Braid invariant related to knot Floer homology and Khovanov homology

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(Joint with Nathan Dowlin)

Khovanov Homology (Khovanov)

- Bigraded $\mathbb{F}$-vector space $\boxed{Kh^{i,j}(K)}$
- **Jones polynomial:**

$$\boxed{J_K(q) = \sum_{i,j} (-1)^i q^j \dim Kh^{i,j}(K)}$$

- Rank detects unknot (Kronheimer–Mrowka)
  Trefoil (Baldwin–Sivek)
- Lower bounds for slice genus
  (e.g. Rasmussen's "s" invariant)
  Unknotting number (A. Dowlin)

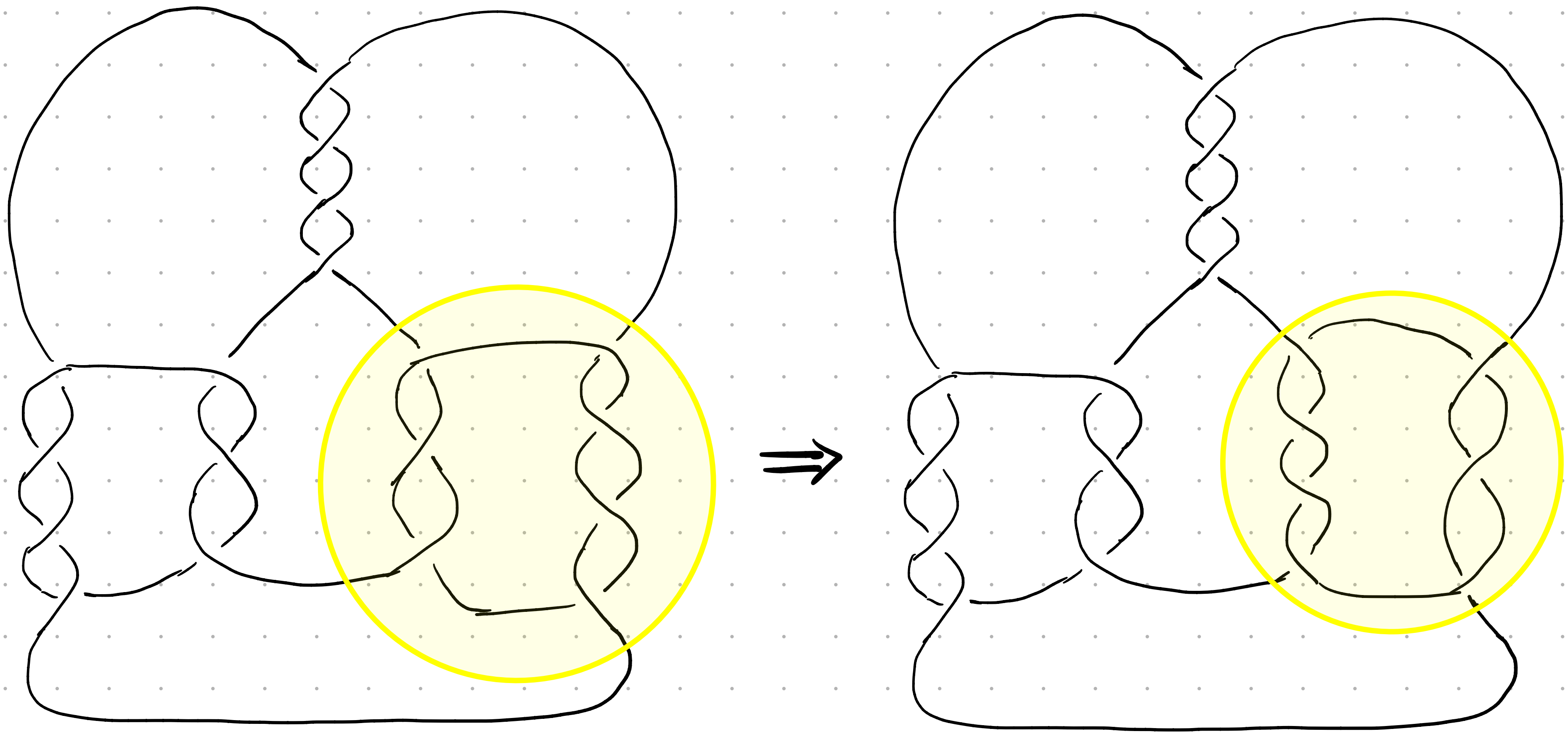
Knot Floer Homology (Ozsváth–Szabó)

