with n punctures. Being a natural object, braids admit generalizations in various directions. Also there are special types of braids defined among all braids by specific properties.

1. Configuration spaces and braids on manifolds

To fix notations let us make the following definitions.
Let M be a topological manifold. Symmetric group Σ_n acts on the Cartesian power M^n of M by the standard formula

$$w(y_1, ..., y_n) = (y_{w(1)}, ..., y_{w(n)}), \quad w \in \Sigma_n. \tag{1}$$

We denote by $F(M, n)$ the space of n-tuples of pairwise different points in M:

$$F(M, n) = \{(p_1, ..., p_n) \in M^n : p_i \neq p_j \text{ for } i \neq j\}.$$

It is called sometimes as *ordered configuration space of n points on* M.

The orbit space of this action $B(M, n) = F(M, n)/\Sigma_n$ is the *(unordered) configuration space of n points on M*. The braid group of the manifold M $Br_n(M)$ is the fundamental group of configuration space

$$Br_n(M) = \pi_1(B(M, n)).$$

The fundamental group of the ordered configuration space is called the pure (or colored) braid group of the manifold M

$$P_n(M) = \pi_1(F(M, n)).$$

The free action of Σ_n on $F(M,n)$ and the projection

$$p : F(M,n) \to F(M,n)/\Sigma_n = B(M,n)$$

defines a covering. The initial segment of the long exact sequence of this covering is as follows:

$$1 \to P_n(M) \xrightarrow{p_*} Br_n(M) \to \Sigma_n \to 1. \tag{2}$$

The canonical example is $M = \mathbb{R}^2$, so $F(\mathbb{R}^2, n)$ and $B(\mathbb{R}^2, n)$ are the configuration spaces (ordered and unordered) of n points on a plane.

2. Artin presentation for braid group

Artin presentation of the braid group Br_n has generators σ_i, $i = 1, ..., n-1$ and relations:

$$\begin{cases} \sigma_i \sigma_j & = \sigma_j \, \sigma_i, \quad \text{if} \quad |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

Define the elements $a_{i,j}$, $1 \leq i < j \leq n$, of Br_n by:

$$a_{i,j} = \sigma_{j-1} ... \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} ... \sigma_{j-1}^{-1}.$$

Geometrically generator of this type is depicted as follows

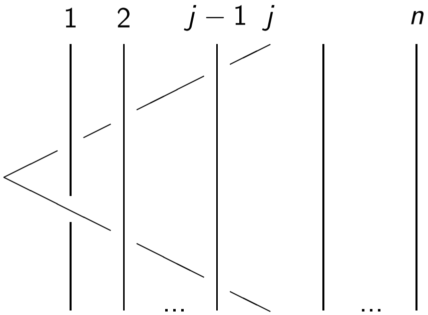


Figure: Generator $a_{1,j}$

They satisfy the Burau relations:

$$\begin{aligned} a_{i,j}a_{k,l} = a_{k,l}a_{i,j} &\text{ for } i<j<k<l \text{ and } i<k<l<j, \\ a_{i,j}a_{i,k}a_{j,k} = a_{i,k}a_{j,k}a_{i,j} &\text{ for } i<j<k, \\ a_{i,k}a_{j,k}a_{i,j} = a_{j,k}a_{i,j}a_{i,k} &\text{ for } i<j<k, \\ a_{i,k}a_{j,k}a_{j,l}a_{j,k}^{-1} = a_{j,k}a_{j,l}a_{j,k}^{-1}a_{i,k} &\text{ for } i<j<k<l. \end{aligned} \tag{3}$$

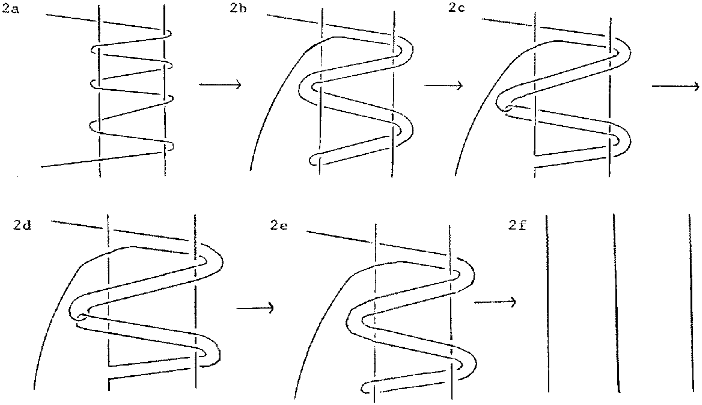
W. Burau proved that this gives a presentation of the pure braid group P_n.

