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DER FORSCHUNG | DER LEHRE | DER BILDUNG
Algebra and Number Theory
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Hopf algebras
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Sheet 11

Problem 1. We consider the $\mathbb{C}$-algebra H generated by C and X with the relations:

$$C^2 = 1, \quad X^2 = 0 \quad \text{and} \quad CX + XC = 0.$$

1. Show that setting $\Delta(C) = C \otimes C$ and $\Delta(X) = 1 \otimes X + X \otimes C$ yields a well-defined Hopf-algebra.

Solution. *We have seen this algebra (or some variation) already a few times. One should check that the definition of* Δ *is compatible with the relation:*

$$\Delta(C)^2 = C^2 \otimes C^2 = 1 \otimes 1 = \Delta(1)$$
$$\Delta(X)^2 = 1 \otimes X^2 + X \otimes CX + X \otimes XC + X^2 \otimes C^2 = X \otimes (CX + XC) = 0 = \Delta(0)$$
$$\Delta(C)\Delta(X) = C \otimes CX + CX \otimes 1 = -\Delta(C)\Delta(X)$$

The counity is given by $\epsilon(X) = 0$ *and* $\epsilon(C) = 1$. *For* S *we set:* $S(C) = C$, $S(X) = CX$ *and therefor* $S(CX) = CXC = -X$. *One easily checks that this gives indeed an antipode for* H*:*

$$S(C)C = 1 = \epsilon(C)1 \qquad S(1)X + S(X)C = 0 = \epsilon(X)1 \qquad S(C)CX + S(CX)1 = 0 = \epsilon(CX)1$$
$$CS(C) = 1 = \epsilon(C)1 \qquad 1S(X) + XS(C) = 0 = \epsilon(X)1 \qquad CS(CX) + CXS(1) = 0 = \epsilon(CX)1$$

□

2. What is the order of S?

Solution. S *has of course order 4.* □

3. Show that

$$R = \frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)$$

is a universal R-matrix.

Solution. *First we need to show that* R *is invertible in* $H \otimes H$. *We remark that:*

$$\frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C)^2 = 1 \otimes 1$$

If we write $a = \frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C)$ *and* $t = \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)$. *We have* $R = a + t = a(1 + at)$ *further more,* at *is clearly nilpotent (actually* $(at)^2 = 0$*). Hence we have:* $(1 - at)a$ *is the inverse of* R.

We first need to show that $\Delta^{opp}(C)R = R\Delta(C)$ and $\Delta^{opp}(X)R = R\Delta(X)$.

$$\begin{aligned}
\Delta^{opp}(C)R &= (C \otimes C)\left(\frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)\right) \\
&= \frac{1}{2}(C \otimes C + C \otimes 1 + 1 \otimes C - 1 \otimes 1) + \frac{1}{2}(CX \otimes CX + CX \otimes X + X \otimes X - X \otimes CX) \\
&= \frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C)(C \otimes C) + \frac{1}{2}(XC \otimes XC - XC \otimes X + X \otimes X + X \otimes XC) \\
&= \frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C)(C \otimes C) + \frac{1}{2}(X \otimes X - X \otimes XC + XC \otimes XC + XC \otimes X)(C \otimes C) \\
&= R\Delta(C)
\end{aligned}$$

$$\begin{aligned}
&\Delta^{opp}(X)R \\
&= (X \otimes 1 + C \otimes X)\left(\frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)\right) \\
&= (X \otimes 1)\left(\frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)\right) \\
&+ (C \otimes X)\frac{1}{2}\left((1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)\right) \\
&= \frac{1}{2}(X \otimes 1 + X \otimes C + XC \otimes 1 - XC \otimes C) + \frac{1}{2}(C \otimes X + C \otimes XC + 1 \otimes X - 1 \otimes XC) \\
&= \frac{1}{2}(X \otimes 1 + X \otimes C - CX \otimes 1 + CX \otimes C) + \frac{1}{2}(C \otimes X - C \otimes CX + 1 \otimes X + 1 \otimes CX)
\end{aligned}$$

$$\begin{aligned}
&R\Delta(X) \\
&= \left(\frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)\right)(1 \otimes X + X \otimes C) \\
&= \left(\frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)\right)(1 \otimes X) \\
&+ \frac{1}{2}\left((1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{1}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)\right)(X \otimes C) \\
&= \frac{1}{2}(1 \otimes X + 1 \otimes CX + C \otimes X - C \otimes CX) + \frac{1}{2}(X \otimes C + X \otimes 1 + CX \otimes C - CX \otimes 1) \\
&= \Delta^{opp}(X)R.
\end{aligned}$$

