Satellite Operators on Knot Concordance

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Topology in dimension n : $n = \underbrace{1, 2, 3}_{\text{geometric techniques}}, 4, \underbrace{5, 6, 7, \ldots}_{\text{algebraic techniques very powerful}}$

Two motivating questions

(1) When does algebra determine topology?

Thm [Freedman]

X^4 a compact, simply-connected 4-mfld w/ $\partial X = S^3$.

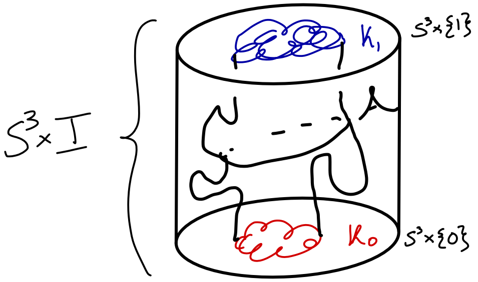
$$H_*(X^4) = H_*(B^4) \implies X \underset{TOP}{\cong} B^4$$

(2) When does topology determine smooth structure?

Thm [Akbulut-Ruberman]

There are compact contractible 4-mflds with multiple distinct smooth structures.

Model setting: Knot Concordance



Given K_0, K_1 in S^3, we say they're $\star$-concordant if they cobound an $\star$-embedded annulus in $S^3 \times I$

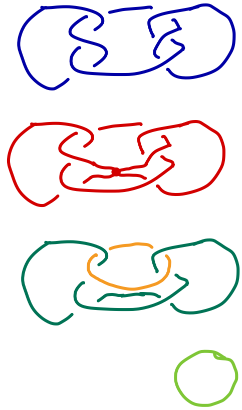
$\star \in \{$smooth, topologically locally flat$\}$

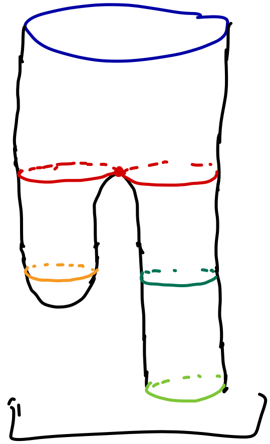
Example → $S^3\times\{1\}$

$S^3\times\{2/3\}$

$S^3\times\{1/3\}$

→ $S^3\times\{0\}$





Note: $K \sim_\star U$ we say K is $\star$-slice.

Why might one care? $\exists$ knots K s.t. $U \not\sim_{sm} K \sim_{TOP} U$ $\Longrightarrow$ exotic smooth structure on $\mathbb{R}^4$!

Theorem (Fox-Milnor) $\mathcal{C}_\star = \{\text{knots in } S^3\}/\sim_\star$ is a countable abelian group, with respect to the operation induced by connected sum.

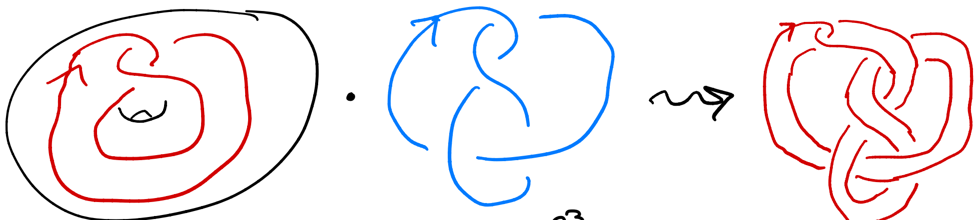
Known: $\mathcal{C}_\star \twoheadrightarrow \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}^\infty$

$\mathcal{C}_\star \supseteq \mathbb{Z}_2^\infty \oplus \mathbb{Z}^\infty$

Unknown:

- torsion of order $\neq 2$?, $\mathbb{Q} \hookrightarrow \mathcal{C}_\star$?

Philosophy: Understand $\mathcal{C}_*$ by looking at geometrically defined $\mathcal{C}_* \to \mathcal{C}_*$



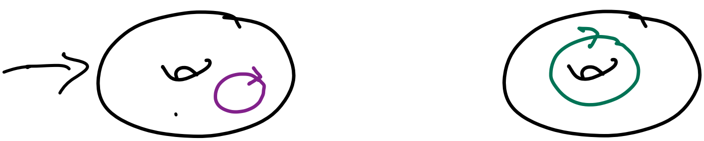
$P: S^1 \to S^1 \times D^2$ $\quad$ $K: S^1 \hookrightarrow S^3$ $\quad$ $P(K): S^1 \hookrightarrow S^3$

Useful Fact: Any pattern gives a well-defined map
$$P: \mathcal{C}_* \to \mathcal{C}_* .$$

$$r(A) = A \times D^2 = (I \times \underbrace{S^1) \times D^2}_{\substack{\cup\!| \\ I \times P}}$$

Examples

Whitehead $\quad$ Mazur $\quad$ $C_{2,1}$

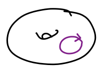


$K \to U$ $\quad$ $K \to K$

Key attribute: w_P: $\quad [P] = w_P [S^1 \times *] \in H_1(S^1 \times D^2)$

winding #

Question 1 What can we say about $P: \mathcal{C}_* \to \mathcal{C}_*$, just from a set-theoretic perspective

Winding Number:	0	± 1	$\omega \notin \{-1,0,1\}$
Surjective?	Never	Sometimes  + (1) Not always (sm) (2)	Never
Injective?	Not always  + (3) Yes?	Sometimes  + (1)	?????? $C_{2,1}(4_1)$ slice?