4. Homotopy braids

4.1 Definitions

Two geometric braids with the same endpoints are called *homotopic* if one can be deformed to the other by simultaneous homotopies of the braid strings in $D^2 \times I$ which fix the endpoints, so that different strings do not intersect.

E. Artin asked the question that if the notion of isotopic and homotopic of braids are the same. The question remained open until 1974, when D. Goldsmith gave an example of a braid which is not trivial in the isotopic sense, but is homotopic to the trivial braid.


2a
2b
2c
2d
2e
2f

This braid is expressed in the canonical generators of the classical braid group in the following form:

$$\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-1}.$$

4.2. Reduced free group

Let $F_n = F(x_1, \ldots, x_n)$ be a free group on generators $x_1, \ldots, x_n$. Consider a subgroup generated by the commutators

$$g x_1 g^{-1} x_1^{-1}, \ \ldots, \ g x_n g^{-1} x_n^{-1},$$

where g is an arbitrary element of $F(x_1, \ldots, x_n)$. It is a normal subgroup of $F(x_1, \ldots, x_n)$; let us denote it by N. The quotient group $K_n = F(x_1, \ldots, x_n)/N$ is called the *reduced free group*. It was introduced by J.Milnor and studied by Habegger & Lin, F.Cohen and F.Cohen & Jie Wu.

Theorem (Habegger and Lin)

K_n is a finitely generated nilpotent group of class $\leq n$.

Let $a_{i,j}$ be the standard (Burau) generators of the pure braid group. The *homotopy braid group* $\widetilde{B}_n$ is the quotient of the braid group B_n by the relations

$$[a_{ik}, a_{ik}^g] = 1, \text{ where } g \in \langle a_{1k}, a_{2k}, \ldots, a_{k-1,k} \rangle, 1 \leq i < k \leq n.$$

The quotient of the pure braid group P_n by the same relations gives the *pure homotopy braid group* $\widetilde{P}_n$ and from the standard short exact sequence for B_n we have the following short exact sequence

$$1 \longrightarrow \widetilde{P}_n \longrightarrow \widetilde{B}_n \longrightarrow S_n \longrightarrow 1,$$

for $\widetilde{B}_n$, where S_n is the symmetric group.

Group $\widetilde{P}_n$ has a decomposition $\widetilde{P}_n = \widetilde{U}_n \leftthreetimes \widetilde{P}_{n-1}$, where $\widetilde{U}_n$ is the quotient of the free group $U_n = \langle a_{1n}, a_{2n}, \dots, a_{n-1,n} \rangle$ of rank $n-1$ by the relations

$$[a_{in}, a_{in}^g] = 1, \text{ where } g \in U_n, 1 \leq i < k \leq n,$$

Note, that $\widetilde{U}_n$ is isomorphic to K_{n-1}. In particular, $\widetilde{U}_2$ is isomorphic to the infinite cyclic group and $\widetilde{U}_3$ is the quotient of $U_3 = \langle a_{13}, a_{23} \rangle$ by the relations

$$a_{13} \cdot a_{23}^{-1} a_{13} a_{23} = a_{23}^{-1} a_{13} a_{23} \cdot a_{13},$$

$$a_{23} \cdot a_{13}^{-1} a_{23} a_{13} = a_{13}^{-1} a_{23} a_{13} \cdot a_{23}.$$

Homotopy braids

F.Cohen & Jie Wu proved that the canonical Artin monomorphism

$$\nu_n : B_n \hookrightarrow \mathrm{Aut}\, F_n$$

generates a homomorphism

$$\tilde{\nu}_n : \tilde{B}_n \to \mathrm{Aut}\, K_n.$$

The homomorphism $\tilde{\nu}_n$ is a monomorphism.

Proposition

The monomorphism $\tilde{\nu}_n$ solves the word problem in $\tilde{B}_n$

Since K_n is a finitely generated nilpotent group of class $\leq n$ then from result of A.I.Mal'cev follows that the word problem is decidable in K_n. From the fact that $\widetilde{B}_n$ is a finite extension of $\widetilde{P}_n$ follows that the word problem is decidable in $\widetilde{B}_n$.