Then we need to compute $(\mathrm{id} \otimes \Delta)(R)$, $(\Delta \otimes \mathrm{id})(R)$, $R_{13}R_{23}$ *and* $R_{13}R_{12}$, *but let us first rewrite* R *in a more convenient way:*

$$\begin{aligned}
R &= 1 \otimes \frac{1+C}{2} + C \otimes \frac{1-C}{2} && + \left(1 \otimes \frac{1+C}{2} - C \otimes \frac{1-C}{2}\right)(X \otimes X) \\
&= 1 \otimes \frac{1+C}{2} + C \otimes \frac{1-C}{2} && + (X \otimes X)\left(1 \otimes \frac{1-C}{2} + C \otimes \frac{1+C}{2}\right) \\
&= \frac{1+C}{2} \otimes 1 + \frac{1-C}{2} \otimes C && + \left(\frac{1+C}{2} \otimes C + \frac{1-C}{2} \otimes 1\right)(X \otimes X) \\
&= \frac{1+C}{2} \otimes 1 + \frac{1-C}{2} \otimes C && + (X \otimes X)\left(-\frac{1-C}{2} \otimes C + \frac{1+C}{2} \otimes 1\right)
\end{aligned}$$

Remarking that $\frac{1+C}{2}$ and $\frac{1-C}{2}$, are orthogonal idempotents, we obtain:

$$\begin{aligned}
R_{13}R_{23} &= a_{13}a_{23} + a_{13}t_{23} + t_{13}a_{23} \\
&= 1 \otimes 1 \otimes \frac{1+C}{2} + C \otimes C \otimes \frac{1-C}{2} \\
&\quad + \left(1 \otimes 1 \otimes \frac{1+C}{2} - C \otimes C \otimes \frac{1-C}{2}\right)(1 \otimes X \otimes X) \\
&\quad + (X \otimes 1 \otimes X)\left(1 \otimes C \otimes \frac{1+C}{2} + C \otimes 1 \otimes \frac{1-C}{2}\right) \\
&= (\Delta \otimes \mathrm{id})(a) \\
&\quad + \left(1 \otimes 1 \otimes \frac{1+C}{2} - C \otimes C \otimes \frac{1-C}{2}\right)(1 \otimes X \otimes X) \\
&\quad + \left(1 \otimes C \otimes \frac{1-C}{2} - C \otimes 1 \otimes \frac{1+C}{2}\right)(X \otimes 1 \otimes X) \\
&= (\Delta \otimes \mathrm{id})(a) \\
&\quad + \left(1 \otimes 1 \otimes \frac{1+C}{2} - C \otimes C \otimes \frac{1-C}{2}\right)(1 \otimes X \otimes X) \\
&\quad + \left(1 \otimes 1 \otimes \frac{1-C}{2} - C \otimes C \otimes \frac{1+C}{2}\right)(X \otimes C \otimes X) \\
&= (\Delta \otimes \mathrm{id})(a) \\
&\quad + \left(1 \otimes 1 \otimes \frac{1+C}{2} - C \otimes C \otimes \frac{1-C}{2}\right)((1 \otimes X + X \otimes C) \otimes X) \\
&= (\Delta \otimes \mathrm{id})(a) + (\Delta \otimes \mathrm{id})(t) = (\Delta \otimes \mathrm{id})(R)
\end{aligned}$$

