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DER FORSCHUNG | DER LEHRE | DER BILDUNG
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Hopf algebras
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Sheet 10

Problem 1. Let H be a finite dimensional Hopf algebra, and let M be a H-modules. Prove that $H \otimes M$ and $M \otimes H$ are free (as H-modules).

Solution. *Let us start with $H \otimes M$. We just remark that the map $\Delta_M = \Delta_H \otimes \mathrm{id}$ endows H-comodule with a structure of H-comodule. Further it is compatible with the structure of H-module in the sense that $H \otimes M$ is a Hopf-module: one only have to check that Δ_M is a H-module map. Indeed we have:*

$$\begin{aligned}
\Delta_M(h \cdot (x \otimes m)) &= \Delta_M(\sum_{(h)} h_{(1)}x \otimes h_{(2)}m) \\
&= \sum_{(h)} \Delta(h_{(1)}x) \otimes h_{(2)}m) \\
&= \sum_{(h),\,(x)} h_{(1)}x_{(1)} \otimes h_{(2)}x_{(2)} \otimes h_{(3)}m) \\
&= h \cdot \left(\sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes m) \right) \\
&= h \cdot \Delta_M(x \otimes m).
\end{aligned}$$

We know that Hopf modules are free. Forgetting the co-module structure we get that $H \otimes M$ is free.

Let us now inspect the case of $M \otimes H$. First remark that H being finite dimensional, the antipode S is therefor inversible and H^{op} is an Hopf algebra with antipode S^{-1}. Thus, $M \otimes H$ can be thought as a module-H^{op}. From the previous case (actually its symmetric), we obtain that $M \otimes H$ is a right Hopf module over H^{op}. We deduce that it is free as a module-H^{op}, this mean that $M \otimes H$ is a free H-module. □

Problem 2. Let G be a finite group and $H = \mathbb{K}G$ its associated Hopf algebra.

1. What are the right and left integrals of H?

Solution. *Let t be equal to $\sum_{g \in G} g$. I claim that $\mathcal{I}_l(H) = \mathcal{I}_r(H) = \mathbb{K}t$. The Hopf algebra H being finite dimensional we know that $\mathcal{I}_l(H)$ and $\mathcal{I}_r(H)$ are vector spaces of dimension 1. We only need to check that t is in $\mathcal{I}_l(H)$ and in $\mathcal{I}_r(H)$. This is clear because for any element g of G we have:*

$$\begin{aligned}
&g \cdot t = t = \epsilon(g)t \quad \text{and} \\
&t \cdot g = t = \epsilon(g)t.
\end{aligned}$$

□

2. What is the distinguished group-like element of H^*?

Solution. *Let us recall that the distinguished group like element α of H^* is determined by:*

$$x \cdot h = \alpha(h)x \quad \text{for } x \in \mathcal{I}_l(H) \text{ and } h \in H.$$

In our case, the Hopf algebra is uni-modular, so that we have $\alpha = \epsilon$. □

3. What is the order of the antipode?

Solution. *We have an explicit formula for the antipode: $S(g) = g^{-1}$, hence it is clear that S has order 2 if $G \not\simeq (\mathbb{Z}/2\mathbb{Z})^{\times n}$ and has order 1 in this last case.* □

4. Compute the right and left integrals of H^*.

Solution. *Let us denote by e the neutral element of G. Let $\phi : \mathbb{K}G \to \mathbb{K}$ be the linear form given by $\phi(e) = 1$ and $\phi(g) = 0$ for $g \in G \setminus \{e\}$. I claim that $\mathcal{I}_r(H^*) = \mathcal{I}_l(H^*) = \mathbb{K}\phi$. The coevaluation on H^* is given by the evaluation of the unit of H (ie by e)*

Let ψ be an element of H^*. We have for every element g of G:

$$\begin{aligned}
\psi\phi(g) &= \sum_{(g)} \psi(g_{(1)})\phi(g_{(2)}) \\
&= \psi(g)\phi(g) \\
&= \begin{cases} \psi(g) & \text{if g= e,} \\ 0 & \text{else,} \end{cases} \\
&= \psi(e)\phi(g).
\end{aligned}$$

Similarly, we have $\phi\psi(g) = \phi(e)\psi(g)$, this proves that ϕ is a left and a right integral. So that we have $\mathcal{I}_r(H^*) = \mathcal{I}_l(H^*) = \mathbb{K}\phi$. □