- Bigraded $\mathbb{F}$-vector space $\boxed{\widehat{HFK}_{i,j}(K)}$
- **Alexander polynomial:**

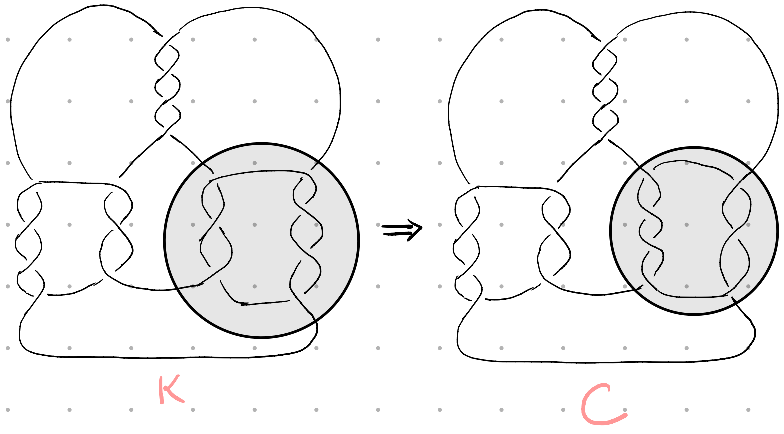
$$\boxed{\Delta_K(t) = \sum_{i,j} (-1)^i t^j \dim \widehat{HFK}_{i,j}(K)}$$

- Rank detects Unknot (Ozsváth–Szabó)
  Trefoil (Hedden–Watson)
- Lower bounds for slice genus
  (e.g. $\tau, \nu, \nu^+, \nu^-, \Upsilon, \ldots$)
  Unknotting number (A. Eftekhary)

Conway mutation



Conway mutation



- Conway mutation does **not** change Jones and Alexander polynomial.

Thm (Wehrli '10, Bloom '10) If K' is obtained from K by Conway mutation then over $\mathbb{Z}/2\mathbb{Z}$

$$Kh^{i,j}(K') \cong Kh^{i,j}(K)$$

Q. What about knot Floer homology?

$$\widehat{HFK}_\delta(K) = \bigoplus_{i-j=\delta} \widehat{HFK}_{i,j}(K)$$

Ex.

$$\widehat{HFK}_{i,j}(K) \underset{j\geq 3}{=} \begin{cases} 0 & j>3 \\ \mathbb{Z}^2 & i=3,4,\ j=3 \\ 0 & i\neq 3,4,\ j=3 \end{cases}$$

$$\widehat{HFK}_{i,j}(C) \underset{j\geq 5}{=} \begin{cases} 0 & j>5 \\ \mathbb{Z}^2 & i=5,6 \quad j=5 \\ 0 & i\neq 5,6 \quad j=5 \end{cases}$$

(Ozsváth–Szabó)

Thm (Zibrowius '19) $\widehat{HFK}_\delta(K)$ is invariant under Conway mutation.

Conj (Rasmussen) There is a spectral sequence from $\widetilde{Kh}(K)$ to $\widehat{HFK}_\delta(K)$.

over $\mathbb{Q}$

Cor $\operatorname{rk} \widetilde{Kh}(K) \geq \operatorname{rk} \widehat{HFK}(K)$

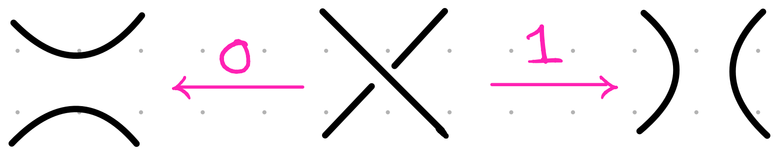
Thm (Dowlin '18) There is a spectral seq. from $Kh(K)$ (with $\mathbb{Q}$ coeff.) to $HFK_2(K)$ (with $\mathbb{Q}$ coeff.)

