A unified knot invariant

Introduction

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- $(U_q(\mathfrak{sl}_2), S_n) \to J_n(q, \mathcal{K})$ (Colored Jones polynomial)
- $(U_{\zeta_{2r}}(\mathfrak{sl}_2), V_A) \to ADO_r(A, \mathcal{K})$ (ADO polynomial).

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There exist $F_\infty(q, A, \mathcal{K}) \in \widehat{\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]}$ *such that:*

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GOAL: Build this invariant and the ambient ring.

How to produce invariants?

The main tool to produce those invariants are ribbon Hopf algebra:

$$(A, \mu, i, \Delta, \epsilon, S, R, v)$$

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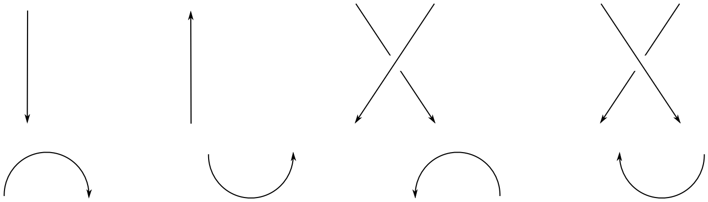
If $R = \sum \alpha \otimes \beta$ we set $u = \sum S(\beta)\alpha$ and $\theta = vu^{-1}$ the pivotal element.

Elementary tangles:



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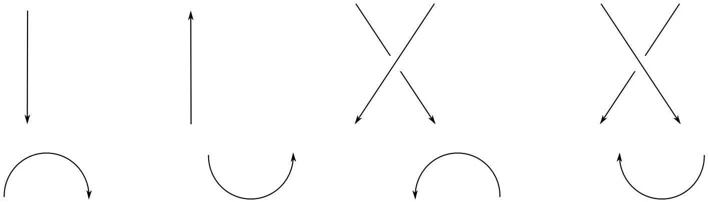
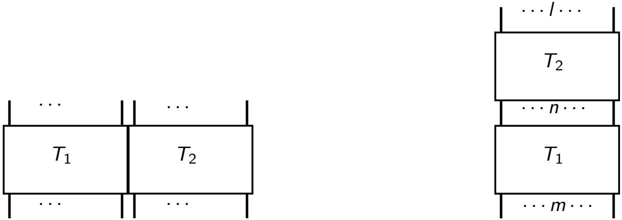
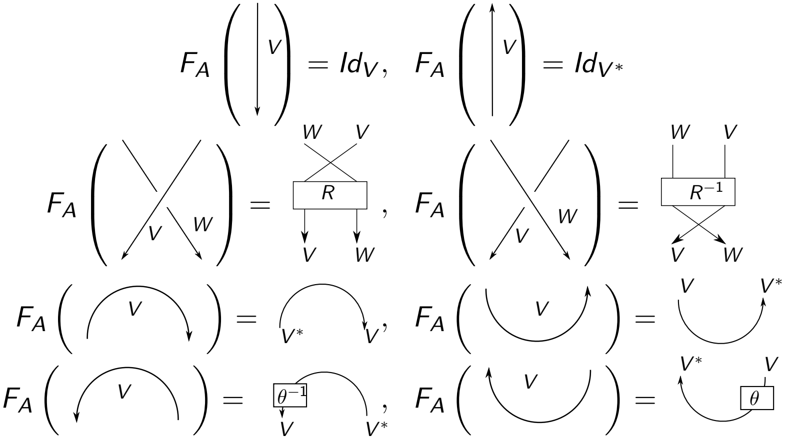


Juxtaposition and concatenation:



RT functor for elementary tangles colored with finite dimensional A-modules:



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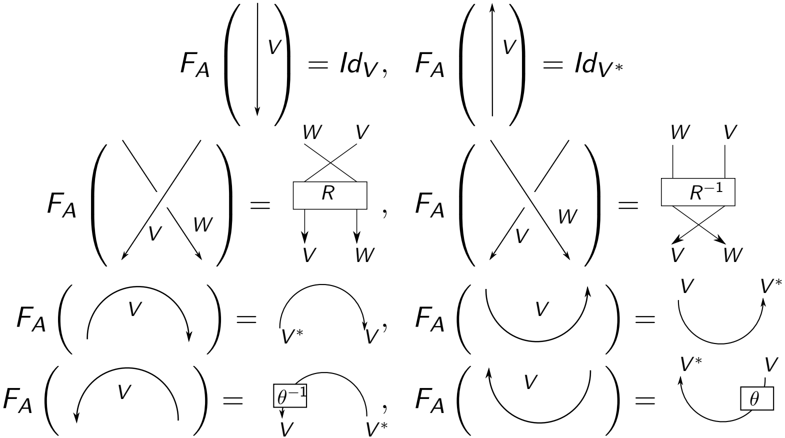

- $F_A(\mathcal{K}, V)$ is a knot invariant.

