Coloured Jones and coloured Alexander polynomials from two Lagrangians intersected in a symmetric power of a surface

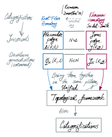
Outline

Motivation
- (I) Homological set-up
- (II) Topological models as state sums of intersections in configuration spaces
- (III) Unified model for $U_q(sl(2))$-invariants as an intersection in a symmetric power

Motivation

- Aim: Define topological models for $U_q(sl(2))$-quantum invariants

Topological model: graded intersection pairing of homology classes in coverings of configuration spaces



Representation theory

$N\in\mathbb{N}$ colour

$(U_q(sl(2)),\ V_N)$ — q parameter, N-dim rep → **Coloured Jones polyn.** $J_N(L,q)\in\mathbb{Z}[q^{\pm1}]$

$(U_{\xi_N}(sl(2)),\ V_\lambda)$ — $\xi_N=e^{2\pi i/2N}$, N-dim rep, $\lambda\in\mathbb{C}$ parameter → **Coloured Alexander polyn.** $\Phi_N(L,\lambda)\in\mathbb{Z}[\xi_N^{\pm1},\xi_N^{\pm\lambda}]$

- Kohno '12, Ito '15, Martel '15

Th (Bigelow '00, Lawrence) Noodles and Forks → skein rel → Jones polyn.

Intersections in configuration spaces

Th (A. '20) ① Unified state sum model via immersed Lagrangians ② Unified state sum model via embedded Lagrangians

Th (A. '21) Unified model from an intersection of two Lagrangians

Intersections in symmetric powers

Main result Fix $N\in\mathbb{N}$ – colour of the quantum invariant. Let L – oriented link; $L=\hat{\beta}_n$ for $\beta_n\in B_n$.

Construction: $\forall\ \bar{i}=(i_1,\dots,i_{n-1})$, $i_1,\dots,i_{n-1}\in\{0,\dots,N-1\}$ ⇝ Two Lagrangians

$\mathbb{Z}\oplus\mathbb{Z}$-covering ($\langle x\rangle$, $\langle d\rangle$) of $\mathrm{Conf}_{(2n-1)(N-1)+2}(\mathbb{D}_{2n})$:

$\mathscr{F}_{\bar{i}}\in H_{2n,(2n-1)(N-1)+1}$, $\mathscr{L}_{\bar{i}}\in H^{\partial}_{2n,(2n-1)(N-1)+1}$

- **Def** (State sum of Lagrangian intersections)

$$\Lambda_N(\beta_n):=u^{-w(\beta_n)}u^{-(n-1)}\sum_{i_1,\dots,i_{n-1}=0}^{N-1}\left\langle(\beta_n\cup\mathbb{I}_n)\,\mathscr{F}_{\bar{i}},\ \mathscr{L}_{\bar{i}}\right\rangle\in\mathbb{Z}[x^{\pm1},d^{\pm1},u^{\pm1}]$$

- **Th 2** (A. '20 Unified model through state sums of Lagrangian intersections)
The polynomial in 3 variables Λ_N recovers the N^{th} Coloured Jones and N^{th} Coloured Alexander polynomials for links.

$J_N(L,q)=\Lambda_N(\beta_n)\big|_{\psi_{1,q,N}}$

$\Phi_N(L,\lambda)=\Lambda_N(\beta_n)\big|_{\psi_{1-N,\xi_N,\lambda}}$

(specialisations of coefficients)

(I) Homological representations

Fix $n,m;k\in\mathbb{N}$. $\mathbb{D}_n:=D^2\setminus\{1,\dots,n\}$ ⇝ $C_{n,m}=\mathrm{Conf}_m(\mathbb{D}_n)$

Def (Local system) Fix $0\le k\le n$

$\pi_1(C_{n,m})\xrightarrow{ab}\mathbb{Z}^n\oplus\mathbb{Z}\xrightarrow{aug}\mathbb{Z}\oplus\mathbb{Z}$ ($\langle x\rangle$, $\langle d\rangle$); $\sigma_i\mapsto x$ if $i\le k$, $-x$ if $i>k$

⇝ $\tilde{C}^{-k}_{n,m}$ covering sp.