5. What is the distinguished group-like element of H?

Solution. *The group like element is e because H^* is unimodular.* □

6. Prove that H is a symmetric algebra.

Solution. *An algebra is symmetric if there exist a linear form φ making $\varphi \circ \mu$ a non degenerate. In our case we can choose $\varphi = \phi$ (see the solution to the previous question). Indeed if $x = \sum_{g \in G} \lambda_g g \neq 0$, we can find an element g of G such that $\lambda_g \neq 0$, and $\varphi(xg^{-1})\lambda_g \neq 0$.* □

Problem 3. We consider the category of super vector space: object are $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, morphisms are linear maps, and the braiding c is given on homogeneous element by:

$$c_{V,W} \quad : \quad \begin{array}{ccc} V \otimes W & \to W \otimes V \\ v \otimes w & \mapsto & (-1)^{|v||w|} w \otimes v \end{array}$$

where $|v|$ and $|w|$ denote the degree of v and w. If V is a vector space, $T(V)$ can be endowed with a natural $\mathbb{Z}/2\mathbb{Z}$-grading by setting:

$$T_0(V) = \bigoplus_n V^{\otimes 2n} \qquad \text{and} \qquad T_1(V) = \bigoplus_n V^{\otimes 2n+1}.$$

Hence $T(V)$ as a natural structure of super-vector space. An algebra A is called *super-commutative* if it is a super vector space and if $m \circ c = m$.

1. Recall the structure of bi-algebra on $T(V)$.

Solution. The multiplication is given by the structure tensor product:

$$\mu_{|V^{\otimes i} \otimes V^{\otimes j}} = \mathrm{id}_{V^{\otimes i+j}}.$$

The comultiplication on V is given by:

$$\Delta_{|V} \quad : \quad \begin{array}{ccc} V & \to & V^{\otimes 0} \otimes V \oplus V \otimes V^{\otimes 0} \subset T(V) \\ v & \mapsto & 1 \otimes v + v \otimes 1. \end{array}$$

As V generate $T(V)$ as an algebra, this determines Δ completely since we want it to be a morphism of algebras. The counit is the canonical isomorphism with $\mathbb{K}$ on $V^{\otimes 0}$ is equal to zero on V. □

2. Prove that $(T(V) \otimes T(V), (\mu \otimes \mu) \circ \tau_{T(V) \otimes T(V)})$ is an algebra. Show that the same definition of Δ on V yields a bialgebra structure on $T(V)$ in the category of super-vector spaces.

Solution. This is true in a more general context: the important here is that c is a braiding. If A and B are two algebras in a braided category then $(A \otimes B, (\mu_A \otimes \mu_B \circ c)$ is an algebra. It is easy to see graphically. Δ is determined by the braiding and its value on V. □

3. Let I be the ideal of $T(V)$ generated by $\{x \otimes y + y \otimes x | x, y \in V\}$. Prove the $\Lambda(V) = T(V)/I$ is a bialgebra in the category of super vector spaces and that as an algebra it is super-commutative. If $x_1, x_2, \ldots x_k$ are elements of V, we write: $x_1 \wedge x_2 \wedge \cdots \wedge x_k := x_1 \otimes x_2 \otimes \cdots \otimes x_k + I$.

Solution. We want to show that I is a two-sided co-ideal. Let us remove the tensor product for the multiplication inside $T(V)$. Any element of I is a sum of elements of the form $t = v(xy + yx)w$ with x and y in V and v and w in $T(V)$. We want to show $\Delta(t)$ is in $I \otimes T(V) + T(V) \otimes I$.

$$\begin{aligned} \Delta(xy + yx) &= \Delta(x)\Delta(y) + \Delta(y)\Delta(x) \\ &= xy \otimes 1 + x \otimes y - y \otimes x + 1 \otimes xy + yx \otimes 1 - x \otimes y + y \otimes x + 1 \otimes yx \\ &= xy \otimes 1 + 1 \otimes xy + yx \otimes 1 + 1 \otimes yx) \\ &= (xy + yx) \otimes 1 + 1 \otimes (xy + yx) \\ &\in I \otimes T(V) + T(V) \otimes I. \end{aligned}$$