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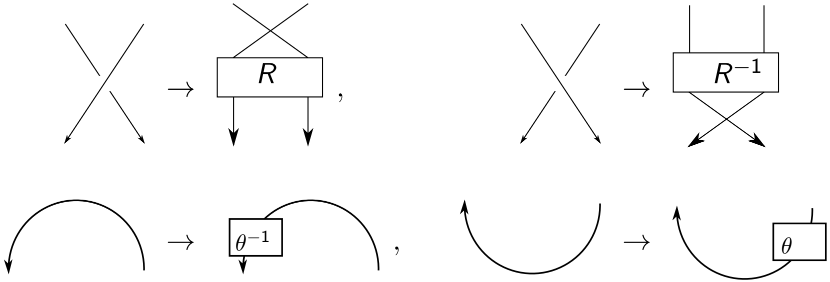

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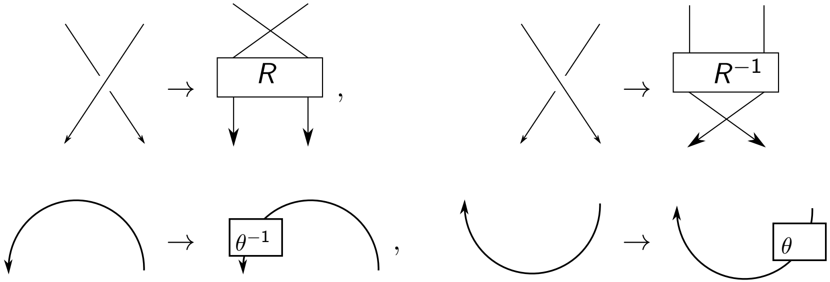

You get $Q^A(\mathcal{K})$ the universal invariant.

The ADO and colored Jones polynomials

Colored Jones example:

- $U_h(\mathfrak{sl}_2)$ is the $\mathbb{Q}((h))$ algebra topologically generated by H, E, F and relations

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

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It's a Hopf algebra.
It has a ribbon structure with

$$R=q^{\frac{H\otimes H}{2}}\sum_n \frac{\{1\}_q^{2n}}{\{n\}_q!}q^{\frac{n(n-1)}{2}}E^n\otimes F^n$$

and $\theta=K^{-1}$.

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- $U_q(\mathfrak{sl}_2) \subset U_h$ is the $\mathbb{Q}(q)$ sub algebra generated by E, F, K. It's also an Hopf algebra. But it's not ribbon since $R \notin U_q \otimes U_q$.

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  Nevertheless, only a finite number of terms in R will have a non zero action on finite dimensional modules.
  - Simples modules: $S_n =< v_0, \ldots, v_n >$ with $Kv_0 = q^n v_0$, $Ev_0 = 0$, $Fv_i = v_{i+1}$.
  - Knot invariant: $(U_q, S_n) \to J_n(q, \mathcal{K})$.

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- More simple modules: V_A the $\mathbb{Q}(\zeta_{2r})[A]$ module freely generated by $(v_0, \dots, v_{r-1})$ endowed with $U_{\zeta_{2r}}$ action: $Kv_0 = Av_0$, $Ev_0 = 0$, $Fv_i = v_{i+1}$.
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The Verma module: Let V^A be the $\mathbb{Q}(q)[A]$ module freely generated by $v_0, v_1, \ldots$ endowed with a U_q action:

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- Problem: It is infinite dimensional, we can't use the quantum invariant machinery.

Algebraic setup to produce the unified invariant.

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- $Q^{U_h} \in U_h$ and V^A the Verma module:

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  - Pb 3: Convergence of R in $U_q \otimes U_q$.

Integral setup:

- Let U the $\mathbb{Z}[q]$ sub algebra of U_h generated by E, $F^{(n)}$ and K with $F^{(n)} = \frac{\{1\}_q^{2n} F^n}{\{n\}_q!}$. (Pb 1 and Pb 2 solved)

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- We build the completed algebra $\hat{U} = \lim\limits_{\overleftarrow{n}} \frac{U}{J_n}$ with $J_n = < F^{(i+k)}\{H+m; n-i\}_q;\ 0 \leq i \leq n,\ m \in \mathbb{Z},\ k \in \mathbb{N} >.$

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- Since $S_n \subset V^A$ with $A = q^n$,

$$J_n(q, \mathcal{K}) v_0 = Q^{U_h} v_0 = F_\infty(q, q^n, \mathcal{K}) v_0.$$

$\rightarrow$ We get back the colored Jones polynomial when $A \rightarrow q^n$.