$$
\begin{aligned}
R_{13}R_{12} &= a_{13}a_{12} + a_{13}t_{12} + t_{13}a_{12} \\
&= \frac{1+C}{2} \otimes 1 \otimes 1 + \frac{1-C}{2} \otimes C \otimes C \\
&\quad + \left(C \otimes 1 \otimes \frac{1+C}{2} + 1 \otimes C \otimes \frac{1-C}{2} \right) (X \otimes X \otimes 1) \\
&\quad + (X \otimes 1 \otimes X) \left(1 \otimes 1 \otimes \frac{1+C}{2} - C \otimes C \otimes \frac{1-C}{2} \right) \\
&= (\mathrm{id} \otimes \Delta)(a) \\
&\quad + \left(C \otimes 1 \otimes \frac{1+C}{2} + 1 \otimes C \otimes \frac{1-C}{2} \right) (X \otimes X \otimes 1) \\
&\quad + \left(1 \otimes 1 \otimes \frac{1-C}{2} + C \otimes C \otimes \frac{1+C}{2} \right) (X \otimes 1 \otimes X) \\
&= (\mathrm{id} \otimes \Delta)(a) \\
&\quad + \left(C \otimes C \otimes \frac{1+C}{2} + 1 \otimes 1 \otimes \frac{1-C}{2} \right) (X \otimes X \otimes C) \\
&\quad + \left(1 \otimes 1 \otimes \frac{1-C}{2} + C \otimes C \otimes \frac{1+C}{2} \right) (X \otimes 1 \otimes X) \\
&= (\mathrm{id} \otimes \Delta)(a) \\
&\quad + \left(C \otimes C \otimes \frac{1+C}{2} + 1 \otimes 1 \otimes \frac{1-C}{2} \right) (X \otimes X \otimes C + X \otimes 1 \otimes X) \\
&= (\mathrm{id} \otimes \Delta)(a) \\
&\quad + \left(C \otimes C \otimes \frac{1+C}{2} + 1 \otimes 1 \otimes \frac{1-C}{2} \right) (X \otimes (X \otimes C + 1 \otimes X)) \\
&= (\mathrm{id} \otimes \Delta)(a) + (\mathrm{id} \otimes \Delta)(t) = (\mathrm{id} \otimes \Delta)(R)
\end{aligned}
$$

This shows that R is a universal R-matrix.

□

4. Deform R_1 into R_q with q in $\mathbb{C}$ to obtain a one parameter family of universal R-matrices.

Solution.

$$R_q = \frac{1}{2}(1 \otimes 1 + 1 \otimes C + C \otimes 1 - C \otimes C) + \frac{q}{2}(X \otimes X + X \otimes CX + CX \otimes CX - CX \otimes X)$$

□

5. Relate R_q^{-1} and R_q.

Solution. $R_q^{-1} = \tau_{A,A}(R_q)$ □

Problem 2. Let H be a quasi-triangular Hopf algebra with R-matrix $R = \sum_{(R)} R_{(1)} \otimes R_{(2)}$. Let X be a right H-module and define $\delta : X \to X \otimes H$ by

$$v \mapsto \sum_R vR_{(1)} \otimes R_{(2)}.$$

1. Show that (X, δ) is a right H-comodule.

Solution. We want to show that $(\delta \otimes \mathrm{id}_H) \circ \delta = (\mathrm{id}_V \otimes \Delta) \circ \delta$ *and* $(\mathrm{id}_V \otimes \epsilon) \circ \delta = \mathrm{id}_V$ *Let* v *be an element of* X*. We have:*

$$\begin{aligned}
(\delta \otimes \mathrm{id}_H) \circ \delta(v) &= \sum_R (\delta(vR_1)) \otimes R_2 \\
&= \sum_{R,R'} (vR_1 R'_1 \otimes R'_2 \otimes R_2 \\
&= \sum_{R,R'} (\rho \otimes \mathrm{id}_H \otimes \mathrm{id}_H)(v \otimes R_1 R'_1 \otimes R'_2 \otimes R_2) \\
&= \sum_{R,R'} (\rho \otimes \mathrm{id}_H \otimes \mathrm{id}_H)(v \otimes R_{13} \otimes R'_{12}) \\
&= \sum_R (\rho \otimes \mathrm{id}_H \otimes \Delta_H)(v \otimes R) \\
&= \sum_R vR_1 \otimes \Delta(R_2) \\
(\mathrm{id}_V \otimes \Delta) \circ \delta(v)
\end{aligned}$$