4.3. Linearity

Existence

Recall that a group G is called *linear* if it has a faithful representation into the general linear group $GL_m(k)$ for some m and a field k.

Theorem

The homotopy braid group $\widetilde{B}_n$ *is linear for all* $n \geq 2$*. Moreover, for every* $n \geq 2$ *there is a natural* m *such that there exists a faithful representation* $\widetilde{B}_n \longrightarrow GL_m(\mathbb{Z})$

Proof. The reduced free group K_n, $n \geq 2$ is nilpotent. Finitely generated nilpotent groups are polycyclic and hence they are represented by integer matrices as was proved by L.Auslender and R.G.Swan. Also the holomorph of every polycyclic group has a faithful representation into $GL_m(\mathbb{Z})$ for some m. Hence, holomorph $Hol(K_n)$ has a faithful representation into $GL_m(\mathbb{Z})$ for some m. The holomorph of the reduced free group $Hol(K_n)$ contains $Aut(K_n)$ as a subgroup and $\widetilde{B}_n$ is embedded into $Aut(K_n)$. $\square$

It is interesting to find a faithful linear representation of $\widetilde{B}_n$ explicitly. One can try to factor through $\widetilde{B}_n$ the known representations of B_n, for example, Burau representation.

4.4. Factorization of the Burau representation through $\widetilde{B}_n$

Let

$$\rho_B : B_n \longrightarrow GL(W_n)$$

be the Burau representation of B_n, where W_n is a free $\mathbb{Z}[t^{\pm 1}]$-module of rank n with the basis $w_1, w_2, \ldots, w_n$.

Let $n = 3$. In this case the automorphisms $\rho_B(\sigma_i)$, $i = 1, 2$, of module W_3 act by the rule

$$\sigma_1 : \begin{cases} w_1 \longmapsto (1-t)w_1 + tw_2, \\ w_2 \longmapsto w_1, \\ w_3 \longmapsto w_3, \end{cases} \qquad \sigma_2 : \begin{cases} w_1 \longmapsto w_1, \\ w_2 \longmapsto (1-t)w_2 + tw_3, \\ w_3 \longmapsto w_2, \end{cases}$$

where we write for simplicity σ_i instead of $\rho_B(\sigma_i)$.

Let us find the action of the generators of P_3 on the module W_3. Recall, that $P_3 = U_2 \rtimes U_3$, where U_2 is the infinite cyclic group with the generator $a_{12} = \sigma_1^2$, U_3 is the free group of rank 2 with the free generators

$$a_{13} = \sigma_2 \sigma_1^2 \sigma_2^{-1}, \quad a_{23} = \sigma_2^2.$$

These elements define the following automorphisms of W_3

$$a_{12} : \begin{cases} w_1 \longmapsto (1 - t + t^2) w_1 + t(1-t) w_2, \\ w_2 \longmapsto (1-t) w_1 + t w_2, \\ w_3 \longmapsto w_3, \end{cases} \tag{4}$$

$$a_{13}:\begin{cases} w_1 \longmapsto (1-t+t^2)w_1 + t(1-t)w_3, \\ w_2 \longmapsto (1-t)^2 w_1 + w_2 - (1-t)^2 w_3, \\ w_3 \longmapsto (1-t)w_1 + tw_3, \end{cases} \tag{5}$$

$$
a_{23} : \begin{cases} w_1 \longmapsto w_1, \\ w_2 \longmapsto (1 - t + t^2) w_2 + t(1 - t) w_3, \\ w_3 \longmapsto (1 - t) w_2 + t w_3, \end{cases} \tag{6}
$$

$$a_{23}^{-1} : \begin{cases} w_1 \longmapsto w_1, \\ w_2 \longmapsto t^{-1} w_2 + (1 - t^{-1}) w_3, \\ w_3 \longmapsto t^{-1}(1 - t^{-1}) w_2 + (1 - t^{-1} + t^{-2}) w_3. \end{cases} \tag{7}$$

Let us denote by $\widetilde{\rho}_B$ the representation

$$\widetilde{\rho}_B : \widetilde{B}_n \longrightarrow GL(W_n)$$

which is the factorization of ρ_B through $\widetilde{B}_n$.