Goal : **Local** framework for proving this Conjecture

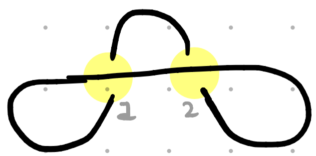
Khovanov homology

Knot diagram : D

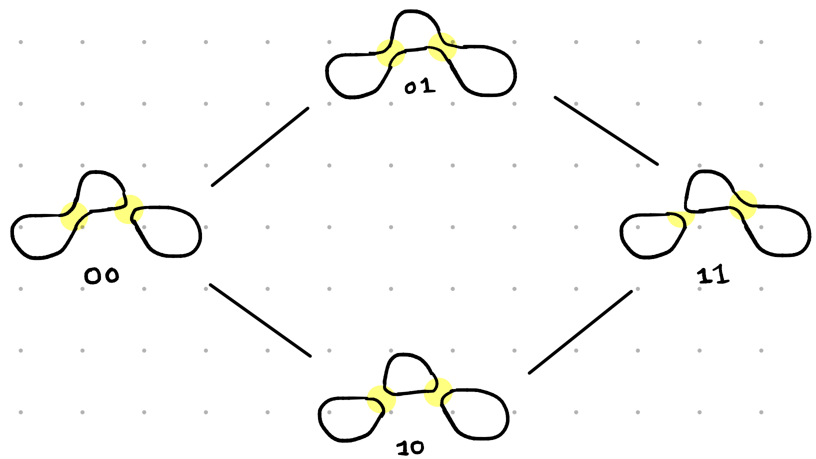
Resolutions :



If $\#(\text{Crossings}) = n \Rightarrow$ A complete resolution for each $v \in \{0,1\}^n$: D_v

Ex. 

Unoriented cube of resolutions



Frobenius algebra : $V = \mathbb{F}[x] / (x^2)$ with $\Delta : V \longrightarrow V \otimes V$

$$1 \longmapsto 1 \otimes x + x \otimes 1$$

$$x \longmapsto x \otimes x$$

$$CKh(D_v) = \overbrace{V \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} V}^{k_v\text{-times}}$$

k_v : # (circles) in D_v

$$CKh(D) = \bigoplus_v CKh(D_v)$$

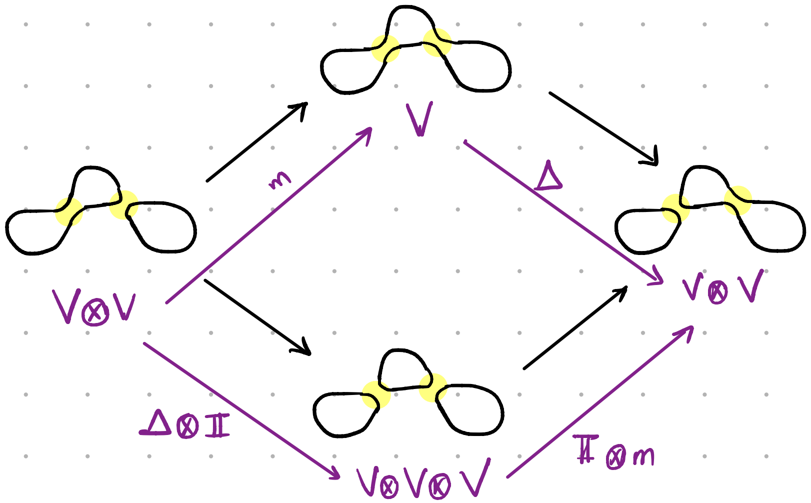
∂

∂ : Splits along the edges i.e.

$$\partial = \sum_{\substack{\text{edges}\\ \text{of the}\\ \text{cube}}} \partial_{v <\cdot v'}$$

$$\partial_{v <\cdot v'} : CKh(D_v) \longrightarrow CKh(D_{v'})$$

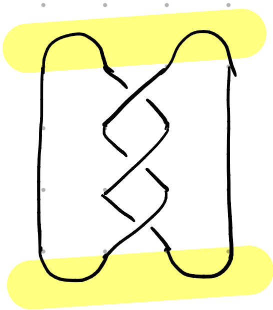
Ex.



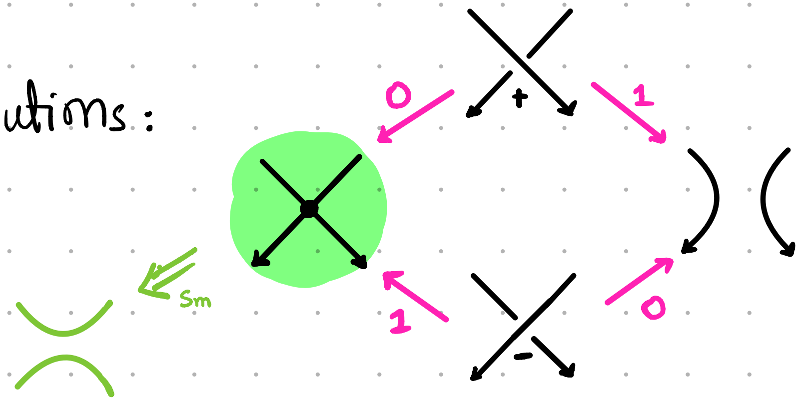
- Product m
- coproduct Δ

Overview

- Plat diagram: D



- Oriented cube of resolutions:



Note. There is a generalization of HFK_2 for singular knots s.t.

$$HFK_2(D_v) \cong CKh(Sm(D_v))$$

w. Dowlin '18 : Filtered Chain Complex $(C_{1\pm1}(D), d_0 + d_1)$

- $C_{1\pm1}(D) = \bigoplus_{v\in\{0,1\}^n} C_{1\pm1}(D_v)$ with d_0 acting on each $C_{1\pm1}(D_v)$, d_1 : splits along the edges of the cube.