where $R = R'$ *(Note that we need to sum twice, that is why we take two different name for the same object) and* ρ *is the structural map of* X *as a module-*H*. Furthermore, we have:*

$$(\mathrm{id}_V \otimes \epsilon) \circ \delta(v) = \qquad = \sum_R vR_1 \epsilon(R_2)$$

$$= v$$

□

2. Show that the right action and the right coaction on X fulfill the (right-right) Yetter-Drinfeld condition:

$$\begin{aligned}
&(\mathrm{id}_X \otimes \mu) \circ (\tau_{H,X} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes (\delta\rho))(\tau_{X,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_X \otimes \Delta) \\
=&(\rho \otimes \mu) \circ (\mathrm{id}_X \otimes \tau_{H,H} \otimes \mathrm{id}_H) \circ (\delta \otimes \Delta).
\end{aligned}$$

Solution. Let v *be an element of* X *and* h *an element of* H*. We compute:*

$$\begin{aligned}
&(\mathrm{id}_X \otimes \mu) \circ (\tau_{H,X} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes (\delta\rho))(\tau_{X,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_X \otimes \Delta)(v \otimes h) \\
&= \sum_{R,(h)} xh_{(2)} R_1 \otimes h_{(1)} R_2 \\
&= \sum_{R,(h)} xR_1 h_{(1)} \otimes R_2 h_{(2)} \\
&= (\rho \otimes \mu) \circ (\mathrm{id}_X \otimes \tau_{H,H} \otimes \mathrm{id}_H) \circ (\delta \otimes \Delta)(x \otimes v)
\end{aligned}$$

□

Problem 3. Let H be a bialgebra in a strict braided category $\mathcal{C}$ with braiding c, i.e. H is equipped with an algebra and a coalgebra structure which are compatible in the following way

$$\Delta\mu = (\mu \otimes \mu)(\mathrm{id} \otimes c_{H,H} \otimes \mathrm{id})(\Delta \otimes \Delta), \quad \Delta \circ \eta = \eta \otimes \eta, \quad \epsilon\mu = \epsilon \otimes \epsilon, \quad \epsilon\eta = \mathrm{id}_1.$$

A right-right Yetter-Drinfeld module over H is an object X in $\mathcal{C}$ together with an (associative, unital) action $\rho : X \otimes H \to X$ and a (coassociative, counital) coaction $\delta : X \to X \otimes H$ such that

$$\begin{aligned}&(\mathrm{id}_X \otimes \mu)(c_{H,X} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes (\delta\rho))(c_{X,H} \otimes \mathrm{id}_H)(\mathrm{id}_X \otimes \Delta)\\ =&(\rho \otimes \mu)(\mathrm{id}_X \otimes c_{H,H} \otimes \mathrm{id}_H)(\delta \otimes \Delta).\end{aligned}$$

1. Assume that H is a Hopf algebra, i.e. there is a morphism $S : H \to H$ such that

$$\mu(S \otimes \mathrm{id})\Delta = \eta\epsilon = \mu(\mathrm{id} \otimes S)\Delta.$$

Show that X is a Yetter-Drinfeld module, if and only if

$$\begin{aligned}\delta\rho =&(\mathrm{id}_X \otimes \mu)(c_{H,X} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \rho \otimes \mu)(S \otimes \mathrm{id}_X \otimes c_{H,H} \otimes \mathrm{id}_H)\\ &(\mathrm{id}_H \otimes \delta \otimes \Delta)(c_{X,H} \otimes \mathrm{id}_H)(\mathrm{id}_X \otimes \Delta)\end{aligned}$$

Solution. Graphically this is immediate: one should not forget the naturality of the braiding. □

2. Let H be a Hopf-algebra. Show that H is a Yetter-Drinfeld module with $\delta := \Delta$ and $\rho := \mu(S \otimes \mu)(c_{H,H} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)$.
Hint: The following equality holds $(S \otimes S) \circ \Delta = c_{H,H}^{-1} \circ \Delta \circ S$.