Proposition

The factorization of the representation $\widetilde{\rho}_B$ *on* $\widetilde{P}_3$ *is trivial. Hence, the image* $\widetilde{\rho}_B(B_3)$ *is isomorphic to the symmetric group* S_3*.*

Proof. To get a representation of $\widetilde{\rho}_B(B_3)$ we must have the following relations among the automorphisms $a_{i,j}$ (4)-(6) of W_3:

$$[a_{13}, a_{13}^{a_{23}}] = 1, \quad [a_{23}, a_{23}^{a_{13}}] = 1,$$

which are equivalent to the relations

$$a_{13} a_{13}^{a_{23}} = a_{13}^{a_{23}} a_{13}, \quad a_{23} a_{23}^{a_{13}} = a_{23}^{a_{13}} a_{23}.$$

From the definitions the automorphisms (4)-(7) we obtain

$$a_{23}^{-1}a_{13}a_{23}:\left\{\begin{array}{l} w_1 \longmapsto (1-t+t^2)w_1 + t(1-t)^2 w_2 + t^2(1-t)w_3, \\ w_2 \longmapsto w_2, \\ w_3 \longmapsto t^{-1}(1-t)w_1 - t^{-1}(1-t)^2 w_2 + tw_3. \end{array}\right.$$

$$
a_{13}a_{13}^{a_{23}}:\begin{cases}
w_1 \longmapsto & (2-4t+4t^2-2t^3+t^4)w_1+\\
& (1-t)^2(-1-t^2+t^3)w_2+\\
& t^2(1-t)(2-t+t^2)w_3,\\
w_2 \longmapsto & (1-t)^2(-t^{-1}+2-t+t^2)w_1+\\
& [(1-t)^4(t+t^{-1})+1]w_2+\\
& t(1-t)^2(-1+t-t^2)w_3,\\
w_3 \longmapsto & (1-t)(2-t+t^2)w_1+\\
& (1-t)^2[-1+t-t^2]w_2+\\
& +t^2(2-2t+t^2)w_3.
\end{cases}
$$

$$a_{13}^{a_{23}} a_{13} : \begin{cases} w_1 \longmapsto & (1 - t + 2t^3 - 2t^4 + t^5) w_1 + t(1-t)^2 w_2 + \\ & + t(1-t)(1 - 2t + 5t^2 - 3t^3 + t^4) w_3, \\ w_2 \longmapsto & (1-t)^2 w_1 + w_2 - (1-t)^2 w_3, \\ w_3 \longmapsto & (1-t)(2 - t + t^2) w_1 - t^{-1}(1-t)^2 w_2 + \\ & + [(1-t)^2(1 + t - 2t^2 + t^3) + t^2] w_3. \end{cases}$$

In order to satisfy relation $a_{13}a_{13}^{a_{23}} = a_{13}^{a_{23}}a_{13}$ the following system of equations should have a solution

$$\begin{cases} 1-3t+4t^2-4t^3+3t^4-t^5=0, \\ (1-t)^2(-1-t-t^2+t^3)=0, \\ t(1-t)^5=0, \\ (1-t)^2(-t^{-1}+1-t+t^2)=0, \\ t^{-1}(1-t)^4(1+t^2)=0, \\ (1-t)^2(1-t+t^2-t^3)=0, \\ (1-t)^2(-1+t-t^2+t^{-1})=0, \\ 1-t-4t^2+8t^3-5t^4+t^5=0. \end{cases}$$

This system has a solution only if $t = 1$. In this case, automorphisms a_{12}, a_{13}, a_{23} are equal to the identity automorphism. $\Box$

4.5. Torsion in $\widetilde{B}_n$

V.Ya.Lin in Kourovka Notebook asked the following

Question

Is there a non-trivial epimorphisms of B_n onto a non-abelian group without torsion?

An answer to this question was done by P. Linnell and T. Schick in 2007.

Homotopy braids

We conjecture that the group $\widetilde{B}_n$, $n \geq 3$, does not have torsion and as there exists the epimorphism $B_n \longrightarrow \widetilde{B}_n$, then $\widetilde{B}_n$ is a good candidate for another solution of the Lin's problem.
We shall prove that $\widetilde{B}_3$ does not have torsion.