How to factorise the ADO invariants?

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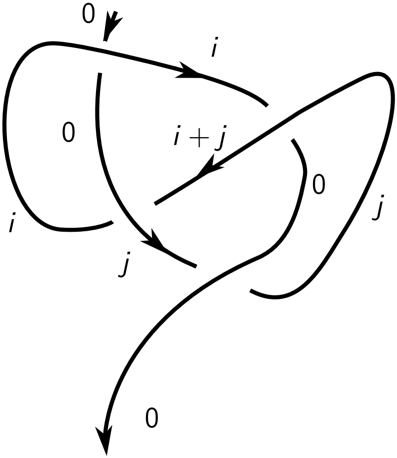

Figure: State diagram of the figure 8 knot

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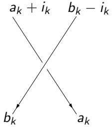

$$\rightarrow q^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_q \{\alpha - a_k; i_k\}_q q^{-(a_k+b_k)\alpha} q^{2(a_k+i_k)(b_k-i_k)}$$

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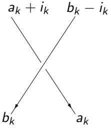

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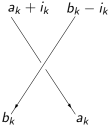
- For ADO, we suppose $a_k, b_k \in \{0, \dots, r-1\}$, let $i_k \in \{0, \dots, r-1\}$:



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with $q = \zeta_{2r}$

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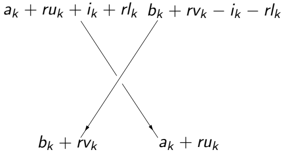

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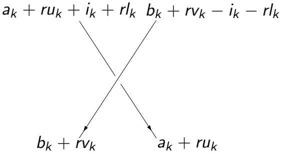

We get the following factorisation:

$$\to \binom{u_k + l_k}{l_k} \{r\alpha\}_{\zeta_{2r}}^{l_k} \zeta_{2r}^{-(u_k+v_k)r\alpha} \times \zeta_{2r}^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_{\zeta_{2r}} \{\alpha - a_k; i_k\}_{\zeta_{2r}} \zeta_{2r}^{-(a_k+b_k)\alpha} \zeta_{2r}^{2(a_k+i_k)(b_k-i_k)}$$

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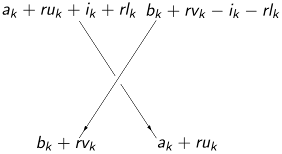

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$$\to \binom{u_k + l_k}{l_k} \{r\alpha\}_{\zeta_{2r}}^{l_k} \zeta_{2r}^{-(u_k+v_k)r\alpha} \times \zeta_{2r}^{\frac{i_k(i_k-1)}{2}} \begin{bmatrix} a_k + i_k \\ i_k \end{bmatrix}_{\zeta_{2r}} \{\alpha - a_k; i_k\}_{\zeta_{2r}} \zeta_{2r}^{-(a_k+b_k)\alpha} \zeta_{2r}^{2(a_k+i_k)(b_k-i_k)}$$

And hence at the level of the invariant:

$$F_\infty(\zeta_{2r}, A, \mathcal{K}) = C_\infty(r, A, \mathcal{K}) \times ADO_r(A, \mathcal{K})$$

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Theorem

There exist $F_\infty(q, A, \mathcal{K}) \in \widehat{\mathbb{Z}[q^{\pm 1}, A^{\pm 1}]}$ *such that:*

$$F_\infty(\zeta_{2r}, A, \mathcal{K}) = \frac{ADO_r(A, \mathcal{K})}{A_{\mathcal{K}}(A^{2r})}, \quad F_\infty(q, q^n, \mathcal{K}) = J_n(q, \mathcal{K})$$

where $\mathcal{K}$ *is a knot in* S^3, *and* $A_{\mathcal{K}}(A)$ *the Alexander polynomial.*

Recap':

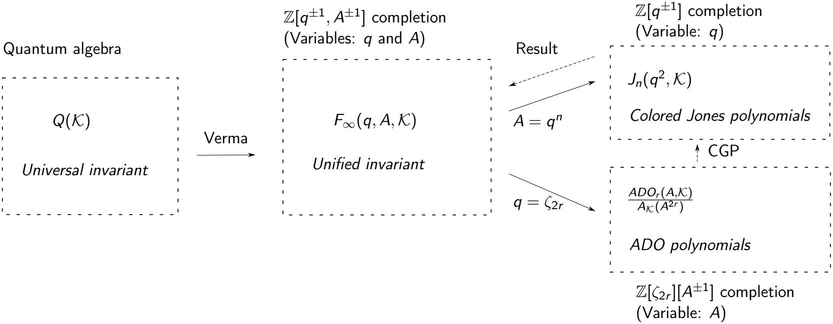

Figure: Visual representation of the unified knot invariant.