Solution. Graphically □

Problem 4. Let $\mathbb{K}$ be a field and let H, L be two bi-algebras over $\mathbb{K}$ and $\phi : H \to L$ a morphism of bialgebras. Denote by H-Mod resp. L-Mod the category of left modules over H resp. L and by Comod-H resp. Comod-L the category of right H resp. L comodules.

1. Show that ϕ induces a functor $\Phi : L\text{-mod} \to H\text{-mod}$.

Solution. This is clear:

$$\begin{aligned}(X, \rho) &\mapsto (X, \rho \circ (\phi \otimes \mathrm{id}_X))\\ (f : X \to Y) &\mapsto (f : X \to Y)\end{aligned}$$

□

2. Show that the functor Φ is strict monoidal.

Solution. *We have to give natural isomorphisms between the H-modules $\Phi(X \otimes Y)$ and $\Phi(X) \otimes \Phi(Y)$. The action on $\Phi(X \otimes Y)$ is given by*

$$(\rho_X \otimes \rho_Y)(\mathrm{id} \otimes \tau_{K,X} \otimes \mathrm{id})((\Delta\phi) \otimes \mathrm{id}).$$

The action on $\Phi(X) \otimes \Phi(Y)$ is given by

$$(\rho_X \otimes \rho_Y)(\phi \otimes \mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \tau_{H,X} \otimes \mathrm{id})(\Delta \otimes \mathrm{id}).$$

These coincide since ϕ commutes with the coproduct. We also have to give give an H-linear isomorphism $\Phi(\mathbb{K}) \cong \mathbb{K}$. But the H-modules $\Phi(\mathbb{K})$ and $\mathbb{K}$ are equal, which follows since $\mathbb{K}$ is the ground field with the trivial action given by the counit and the counit is preserved by ϕ.
Since the H-modules $\Phi(X \otimes Y)$ and $\Phi(X) \otimes \Phi(Y)$ and $\Phi(\mathbb{K})$ and $\mathbb{K}$ are equal we can take the identity linear maps as the needed isomorphisms. □

3. Show that ϕ induces a functor Ψ : comod-$H \to$ comod-L. Is this functor monoidal?

Solution. *Define Ψ : comod-$H \to$ comod-K as follows*

$$(X, \delta) \mapsto (X, (\mathrm{id}_X \otimes \phi) \circ \delta)$$
$$(f : X \to Y) \mapsto (f : X \to Y)$$

One has to check that $(\mathrm{id}_X \otimes \phi) \circ \delta$ is a K-coaction on X, which follows since ϕ commutes with the comultiplications of H and K.
Since ϕ also commutes with the multiplications of H and K this functor is again strict monoidal. (Remember that the coaction of H resp. K on $X \otimes Y$ involves the multiplication of H resp. K.) □

4. Let H, L be quasi-triangular with R-matrices R, R'. Show that in this case the functor Φ is braided, if and only if $(\phi \otimes \phi)(R) = R'$.

Solution.

$$(X, \delta) \mapsto (X, (\mathrm{id}_X \otimes \phi) \circ \delta)$$
$$(f : X \to Y) \mapsto (f : X \to Y)$$

One has to check that $(\mathrm{id}_X \otimes \phi) \circ \delta$ is a K-coaction on X, which follows since ϕ commutes with the comultiplications of H and K.
Since ϕ also commutes with the multiplications of H and K this functor is again strict monoidal. (Remember that the coaction of H resp. K on $X \otimes Y$ involves the multiplication of H resp. K.)