Let $\widetilde{P}_3$, $\widetilde{U}_2$, $\widetilde{U}_3$ be the images of P_3, U_2, U_3 by the canonical epimorphism $B_3 \longrightarrow \widetilde{B}_3$. Denote by b_{ij}, $1 \leq i < j \leq 3$ the images of a_{ij}, $1 \leq i < j \leq 3$ by this epimorphism. Then $\widetilde{U}_2 = \langle b_{12} \rangle$ is the infinite cyclic group and

$$\widetilde{U}_3 = \langle b_{13}, b_{23} \mid\mid [b_{13}, b_{13}^{b_{23}}] = [b_{23}, b_{23}^{b_{13}}] = 1 \rangle =$$

$$= \langle b_{13}, b_{23} \mid\mid [b_{13}, b_{13}[b_{13}, b_{23}]] = [b_{23}, b_{23}[b_{23}, b_{13}]] = 1 \rangle.$$

Using commutator identities or direct calculations we see that the last two relations are equivalent to the following relation

$$[[b_{23}, b_{13}], b_{23}] = [[b_{23}, b_{13}], b_{13}] = 1.$$

Hence, $\widetilde{U}_3$ is a free 2-step nilpotent group of rank 2 and so, every element $g \in \widetilde{U}_3$ has a unique presentation of the form

$$g = b_{13}^{\alpha} b_{23}^{\beta} [b_{23}, b_{13}]^{\gamma}$$

for some integers α, β, γ.

The same way as in the case of classical braid group, $\widetilde{U}_3$ is a normal subgroup of $\widetilde{P}_3$ and the action of $\widetilde{U}_2$ is defined in the following lemma.

Lemma

The action of $\widetilde{U}_2$ on $\widetilde{U}_3$ is given by the formulas

$$b_{13}^{b_{12}^k} = b_{13}[b_{23}, b_{13}]^k, \quad b_{23}^{b_{12}^k} = b_{23}[b_{23}, b_{13}]^{-k},$$

$$[b_{23}, b_{13}]^{b_{12}^k} = [b_{23}, b_{13}], \quad k \in \mathbb{Z}. \ \square$$

The action of the generators σ_1 and σ_2 of $\widetilde{B}_3$ on $\widetilde{P}_3$ is given in the next lemma.

Lemma

The following conjugation formulas hold in $\widetilde{B}_3$

$$b_{12}^{\sigma_1^{\pm 1}} = b_{12}, \quad b_{13}^{\sigma_1} = b_{23}[b_{23}, b_{13}]^{-1}, \quad b_{23}^{\sigma_1} = b_{13}, b_{13}^{\sigma_1^{-1}} = b_{23},$$

$$b_{23}^{\sigma_1^{-1}} = b_{13}[b_{23}, b_{13}]^{-1}, [b_{23}, b_{13}]^{\sigma_1^{-1}} = [b_{23}, b_{13}]^{-1},$$

$$b_{12}^{\sigma_2} = b_{13}[b_{23}, b_{13}]^{-1}, \quad b_{13}^{\sigma_2} = b_{12}, \quad b_{23}^{\sigma_2^{\pm 1}} = b_{23}, \quad b_{12}^{\sigma_2^{-1}} = b_{13},$$

$$b_{13}^{\sigma_2^{-1}} = b_{12}[b_{23}, b_{13}]^{-1}, [b_{23}, b_{13}]^{\sigma_2^{-1}} = [b_{23}, b_{13}]^{-1}. \quad \square$$

Let us denote by $\Lambda_3 = \{e, \sigma_1, \sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1\}$ the set of representatives of $\widetilde{P}_3$ in $\widetilde{B}_3$. Then every element in $\widetilde{B}_3$ can be written in the form

$$b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^{\delta} \lambda, \text{ where } \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \quad z = [b_{23}, b_{13}], \quad \lambda \in \Lambda_3.$$

Theorem

The group $\widetilde{B}_3$ does not have torsion.

Proof. The group $\widetilde{P}_3$ does not have torsion. Hence, if $\widetilde{B}_3$ has elements of finite order, then they have the form

$$b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^{\delta} \lambda, \quad \lambda \in \Lambda_3 \setminus \{e\}.$$

Every element which is conjugate with an element of finite order has a finite order. Taking into account the following formulas

$$\sigma_1^{-1}{\cdot}\sigma_2{\cdot}\sigma_1 = b_{12}^{-1}\sigma_1\sigma_2\sigma_1,\ \ \sigma_2\sigma_1{\cdot}\sigma_2{\cdot}\sigma_1^{-1}\sigma_2^{-1} = \sigma_1,\ \ \sigma_1^{-1}{\cdot}\sigma_1\sigma_2{\cdot}\sigma_1 = \sigma_2\sigma_1,$$

it is sufficient to consider only two cases: $\lambda = \sigma_2$ and $\lambda = \sigma_1\sigma_2$.

