Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
Mathematics department

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Hopf algebras
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Sheet 10

Problem 1. Let H be a finite dimensional Hopf algebra, and let M be a H-modules. Prove that $H \otimes M$ and $M \otimes H$ are free (as H-modules).

Problem 2. Let G be a finite group and $H = \mathbb{K}G$ its associated Hopf algebra.

1. What are the right and left integrals of H?
2. What is the distinguished group-like element of H^*?
3. What is the order of the antipode?
4. What is the right and left integrals of H^*?
5. What is the distinguished group-like element of H?
6. Prove that H is a symmetric algebra.

Problem 3. We consider the category of super vector space: object are $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, morphisms are linear maps, and the braiding c is given on homogeneous element by:

$$c_{V,W} \quad : \quad \begin{array}{ccc} V \otimes W & \to W \otimes V \\ v \otimes w & \mapsto & (-1)^{|v||w|} w \otimes v \end{array}$$

where $|v|$ and $|w|$ denote the degree of v and w. If V is a vector space, $T(V)$ can be endowed with a natural $\mathbb{Z}/2\mathbb{Z}$-grading by setting:

$$T_0(V) = \bigoplus_n V^{\otimes 2n} \qquad \text{and} \qquad T_1(V) = \bigoplus_n V^{\otimes 2n+1}.$$

Hence $T(V)$ as a natural structure of super-vector space. An algebra A is called *super-commutative* if it is a super vector space and if $m \circ c = m$.

1. Recall the structure of bi-algebra on $T(V)$.
2. Prove that $(T(V) \otimes T(V), (\mu \otimes \mu) \circ \tau_{T(V) \otimes T(V)})$ is an algebra. Show that the same definition of Δ on V yields a bialgebra structure on $T(V)$ in the category of super-vector spaces.
3. Let I be the ideal of $T(V)$ generated by $\{x \otimes y + y \otimes x | x, y \in V\}$. Prove the $\Lambda(V) = T(V)/I$ is a bialgebra in the category of super vector spaces and that as an algebra it is super-commutative. If $x_1, x_2, \ldots x_k$ are elements of V, we write: $x_1 \wedge x_2 \wedge \cdots \wedge x_k := x_1 \otimes x_2 \otimes \cdots \otimes x_k + I$.
4. Prove that $\Lambda(V)$ is an Hopf algebra in the category of super vector spaces.
5. Let $(v_1, \ldots, v_n)$ be a basis of V. For a k-tuple $I := (i_1, i_2, \ldots, i_k)$ with $1 \leq i_\ell \leq k$ we define $v_I := v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k}$. Prove that the set $\{v_J\}$ is a basis of $\Lambda(V)$ where J runs over the set of all strictly ordered multi-indices, i.e. $i_1 < i_2 < \ldots < i_k$. What is the dimension of $\Lambda(V)$?
6. Compute $\Delta(v_J)$ for all multi-indices $J = (1, 2, \ldots, k)$ with $1 \leq k \leq n$.
7. Let $(v^1, \ldots, v^n)$ be the dual base of $(v_1, \ldots, v_n)$. Show that $\lambda := v^1 \wedge \ldots \wedge v^n \in \Lambda(V^*)$ is a two-sided cointegral for $\Lambda(V)$.

Problem 4. Let $n \geq 2$ and $\lambda \in \mathbb{C}$ be a primitve nth rooth of unity. We consider the $\mathbb{C}$-algebra $H_{n^2}(\lambda)$ generated by C and X and subected to the relations

$$C^n = 1, \qquad X^n = 0, \qquad \text{and} \qquad XC = \lambda CX.$$

We define a comultiplication Δ by setting:

$$\Delta(C) = C \otimes C \quad \Delta(X) = C \otimes X + X \otimes 1$$

1. Prove that $H_{n^2}(\lambda)$ is a Hopf algebra. What is its dimension?
2. Give a list of isomorphism classes of simple modules of $H_{n^2}(\lambda)$.
3. Give a list of isomorphism classes of projective indecomposable modules of $H_{n^2}(\lambda)$.
4. What is the left integral of $H_{n^2}(\lambda)$?
5. What is the distinguished group-like element of $H_{n^2}(\lambda)^*$?