Monodromy of the local system φ^{-k}

- Let $w\in\partial\mathbb{D}_n$; $\tilde{d}\in\pi^{-1}(d)$

- **Tools** Homology of this covering sp.
  1. $H^{-k}_{n,m}\subseteq H^{lf}_m(\tilde{C}^{-k}_{n,m},\pi^{-1}(w);\mathbb{Z})$ — B_n ↺ MCG; Borel-Moore w.r.t. punctures, collisions
  2. $H^{-k,\partial}_{n,m}\subseteq H^{lf}_m(\tilde{C}^{-k}_{n,m},\partial;\mathbb{Z})$
  3. Prop (A.–Palmer) Intersection pairing: $\langle\,,\rangle: H^{-k}_{n,m}\otimes H^{-k,\partial}_{n,m}\to\mathbb{Z}[x^{\pm1},d^{\pm1}]$

Construction of homology classes

$E_{m,n}=\{n\text{-partitions of }m\}$; $e=(e_1,\dots,e_n)$ ⇝ $\mathscr{F}_e\in H^{-k}_{n,m}$ (lift through $\tilde{\eta}(\tilde{d})$)

(II) A. Topological model with immersed Lagrangians

Context: L–oriented link; $L=\hat{\beta}_n$, $\beta_n\in B_n$; Fix $N\in\mathbb{N}$

Def (Main classes) $\bar{i}=(i_1,\dots,i_{n-1})$, $i_j\in\{0,\dots,N-1\}$

$\tilde{\mathscr{U}}_{\bar{i}}\in H_{2n-1,(n-1)(N-1)}$, $\mathscr{G}^N_n\in H^{\partial}_{2n-1,(n-1)(N-1)}$

Def (Homology Classes)

$$\left(\mathscr{E}^N_n:=\sum_{\bar{i}=0}^{N-1}d^{-\Sigma i_k}\,\tilde{\mathscr{U}}_{\bar{i}},\ \mathscr{G}^N_n\right)$$

Not. (Specialisation of coefficients) Let $c\in\mathbb{Z}$

$\psi_{c,q,\lambda}:\mathbb{Z}[u^{\pm1},x^{\pm1},d^{\pm1}]\to\mathbb{Z}[q^{\pm1},q^{\pm\lambda}]$; $\psi_{q,\lambda}:\mathbb{Z}[x^{\pm1},d^{\pm1}]\to$ …

- **Th 1** (A. '20 Topological model via immersed Lagrangians) $L=\hat{\beta}_n$

Let $I_N(\beta_n):=\left\langle(\beta_n\cup\mathbb{I}_{n-1})\,\mathscr{E}^N_n,\ \mathscr{G}^N_n\right\rangle\in\mathbb{Z}[x^{\pm1},d^{\pm1}]$

Then, I_N recovers the N^{th} col. Jones and N^{th} col. Alexander polyn.:

$J_N(L,q)=q^{(N-1)w(\beta_n)}q^{-(N-1)(n-1)}\cdot I_N(\beta_n)\big|_{\psi_{q,N-1}}$

$\Phi_N(L,\lambda)=\xi_N^{(N-1)\lambda w(\beta_n)}\xi_N^{(N-1)(n-1)}\cdot I_N(\beta_n)\big|_{\psi_{\xi_N,\lambda}}$

- **Corollary 1** (Relation to Bigelow's model for the Jones polynomial)
Th 1 for $N=2$ recovers Bigelow's classes: $\mathscr{F}^2_n\to$ forks, $\mathscr{G}^2_n\to$ noodles

Proof For $N=2$ Jones polyn. $i_1,\dots,i_{n-1}\in\{0,1\}$

$\mathscr{E}^2_n=\sum d^{-\Sigma i_k}\tilde{\mathscr{U}}_{\bar{i}}$, $\mathscr{G}^2_n$

(up to homotopy and d-coefficients)

$\mathscr{E}^2_n$ fork and $\mathscr{G}^2_n$ noodle

- **Corollary 2** (ADO invariants from $\mathbb{Z}\oplus\mathbb{Z}_{2N}$-covering spaces)
The N^{th} coloured Alexander invariant comes from an intersection pairing in a $\mathbb{Z}\oplus\mathbb{Z}_{2N}$-covering of a conf. sp. in the punctured disk.

Questions ① Model with embedded Lagrangians ② Simple lifts (paths up to the base point) ③ Suitable for computations

(II) B. Model with embedded Lagrangians

Construction Ideas

①② Replace (figure eight, φ^0) by (arc, φ^{-n}) (change the local system). Add an extra puncture.

Def (Homology classes) $\bar{i}=(i_1,\dots,i_{n-1})$, $i_j\in\{0,\dots,N-1\}$

$\mathscr{F}_{\bar{i}}\in H^{-n}_{2n,(n-1)(N-1)+1}$, $\mathscr{L}_{\bar{i}}\in H^{-n,\partial}_{2n,(n-1)(N-1)+1}$