Let $\lambda = \sigma_2$, take $g = b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^{\delta} \sigma_2$. Then we have

$$g^2 = b_{12}^{\alpha+\beta} b_{13}^{\alpha+\beta} b_{23}^{2\gamma+1} z^{\alpha\gamma+\beta(\beta-\gamma+\alpha-1)}.$$

If $g^2 = 1$, then $\alpha + \beta = 0$ and we have

$$g^2 = b_{23}^{2\gamma+1} z^{2\alpha\gamma+\alpha}.$$

Since $2\gamma + 1$ cannot be zero for integer γ, the elements of this form cannot be of finite order.

Let $\lambda = \sigma_1\sigma_2$. Then we have

$$(\sigma_1\sigma_2)^2 = b_{12}\sigma_2\sigma_1, \quad (\sigma_1\sigma_2)^3 = b_{12}b_{13}b_{23}.$$

We calculate

$$g^3 = (b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^{\delta} \sigma_1 \sigma_2)^3 =$$
$$b_{12}^{\alpha+\beta+\gamma+1} b_{13}^{\alpha+\beta+\gamma+1} b_{23}^{\alpha+\beta+\gamma+1} z^{\alpha(\alpha+2\gamma-\beta)+\beta^2+\gamma^2-\beta\gamma+3\delta+3\beta}.$$

If $g^3 = 1$, then the following system of linear equations has a solution over $\mathbb{Z}$

$$\begin{cases} \alpha + \beta + \gamma + 1 = 0, \\ \alpha(\alpha + 2\gamma - \beta) + \beta^2 + \gamma^2 - \beta\gamma + 3\delta + 3\beta = 0. \end{cases}$$

From the first equation one gets: $\alpha = -1 - \beta - \gamma$. Inserting this equality into the second equation, we have

$$3(\beta^2 + 2\beta + \delta) + 1 = 0.$$

However, this equation does not have integer solutions. $\square$

5. Brunnian Braids

5.1. Operations d_i

The operation $d_i\colon B_n(M) \to B_{n-1}(M)$ is obtained by forgetting the i-th strand.

Proposition

Let M be a surface. The operations

$$d_i \colon B_n(M) \to B_{n-1}(M),\ 1 \leq i \leq n,$$

satisfy the following identities:

1) $d_i d_j = d_j d_{i+1}$ *for* $i \geq j$*;*
2) $d_i(\beta\beta') = d_i(\beta) d_{i\cdot\beta}(\beta')$.

Corollary

The map d_i is homomorphism when restricted to the pure braid group $P_n(M)$.

5.2. Brunnian braids

Definition

A braid $\beta \in B_n(M)$ is called Brunnian if it is a solution of system of n equations

$$\begin{cases} d_1(\beta) = 1, \\ \dots \\ d_n(\beta) = 1. \end{cases} \tag{8}$$

The set of n-strand Brunnian braids is denoted by $\mathrm{Brun}_n(M)$.

Intuitively a Brunnian braid means a braid that becomes trivial after removing any one of its strands.
If $\beta, \beta' \in \mathrm{Brun}_n(M)$, then

$$d_i(\beta\beta') = d_i(\beta) d_{i \cdot \beta}(\beta') = 1$$

for $1 \leq i \leq n$ and so the product $\beta\beta' \in \mathrm{Brun}_n(M)$. Similar β^{-1} is Brunnian provided β is. Thus $\mathrm{Brun}_n(M)$ is a subgroup of $B_n(M)$.

6. Lie algebras and braids

 6.1. Lie algebras from descending central series of groups

For a group G the descending central series

$$G = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_i \geq \Gamma_{i+1} \geq \ldots .$$

is defined by the formulae

$$\Gamma_1 = G, \quad \Gamma_{i+1} = [\Gamma_i, G].$$

The descending central series of a discrete group G gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $L(G)$

$$L_i(G) = \Gamma_i(G)/\Gamma_{i+1}(G).$$

Let K be a commutative ring with unit.

Definition. An algebra L over K is called a Lie algebra over K if its multiplication (denoted by $(x, y) \mapsto [x, y]$) verifies identities:

(1) $[x, x] = 0$

(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all x, y, z in L.