- $(C_{1\pm1}(D_v), d_0)$: algebraic model for $CFK_2(D_v)$.

Thm. $H_*(C_{1\pm1}(D_v), d_0) \cong Kh(sm(D_v)) \cong HFK_2(D_v)$.

$H_*(H_*(C_{1\pm1}(D_v), d_0), d_1^*) \cong Kh(K)$.

$H_{1-1}(K) := H_*(C_{1\pm1}(D), d_0 + d_1)$ is a knot invariant.

$\boxed{\text{Cor}}$ $Kh(K) \Rightarrow H_{1-1}(K)$

$\boxed{\text{Conj}}$ $H_{1-1}(K) \cong HFK_2(K)$

$\Downarrow$ "reduced"

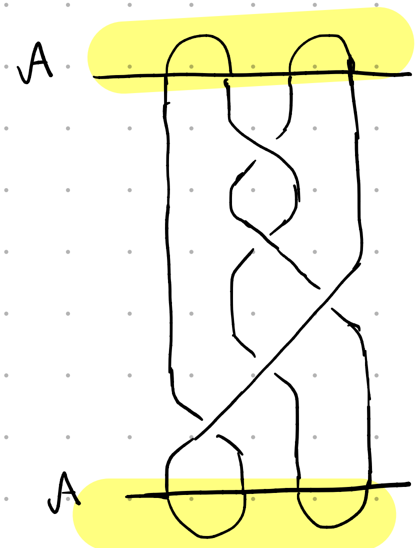
Rasmussen's Conj. over $\mathbb{Q}$

Idea: (1) Define local/gluable invariants for braids

(2) Compare it with Ozsváth–Szabó's tangle invariants

$\Downarrow$

Computer program for computing $\widehat{HFK}$ which is much faster than the previous ones!



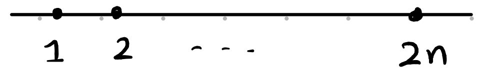
diff. bimodule $M[b]$ over $\mathcal{A}$ s.t. $M(b_1 \circ b_2) = M(b_1) \otimes_{\mathcal{A}} M(b_2)$

and "closing up" $C_{1\pm1}(\mathcal{D})$

$\uparrow$ plat closure of b

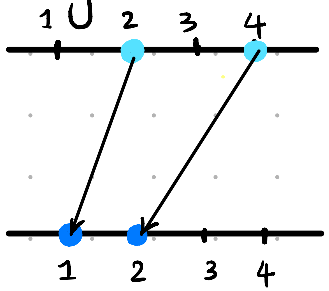
Algebra : $\mathcal{A}(n,K)$

- $\mathbb{Q}[u_1, \dots, u_{2n}]$ – algebra associated with $[2n] = \{1, \dots, 2n\}$



- **Generators** : monotone bijection $P: S \longrightarrow S'$ where $S, S' \subset [2n]$ with $|S| = |S'| = k$.

Ex. $n=2$, $K=2$



- **Product** : "almost" Concatenation i.e.

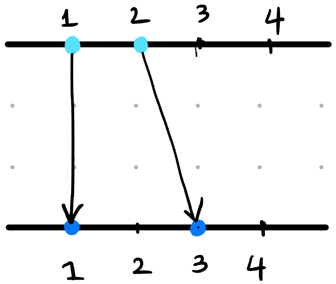
$$P_1 P_2 = \prod_{i=1}^{2n} u_i^{\alpha_i}\, P \quad \text{where } P = P_2 \circ P_1 \text{ if defined otherwise } 0.$$

- **Relations** : $R_{i+1} R_i = 0 \qquad L_i L_{i+1} = 0$

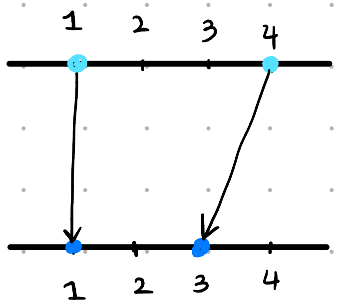
Smaller set of generators :