Assume $(\phi \otimes \phi)(R) = R'$, then $\Phi(c^{R'}_{X,Y}) = c^{R}_{\Phi(X),\Phi(Y)}$. Thus Φ is a braided functor.
Now suppose Φ is a braided functor. We take for X and Y the regular left K-module, i.e. K with left multiplication. Since Φ is strict and braided we get the equality

$$\Phi(c^{R'}_{K,K}) = c^{R}_{\Phi(K),\Phi(K)}.$$

If we apply these morphisms to $\eta_K \otimes \eta_K$ we get the equality $R' = (\phi \otimes \phi)(R)$. □

Problem 5. Let H be a quasi-triangular Hopf algebra, with antipode S, R-matrix $R = R_{12}$ and Drinfeld element $u = \sum_R S(R_{(2)})R_{(1)}$. We denote $\Delta' = \tau \circ \Delta$.

1. Show that the following formula endow $H \otimes H$ with a structure of module-$H^{\otimes 4}$:

$$(x \otimes y) \bullet (a \otimes b \otimes c \otimes d) = S(b)xa \otimes S(d)yc.$$

Solution. *We have:*

$$(x \otimes y) \bullet (1 \otimes 1 \otimes 1 \otimes 1) = S(1)x1 \otimes S(1)y1 = x \otimes y$$

and

$$\begin{aligned}
&(x \otimes y) \bullet ((a \otimes b \otimes c \otimes d) \cdot (e \otimes f \otimes g \otimes h)) \\
&\quad = (x \otimes y) \bullet (ae \otimes bf \otimes cg \otimes dh) \\
&\quad = S(bf)xae \otimes S(dh)ycg \\
&\quad = S(f)S(b)xae \otimes S(h)S(d)ycg \\
&\quad = S(f)(S(b)xa)e \otimes S(h)(S(d)yc)g \\
&\quad = (S(b)xa \otimes S(d)yc) \bullet (e \otimes f \otimes g \otimes h) \\
&\quad = ((x \otimes y) \bullet (a \otimes b \otimes c \otimes d)) \bullet (e \otimes f \otimes g \otimes h)
\end{aligned}$$

□

2. Compute $R_{21} \bullet R_{23}$ and $R_{21} \bullet (R_{23}R_{13}R_{12}R_{14})$.

Solution. $R_{12} \bullet R_{23} = 1 \otimes 1$, $R_{21} \bullet (R_{23}R_{13}) = u \otimes 1$ *and* $R_{21} \bullet (R_{23}R_{13}R_{12}R_{14}) = u \otimes 1$ □

3. Prove the following equality in $H^{\otimes 4}$: $R_{12}(\Delta \otimes \Delta')(R) = R_{23}R_{13}R_{12}R_{14}R_{24}$.

Solution. *We have:*

$$\begin{aligned}
(\Delta \otimes \Delta')(R) &= (\mathrm{id}_{H^{\otimes 2}} \otimes \tau) \circ (\Delta \otimes \mathrm{id}_{H^{\otimes 2}}) \circ (\mathrm{id}_H \otimes \Delta)(R) \\
&= (\mathrm{id}_{H^{\otimes 2}} \otimes \tau) \circ (\Delta \otimes \mathrm{id}_{H^{\otimes 2}})(R_{13}R_{12}) \\
&= (\mathrm{id}_{H^{\otimes 2}} \otimes \tau)((\Delta \otimes \mathrm{id}_{H^{\otimes 2}})(R_{13})(\Delta \otimes \mathrm{id}_{H^{\otimes 2}})(R_{12})) \\
&= (\mathrm{id}_{H^{\otimes 2}} \otimes \tau)(R_{14}R_{24}R_{13}R_{23})) \\
&= (\mathrm{id}_{H^{\otimes 2}} \otimes \tau)(R_{13}R_{23}R_{14}R_{24}))
\end{aligned}$$

Multiplying on the left by R_{12}*, we obtain:*

$$\begin{aligned}
R_{12}(\Delta \otimes \Delta')(R) &= R_{12}R_{13}R_{23}R_{14}R_{24} \\
&= R_{23}R_{13}R_{12}R_{14}R_{24}
\end{aligned}$$

□

4. Prove that:

$$\Delta(u) = (R_{21}R)^{-1}(u \otimes u) = (u \otimes u)(R_{21}R)^{-1}$$

5. Prove that $g = u(Su^{-1})$ is group like, and that S^4 is an inner automorphism.