6.2. Presentation of the Lie algebra $L(P_n)$

This presentation was given by Toshitake Kohno. It is the quotient of the free Lie algebra $L[A_{i,j} \mid 1 \leq i < j \leq n]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$ modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given by the following three relations:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0, \text{ if } i < j < k, \\ [A_{i,k}, A_{i,j} + A_{j,k}] = 0, \text{ if } i < j < k. \end{cases} \tag{9}$$

Where $A_{i,j}$ is the projection of the $a_{i,j}$ to $L(P_n)$.

6.3. Lie algebra $\mathrm{L}^{\mathrm{P}}(\mathrm{Brun_n})$

We consider the restriction $\{\Gamma_q(P_n) \cap \mathrm{Brun}_n\}$ of the lower central series of P_n to Brun_n. This gives a relative Lie algebra

$$L^P(\mathrm{Brun}_n) = \bigoplus_{q=1}^{\infty} (\Gamma_q(P_n) \cap \mathrm{Brun}_n)/(\Gamma_{q+1}(P_n) \cap \mathrm{Brun}_n),$$

which is a two-sided Lie ideal of $L(P_n)$. The purpose is to study the Lie algebra $L^P(\mathrm{Brun}_n)$.
This is a Lie subalgebra of $L(P_n)$, we call it the *relative Lie algebra associated with Brunnian subgroup* of the pure braid group.

Lie algebra $\mathrm{L^P(Brun_n)}$

Proposition

$\mathrm{L^P(Brun_n)}$ *is a two-sided Lie ideal in* $L(P_n)$.

6.4. Definition and properties of $\mathrm{L^P(Brun_n)}$

The removing-strand operation on braids induces an operation

$$d_k\colon L(P_n) \longrightarrow L(P_{n-1})$$

formulated by

$$d_k(A_{i,j}) = \begin{cases} A_{i,j} & \text{if} \quad i < j < k \\ 0 & \text{if} \quad k = j \\ A_{i,j-1} & \text{if} \quad i < k < j \\ 0 & \text{if} \quad k = i \\ A_{i-1,j-1} & \text{if} \quad k < i < j. \end{cases} \tag{10}$$

Proposition

The relative Lie algebra $\mathrm{L}^{\mathrm{P}}(\mathrm{Brun_n})$ *is the Lie subalgebra* $\bigcap_{i=1}^{n} \ker(d_i : L(P_n) \to L(P_{n-1}))$.

6.5. Generators for the Lie algebra $\mathrm{L^P(Brun_n)}$

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The following fact is a Lie algebra analogue of the theorem proved by A. A. Markov for the pure braid group.

Proposition

The kernel of the homomorphism $d_n : L(P_n) \to L(P_{n-1})$ is a free Lie algebra, generated by the free generators $A_{i,n}$, for $1 \leq i \leq n-1$.

$$\mathrm{Ker}(d_n : L(P_n) \to L(P_{n-1})) = L[A_{1,n}, \dots, A_{n-1,n}].$$

For a set Z, let $L[Z]$ denote the free Lie algebra freely generated by Z. Let X and Y be non-empty sets with $X \cap Y = \emptyset$, $X \cup Y = Z$. Let π be the Lie homomorphism

$$\pi \colon L[Z] \longrightarrow L[Y]$$

such that $\pi(x) = 0$ for $x \in X$ and $\pi(y) = y$ for $y \in Y$.

Proposition

The kernel of π is a free Lie algebra, generated by the following family of free generators:

$$x, [\cdots[x, y_1], \ldots, y_t] \tag{11}$$

for $x \in X, y_i \in Y$ for $1 \le i \le t$.

Proposition

The intersection of the kernels of the homomorphisms d_n and d_k, $k \neq n$, is a free Lie algebra, generated by the following infinite family of free generators:

$$A_{k,n}, [\cdots [A_{k,n}, A_{j_1,n}], \dots, A_{j_m,n}] \tag{12}$$

for $j_i \neq k, n$; $j_i \leq n-1$; $i \leq m$; $m \geq 1$:

Another set of free generators of $\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k)$ can be obtained using Hall bases.