• **Idempotents** : $\iota_S = id_S$

• **Right / Left shifts** : $R_i = \sum_{S\cap\{i,i+1\}=\{i\}} R_i^S$ $\quad L_i = \sum_{S\cap\{i,i+1\}=\{i+1\}} L_i^S$



: $R_2^{\{1,2\}}$

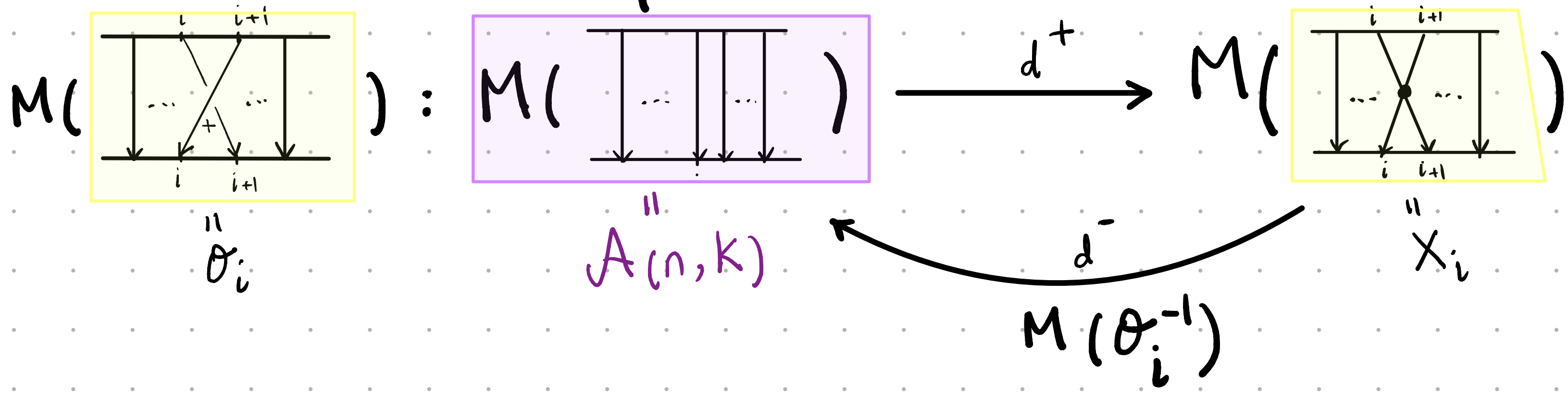


: $L_3^{\{1,4\}}$

• **"almost"** : $R_i L_i = u_i \iota_i$ $\qquad L_i R_i = u_i \iota_{i+1}$ $\qquad \iota_i = \sum_{i\in S} \iota_S$

$\boxed{\textbf{Thm.}\ \mathcal{A}(n,k) \cong B(2n,n-k,\emptyset) \text{ where } B(2n,n-k,\emptyset) \text{ is OS's algebra.}}$

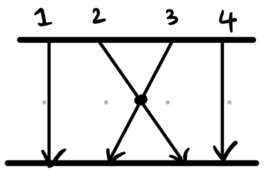
Bimodule for elementary braid :



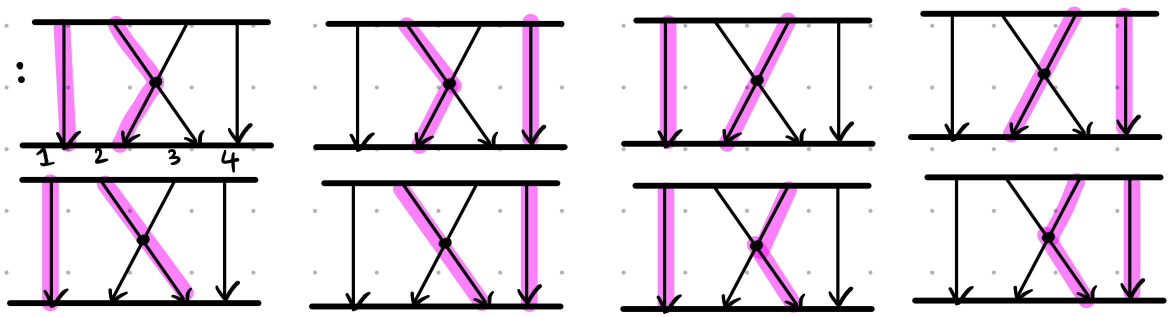
$M(X_i)$:

Generators : K pairwise disjoint from top to bottom, called a **cycle**

Ex: $n = 2$, $i = 2$ $\Rightarrow$
$K = 2$



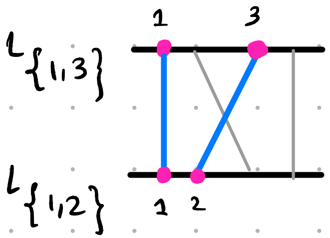
cycles:



Relations:

(1) For any cycle x :

$L_b \, x \, L_t = x$

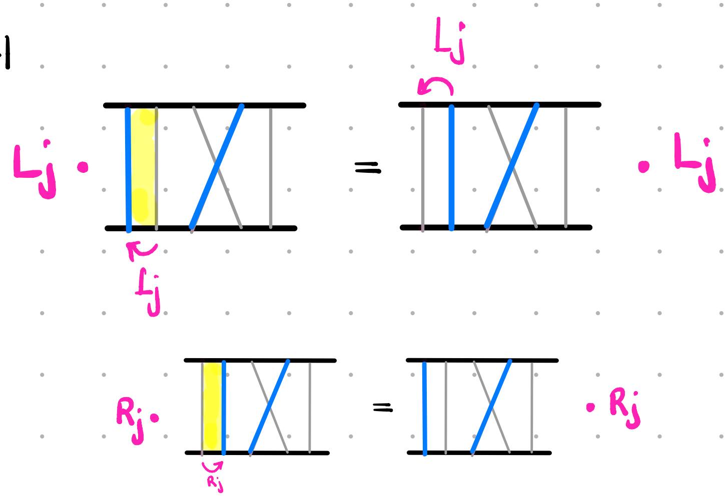


(2) For $j \neq i, i+1$ $\quad u_j \, x = x \, u_j$

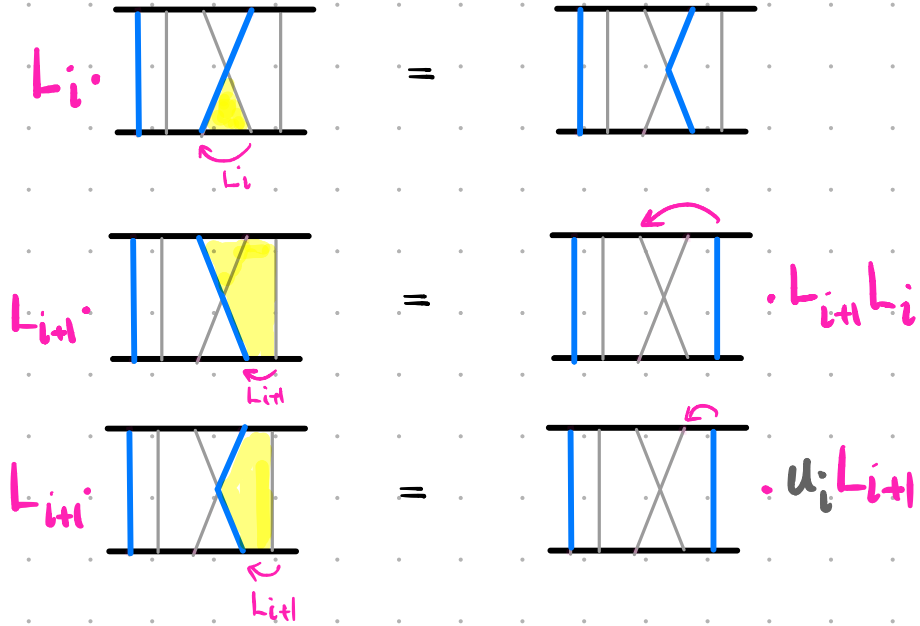
$(u_i + u_{i+1}) \, x = x \, (u_i + u_{i+1})$

$(u_i u_{i+1}) \, x = x \, (u_i u_{i+1})$

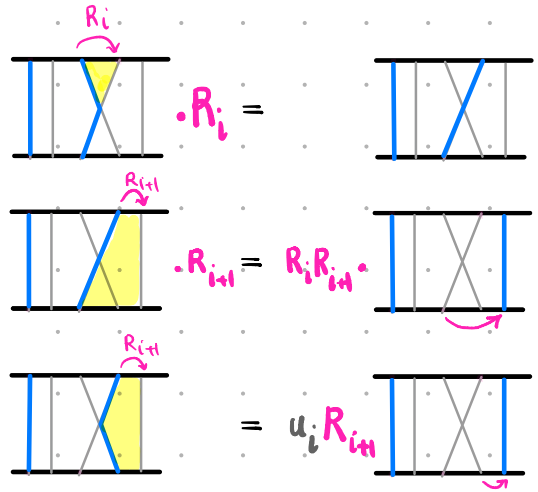
(3) $j \neq i, i+1$



(4)

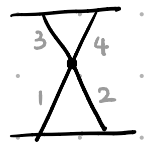


(5)



Edge map: $\bar{d} : M(X_i) \longrightarrow \mathcal{A}(n,k)$

Notation:



$Z = \phi, 13, 23, 14, 24$

Let x_Z denote the sum of all cycles that locally contain the edges specified by Z.

