Concordance invariants from $E(-1)$

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Concordance homomorphisms from algebraic knot homologies

- Rasmussen's $s(K)$
  - From Lee deformation of Khovanov homology
  - Strong bounds on $g_4(K)$
- Extended by Wu, Gornik, Lobb, Lewark (among others) to the Khovanov-Rozansky sl(n) homologies
  - One concordance homomorphism coming from each polynomial and choice of root
- According to Dunfield-Gukov-Rasmussen superpolynomial conjectures, these use differentials d_N for $N > 0$

Invariants from knot Floer homology

- Ozsváth-Szabó's $\tau(K)$
- Ozsváth-Stipsicz-Szabó's $\Upsilon_K(t)$
  - A one-parameter family of invariants
  - Value $\upsilon(K) = \Upsilon_K(1)$ gives nonorientable information
- Many other invariants (most aren't additive)
- Gives information from d_1, d_0, d_{-1}

$\mathcal{F}_3$ Khovanov homology

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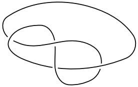
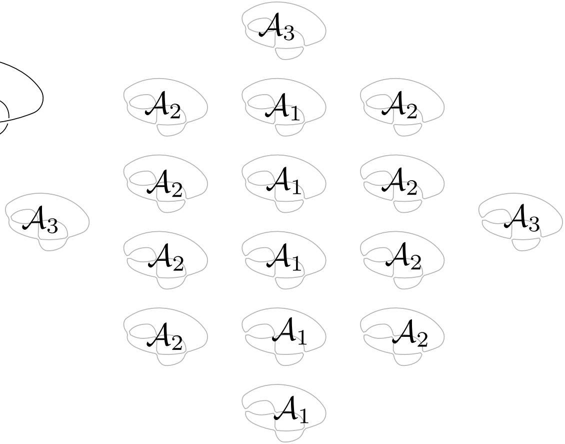




$\mathcal{F}_3$ edge maps

Then insert edge maps between resolutions differing at only one crossing.

- If two circles merge,

$$\mathcal{A}_n \to \mathcal{A}_n/(x_i = x_j) \cong \mathcal{A}_{n-1}$$

- If one circle splits,

$$\mathcal{A}_{n-1} \cong \mathcal{A}_n/(x_i = x_j) \xrightarrow{x_i + x_j} \mathcal{A}_n$$

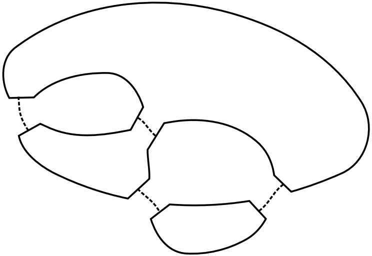
Reduced homologies

From this $\mathcal{F}_3$ Khovanov homology, can recover other versions.

- The quotient by x_p^2 gives standard Khovanov homology
- The quotient by x_p gives reduced Khovanov homology
- The quotient by $x_p^2 - 1$ gives Lee homology
- The quotient by $x_p - 1$ gives a reduced version of Lee homology

(x_p acts by the variable of the circle containing a basepoint p on each vertex)

A double complex computing Khovanov homology

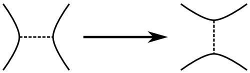


Can write down a chain complex with homology $\mathcal{A}_n$

- A tensor product of copies of a matrix factorization for each resolved crossing
- Over a ground ring specifying how they connect

Saddle maps of matrix factorizations

The edge maps in the Khovanov complex come from a matrix factorization homomorphism



- No need to distinguish merge and split!
- Can build a double complex $C(K)$ out of the matrix factorizations of all resolutions of K, with these saddle maps as the horizontal differential

Properties of $C(K)$

- $C(K)$ has a grading $q - 3h$ that the differential lowers by 3, and a filtration h
- The homology of $C(K)$ is $\mathbb{Z}[x]$
  - The E_2 page of the associated spectral sequence is $\mathcal{F}_3$ Khovanov homology
- $\mathbb{Z}[x]$ action coming from an oriented basepoint
  - Gives reduced $(C(K)/x)$ and reduced Lee $(C(K)/(x-1))$ versions

Invariants from unreduced homology

- Homology of $C(K)$ is $\mathbb{Z}[x]$, so filtration of $C(K)$ gives a filtration of $\mathbb{Z}[x]$.

Definition

$T_i(K)$ is the deepest level of the filtration supporting the homology class $(2x)^i$

Theorem

$T_i(K)$ is an increasing sequence of even integers, and $T_{g_4(K)}(K) = 0$.