We suppose that all Lie monomials on $B_1, \dots, B_k$ are ordered lexicographically.
Lie monomials $B_1, \dots, B_k$ are the *standard* monomials of degree 1.
If we have defined standard monomials of degrees $1, \dots, n-1$, then $[u, v]$ is a *standard* monomial if both of the following conditions hold:
(1) u and v are standard monomials and $u > v$.
(2) If $u = [x, y]$ is the form of the standard monomial u, then $v \geq y$.
Standard monomials form the *Hall basis* of a free Lie algebra (also over $\mathbb{Z}$).

Examples of standard monomials are the products of the type:

$$[\cdots[B_{j_1}, B_{j_2}], B_{j_3}], \ldots, B_{j_t}], \; j_1 > j_2 \leq j_3 \leq \cdots \leq j_t. \tag{13}$$

Proposition

The intersection $\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k)$, $k \neq n$, *is a free Lie algebra, generated by the standard monomials on* $A_{i,n}$ *where the letter* $A_{k,n}$ *has only one enter. In other words the free generators are standard monomials which are products of monomials of type (13) where only one such monomial contains one copy of* $A_{k,n}$.

We recursively define the sets $\mathcal{K}(n)_k$, $1 \leq k \leq n$, in the reverse order as follows:

1) Let $\mathcal{K}(n)_n = \{A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}\}$.
2) Suppose that $\mathcal{K}(n)_{k+1}$ (with $k \leq n-1$) is defined as a subset of Lie monomials on the letters

$$A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}.$$

Let

$$\mathcal{A}_k = \{W \in \mathcal{K}(n)_{k+1} \mid W \text{ does not contain } A_{k,n} \text{ in its entries}\}.$$

3) Define

$$\mathcal{K}(n)_k = \{W' \text{ and } [\cdots[[W', W_1], W_2], \ldots, W_t]\}$$

for $W' \in \mathcal{K}(n)_{k+1} \setminus \mathcal{A}_k$ and $W_1, W_2, \ldots, W_t \in \mathcal{A}_k$ with $t \geq 1$. Note that $\mathcal{K}(n)_k$ is again a subset of Lie monomials on letters $A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}$.

Example

Let $n = 3$. The set $\mathcal{K}(3)_1$ is constructed by the following steps:

1) $\mathcal{K}(3)_3 = \{A_{1,3}, A_{2,3}\}$.
2) $\mathcal{A}_2 = \{A_{1,3}\}$,
$$\mathcal{K}(3)_2 = \{A_{2,3}, [\cdots[A_{2,3}, A_{1,3}], \ldots, A_{1,3}]\}.$$
3) $\mathcal{A}_1 = \{A_{2,3}\}$,
$\mathcal{K}(3)_1 = \{[\cdots[A_{2,3}, A_{1,3}], \ldots, A_{1,3}], A_{2,3}], \ldots, A_{2,3}]\}$.

Theorem

The Lie algebra $\mathrm{L^P(Brun_n)}$ *is a free Lie algebra generated by* $\mathcal{K}(n)_1$ *as a set of free generators.*

6.6. The Rank of $L_q^P(\mathrm{Brun}_n)$

Observe that the Lie algebra $L(P)$ is of finite type in the sense that each homogeneous component $L_k(P_n)$ is a free abelian group of finite rank. Thus the subgroup

$$\mathrm{L}^{\mathrm{P}}(\mathrm{Brun_n}) \cap L_k(P_n)$$

is a free abelian group of finite rank. We give now a formula for the rank of $L_q^P(\mathrm{Brun}_n)$

Corollary

There is a formula

$$\mathrm{rank}(L_q(P_n)) = \sum_{k=0}^{n-1} \binom{n}{k} \mathrm{rank}(L_q^P(\mathrm{Brun}_{n-k}))$$

for each n and q.

Theorem

$$\mathrm{rank}(L_q^P(\mathrm{Brun}_n)) = \sum_{k=0}^{n-1}(-1)^k \binom{n}{k} \mathrm{rank}(L_q(P_{n-k}))$$

for each n and q, where $P_1 = 0$ and, for $m \geq 2$,

$$\mathrm{rank}(L_q(P_m)) = \frac{1}{q}\sum_{k=1}^{m-1}\sum_{d|q}\mu(d)k^{q/d}$$

with μ the Möbis function.

Lie algebra $\mathrm{L}^{\mathrm{P}}(\mathrm{Brun_n})$

Results about Lie algebra $\mathrm{L}^{\mathrm{P}}(\mathrm{Brun_n})$ are from a joint work with Jingyan Li and Jie Wu.