- $x_\phi \longmapsto \sum_{S \cap \{i,i+1\} = \phi} \iota_S$
- $x_{13} \longmapsto \sum_{S \cap \{i,i+1\} = i} \iota_S$
- $x_{14} \longmapsto R_i$
- $x_{23} \longmapsto L_i$
- $x_{24} \longmapsto u_i \left(\sum_{S \cap \{i,i+1\} = i+1} \iota_S \right)$

The edge map for $M(\theta_i)$ is defined similarly.

Note : Both $M(\theta_i^{-1})$ and $M(\theta_i)$ can be reinterpreted as <u>DA bimodules</u> with 6 generators, and δ_1^1 : diff. δ_2^1 : right multi and $\delta_i^1 = 0$ for $i > 2$.

Generators of $M_{DA}(\sigma_i^{-1})$:

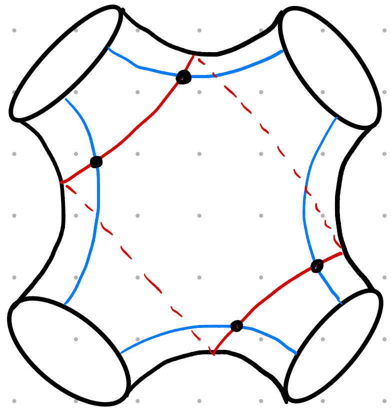
$$\left.\begin{aligned} N_+ &= x_\emptyset + x_{13} & N_- &= N_+ u_i - u_{i+1} N_+ \\ E &= N_+ L_{i+1} & W &= N_+ R_{i+1} \end{aligned}\right\} \text{ in } M(X_i)$$

$$N_0 = \sum_{i+1 \notin S} \iota_S \qquad S = \iota_{i+1} \Big\} \text{ in } M(\mathrm{II}) = \mathcal{A}$$

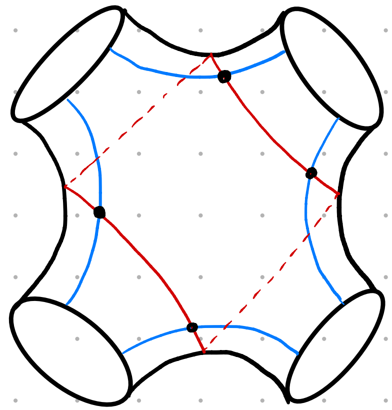
Thm. DA bimodules $M_{DA}(\sigma_i)$ (resp. $M_{DA}(\sigma_i^{-1})$) and $OS_{DA}(\sigma_i^{-1})$ (resp. $OS_{DA}(\sigma_i)$) defined over the isomorphic algebras $A(n,k)$ and $B(2n, n-k)$ are chain homotopic.