(1) M.-Piccirillo, 2018

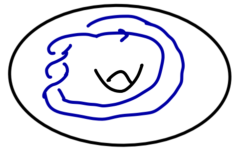
(2) A. Levine, 2016

(3) M., 2019

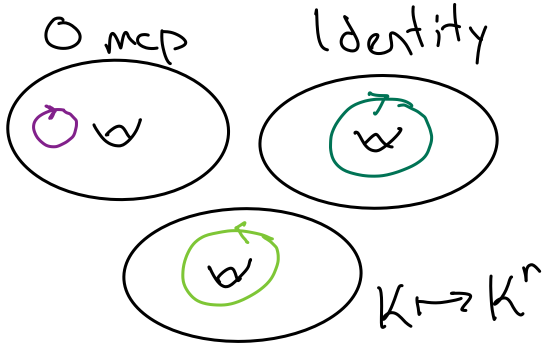
Question 2 How do satellite operators interact with additional structure on $\mathcal{C}_*$?

More precisely: When is $P : \mathcal{C}_* \to \mathcal{C}_*$ a homomorphism?

(1) First obstruction:
$P(U)$ must be slice.



(2) Sometimes!

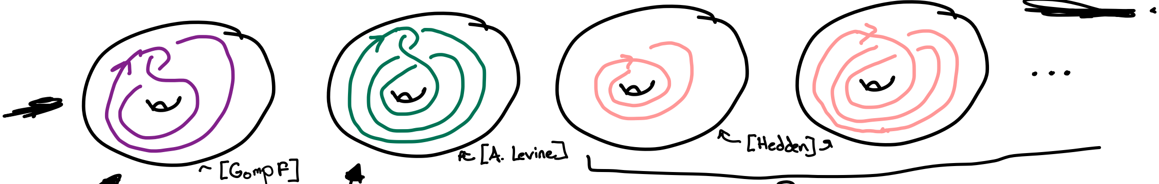


(3) Conjecture [Hedden]:
The only satellite induced maps on $\mathcal{C}_*$ are $[K] \underset{w\#0}{\longmapsto} 0$, $[K] \underset{w\#1}{\longmapsto} [K]$, and $[K] \underset{w\#-1}{\longmapsto} [K^r]$.

Technical Challenge:
If $P(U) \sim U$, then algebraically P "looks like" a homomorphism.

Some results:

(1) These do not induce homorphisms! on $\mathcal{C}_{sm}$.



Top: 0 map Id? homs...?

[Work in progress w/ T. Lidman + J. Pinzón-Caicedo: more broadly applicable criteria]

(2) [M. '19]

For every winding # except ± 1, there exist (many) patterns P of that winding # such that

- $P(U)$ is slice
- $P : \mathcal{C}_{TOP} \to \mathcal{C}_{TOP}$ is not a hom.

Def'n

$$g_3(K) = \min\{ g(F) : F \hookrightarrow S^3,\ \partial F = K \}$$

$$g_4^{*}(K) = \min\{ g(F) : F \overset{*}{\hookrightarrow} B^4,\ \partial F = K \}$$

Thm [Schubert, 1952]

If P pattern, $K \neq U$

Then

$$g_3(P(K)) = \text{``}g_3(P)\text{''} + |\omega_P|\, g_3(K)$$

not always $g_3(P(U))$

Corollaries

(a) $g_3(C_{n,1}\, T_{2,3}) = n$

(b) $\displaystyle\lim_{k\to\infty} \frac{g_3(P(T_{2,2k+1}))}{g_3(T_{2,2k+1})} = |\omega_P|$

What happens w/ smooth 4-genus?

- $g_4^{sm}(C_{n,1}\, T_{2,3}) = n$

- $\displaystyle\lim_{k\to\infty} \frac{g_4^{sm}(P(T_{2,2k+1}))}{g_4^{sm}(T_{2,2k+1})} = |\omega_P|.$

What happens w/ topological 4-genus?

Thm [Feller - M - Pinzón-Caicedo, '19]

- $g_4^{top}(C_{n,1}\, T_{2,3}) = 1$

- $\displaystyle\lim_{k\to\infty} \frac{g_4^{top}(P(T_{2,2k+1}))}{g_4^{top}(T_{2,2k+1})} = \begin{cases} 0 & \omega_P = 0 \\ 1 & \text{else.} \end{cases}$

Open: $g_4^{top}(P(K)) \leq g_4^{top}(P(U)) + g_4^{top}(K)$?