Δ being a morphism of algebra, $\Delta(t)$ is as well in $I \otimes T(V) + T(V) \otimes I$. Furthermore, the restriction of the counity on I is equal to zero. All together, this means that the comultiplication on $T(V)$ induces a well-defined comultiplication on $T(V)/I$, hence $T(V)/I$ is a bialgebra. We remark that I is generated by homogeneous element, this implies, that the grading is preserved and hence the $\mathbb{Z}/2\mathbb{Z}$-grading as well. We now want to show that $T(V)/I$ is super-commutative. Let $x = x_1 \otimes \cdots \otimes x_i \in V^{\otimes i}$ and $y = y_1 \otimes \cdots \otimes y_j \in V^{\otimes j}$. We have:

$$\begin{aligned} x \wedge y &= x_1 \wedge x_2 \cdots \wedge x_i \wedge y_1 \wedge y_2 \wedge \cdots \wedge y_j \\ &= (-1)^i y_1 \wedge x_1 \wedge x_2 \cdots \wedge x_i \wedge y_2 \wedge \cdots \wedge y_j \\ &= (-1)^{2i} y_1 \wedge y_2 \wedge x_1 \wedge x_2 \cdots \wedge x_i \wedge y_3 \wedge \cdots \wedge y_j \\ &= (-1)^{ij} y \wedge x \end{aligned}$$

This is the relation we wanted. □

4. Prove that $\Lambda(V)$ is an Hopf algebra in the category of super vector spaces.

Solution. One can check that setting $S(x) = -x$ for $x \in V$ fulfills the requirements. □

5. Let $(v_1, \ldots, v_n)$ be a basis of V. For a k-tuple $I := (i_1, i_2, \ldots, i_k)$ with $1 \leq i_\ell \leq k$ we define $v_I := v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k}$. Prove that the set $\{v_J\}$ is a basis of $\Lambda(V)$ where J runs over the set of all strictly ordered multi-indices, i.e. $i_1 < i_2 < \ldots < i_k$. What is the dimension of $\Lambda(V)$?

Solution. The fact that $v_i v_j = -v_j v_i$ implies that $\Lambda(V)$ is spanned by $\{v_J\}$. We want to show that this family is free Let $\widetilde{V}$ be the vector space spanned by the symbols v_J where J runs over the set of all strictly ordered multi-indices. We consider the linear map ϕ from $T(V)$ to $\widetilde{V}$ which sends an element $v_{i_1} \otimes \ldots v_{i_k}$ to 0 if the i_j are not distinct or to $(-1)^{|\sigma|} v_I$ where I is the only ordered multi-index consisting of the i_j and σ is the permutation mapping $(i_1, \ldots i_k)$ to I. We clearly have $I \subset \ker \phi$, which proves that the family (v_J) is free. The dimension of $\Lambda(V)$ is equal to the number of ordered multi-index, this is equal to 2^n. □

6. Compute $\Delta(v_J)$ for all multi-indices $J = (1, 2, \ldots, k)$ with $1 \leq k \leq n$.

Solution. One shows by induction on k that:

$$\Delta(v_J) = \sum_{i=0}^{k} \sum_{\sigma \in S_{i,k}} s(\sigma) v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(i)} \otimes v_{\sigma(i+1)} \wedge v_{\sigma(i+2)} \wedge \cdots \wedge v_{\sigma(k)}$$

where $S_{i,k}$ is the set of permutations[1] which as applications are increasing on $[1, i]$ and on $[i+1, k]$, and $s(\sigma)$ is the signature of σ. The sign comes from the braiding of the category of super vector spaces. □

7. Let $(v^1, \ldots, v^n)$ be the dual base of $(v_1, \ldots, v_n)$. Show that $\lambda := v^1 \wedge \ldots \wedge v^n \in \Lambda(V^*)$ is a two-sided cointegral for $\Lambda(V)$.

[1]Such permutations are called a $(i, k-i)$-shuffle.

Solution. *First note that if J_1 and J_2 are two ordered multi-indices, we have $v^{J_1}(v_{J_2}) = 0$ if $J_1 \neq J_2$ and $v^{J_1}(v_{J_1}) = (-1)^{(|J_1||J_1|-1)/2}$. We just have to compute $(\mathrm{id}_{\Lambda(V)} \otimes \lambda)\Delta_{\Lambda(V)}(v_J)$ and $(\lambda \otimes \mathrm{id}_{\Lambda(V)})\Delta_{\Lambda(V)}(v_J)$: if $J \neq (1, 2, \ldots, n)$ then*

$$(\mathrm{id}_{\Lambda(V)} \otimes \lambda)\Delta_{\Lambda(V)}(v_J) = (\lambda \otimes \mathrm{id}_{\Lambda(V)})\Delta_{\Lambda(V)}(v_J) = 0 = \lambda(v_J) \cdot 1.$$