Cor. For any braid b $M_{DA}(b) \simeq OS_{DA}(\bar{b})$

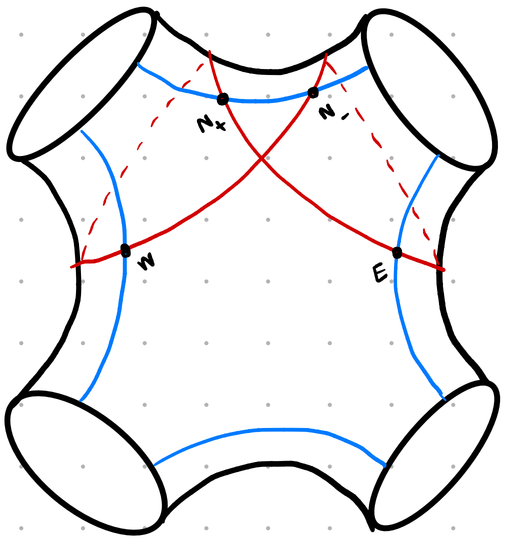
mirror



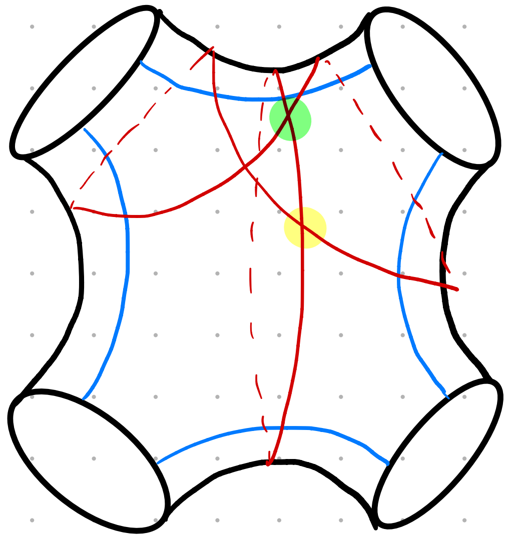
σ_i

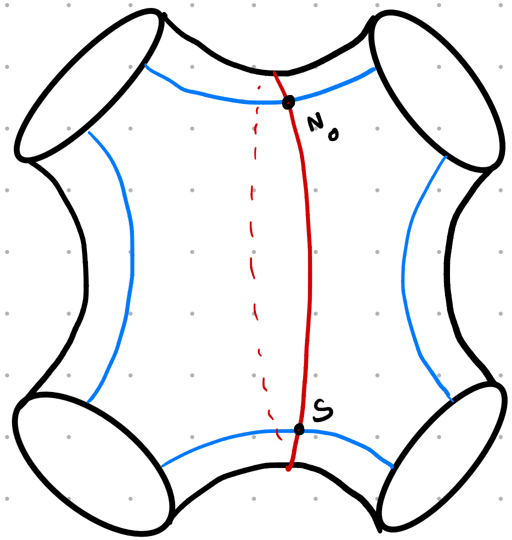


σ_i^{-1}



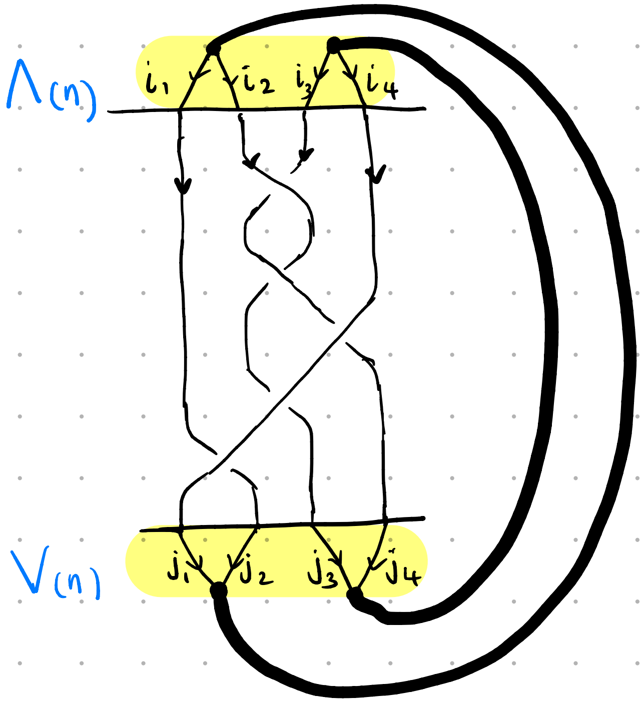
X_i





Id

Knot invariant :



$$M(\Lambda(n))_{A(n,n)} \simeq OS_D(\Lambda(n))$$

$$\boxtimes$$

$$M(b)$$

$$\boxtimes$$

$${}^{A(n,n)}M(V(n)) \longleftarrow$$ will be $OS_A(\Lambda(n))$ if we treat all minima the same.

$$\|$$

$$M(D)$$

- $M(D)$: curved complex $R = \mathbb{Q}[u_1, \dots, u_m]$

$$\partial^2 = \sum_{k=1}^{n} u_{i_{2k-1}} u_{i_{2k}} - \sum_{k=1}^{n} u_{j_{2k-1}} u_{j_{2k}}$$

- $M(D) \otimes \left(R \underset{u_{i_1}+u_{i_2}+u_{j_1}+u_{j_2}}{\overset{u_{i_1}+u_{i_2}-u_{j_1}-u_{j_2}}{\rightleftarrows}} R \right) \otimes \cdots \otimes \left(R \underset{u_{i_{2n-1}}+u_{i_{2n}}+u_{j_{2n-1}}+u_{j_{2n}}}{\overset{u_{i_{2n-1}}+u_{i_{2n}}-u_{j_{2n-1}}-u_{j_{2n}}}{\rightleftarrows}} R \right)$

They turn their curved complex into a chain complex by adding new elements to the algebra.

Thank you