If $J = (1, 2, \ldots, n)$, we have:

$$(\mathrm{id}_{\Lambda(V)} \otimes \lambda)\Delta_{\Lambda(V)}(v_J) = (\lambda \otimes \mathrm{id}_{\Lambda(V)})\Delta_{\Lambda(V)}(v_J) = (-1)^{n(n-1)/2} = \lambda(v_J) \cdot 1.$$

□

Problem 4. Let $n \geq 2$ and $\lambda \in \mathbb{C}$ be a primitve nth rooth of unity. We consider the $\mathbb{C}$-algebra $H_{n^2}(\lambda)$ generated by C and X and subected to the relations

$$C^n = 1, \qquad X^n = 0, \qquad \text{and} \qquad XC = \lambda CX.$$

We define a comultiplication Δ by setting:

$$\Delta(C) = C \otimes C \quad \Delta(X) = C \otimes X + X \otimes 1$$

1. Prove that $H_{n^2}(\lambda)$ is a Hopf algebra. What is its dimension?

Solution. *We have to prove that Δ is compatible with the relations and that their exists an antipode. Let us start with Δ. We have:*

$$\begin{aligned}
&\Delta(C)^n = (C \otimes C)^n = C^n \otimes C^n = 1 \otimes 1 = \Delta(1). \\
&\Delta(X)\Delta(C) = \lambda(C^2 \otimes CX + CX \otimes C) = \lambda\Delta(C)\Delta(X) \\
&\Delta(X)^n = \sum_{k=0}^{n} \binom{n}{k}_\lambda C^k X^{n-k} \otimes X^k = X^n \otimes 1 + C^n \otimes X^n = 0
\end{aligned}$$

In the last equality, $\binom{n}{k}_\lambda$ is the polynomial in λ analogue to the binomial coefficient. For $k \in [1, n-1]$, this polynomial is 0 because λ is a primitive root of 1. Of course we have $\epsilon(C) = 1$ and $\epsilon(X) = 0$. The formulas for Δ suggest $S(C) = C^{n-1}$ and $S(X) = -C^{n-1}X$. A base of $H_{n^2}(\lambda)$ is clearly given by $(C^i X^j)_{0 \neq i,j \neq n}$. So that it has dimension n^2. □

2. What is the order of the antipode?

Solution. *We have $S^2(X) = \lambda X$, this show that S has order $2n$.* □

3. Give a list of isomorphism classes of simple modules of $H_{n^2}(\lambda)$.

Solution. *If M is a module, $X \cdot M$ is a sub-module, furthermore, $X^n \cdot M = \{0\}$, this shows that $X \cdot M$ is a strict sub-modules of M. If M is simple, this implies that X acts trivially on M. The action of C is diagonalisable. But on the other hand C preserves its own eigenspace, this implies that the action of C on M is a multiple (a power of λ) of the identity. For ω nth-root of 1, we consider the 1-dimensional $H_{n^2}(\lambda)$-module V_w, where C acts by multiplication by ω and X acts by zero. This gives a list of all the n isomorphism classes of simple $H_{n^2}(\lambda)$-modules.* □

4. Give a list of isomorphism classes of projective indecomposable modules of $H_{n^2}(\lambda)$.

Solution. *We consider the polynomial* $P(t) = 1+t+\cdots+t^{n-1}$ *For* $i \in [0, n-1]$ *we consider* P_i *the sub-vector space of* $H_{n^2}(\lambda)$ *spanned by* $(X^j P(\lambda^i C))_{j\in[0,n-1]}$*. The* P_i*'s are clearly projective and non-isomorphic. On the other hand, every* P_i *contains only one simple modules, so that the* P_i*'s are indecomposable. They form a full list because their sum is isomorphic to* $H_{n^2}(\lambda)$ *as a left module.* □

5. What is the left integral of $H_{n^2}(\lambda)$?

Solution. *Let* $t = P(C)X^{n-1}$*. We claim that* $\mathbb{C}t$ *is the left integral of* $H_{n^2}(\lambda)$*. Indeed:*

$$Ct = t = \epsilon(C)t \quad \textit{and} \quad Xt = 0 = \epsilon(X)t.$$

□

6. What is the distinguished group-like element of $H_{n^2}(\lambda)^*$?

Solution. *Let us denote by* α *the distinguished group-like element of* $H_{n^2}(\lambda)^*$*. We have:*

$$tX = 0 = \alpha(X)t \quad \textit{and} \quad tC = \lambda^{n-1}t = \lambda^{-1}t = \alpha(C)t$$

Moreover we know that α *is group like. We can write the closed formula:*

$$\alpha(C^i X^j) = 0^j \lambda^{-i}.$$

□