Invariant from reduced homology

Definition

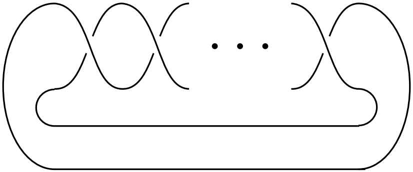
$t(K)$ is the deepest level of the filtration on $\overline{C}(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ supporting a nonzero homology class

- t is a concordance homomorphism
- $t(K)$ obstructs the existence of a nonorientable surface bounding K
- For torus and alternating knots, $t(K) = -2\upsilon(K)$.

Nonorientable surfaces in B^4

Definition

The nonorientable slice genus $\gamma_4(K)$ is the minimum of $b_1(F)$ over (possibly) nonorientable surfaces $F \subset B^4$ with $\partial F = K$



$$\gamma_4(T_{2,2n+1}) = 1$$

In addition to b_1, a nonorientable surface has another fundamental invariant.

Definition

If $F \subset B^4$ is properly embedded, the normal euler number $e(F)$ is the difference between the Seifert framing of K and the framing that extends over F.

- Can be calculated as a change in writhe
- e.g. $e(F) = -4n - 2$ if F is the Möbius band with boundary $T_{2,2n+1}$.

The signature bound

Theorem (Gordon, Litherland 1978)

If K is the boundary of F,

$$\left|\sigma(K) - \frac{1}{2}e(F)\right| \leq b_1(F)$$

Proven by showing that the right hand side is the signature of the double cover of B^4 branched over F.

Batson's bound

Theorem (Batson 2012)

If K is the boundary of F,

$$\frac{1}{2}e(F) - 2d(S^3_{-1}(K)) \leq b_1(F)$$

Corollary

$$\gamma_4(K) \geq \frac{1}{2}\sigma(K) - d(S^3_{-1}(K))$$

Example

$$\gamma_4(T_{2k,2k-1}) = k - 1$$

Bound from Upsilon

Ozsváth, Stipsicz, and Szabó proved a similar bound involving the invariant $\upsilon(K) = \Upsilon_K(1)$.

Theorem (Ozsváth, Stipsicz, Szabó 2015)

$$\left|\upsilon(K) - \frac{1}{2}\sigma(K)\right| \leq \gamma_4(K)$$

Bound from t

Theorem

If K is the boundary of F,

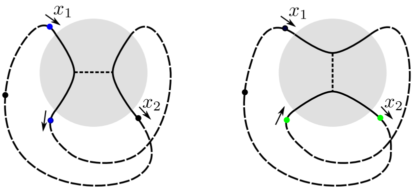
$$\left| t(K) + \frac{1}{2} e(F) \right| \leq b_1(F)$$

Theorem

$$|t(K) + \sigma(K)| \leq 2\gamma_4(K)$$

The nonorientable genus bound

Need to relate $\overline{C}(K_1)$ and $\overline{C}(K_1)$ on either side of a nonorientable band.



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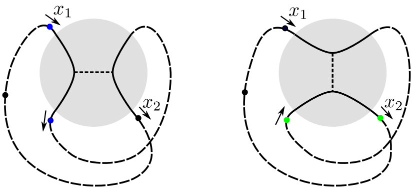

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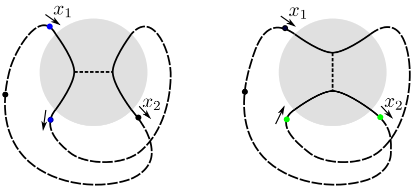


- Have the saddle map s, but this is often 0.
- In nonorientable case, $xs \sim sx$ and $xs \sim -sx$, so $2xs \sim 0$
- On reduced homology, $2xs = 0$, so this homotopy is a chain map.

A surprise

- This chain map induces an isomorphism on homology!
- It raises the filtration by at least $-e/2 - 1$, proving the bound

Limitations of current techniques

Proposition

If f_1 and f_2 are knot invariants with

$$\left| f_i(K) - \frac{1}{2}e(F) \right| \leq b_1(F),$$

then $|f_1(K) - f_2(K)| \leq 2g_T(K)$.

Can't see if $\gamma_4(K) > g_T(K)$

Values of $t(K)$

Proposition

If K is homologically thin, then $t(K) = s(K)$

Proposition

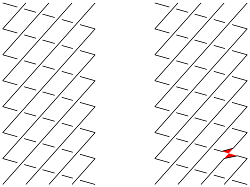
$t(T_{p,q}) = -2\upsilon(T_{p,q})$ *and is determined by the recurrence*

$$t(T_{p,q+p}) = t(T_{p,q}) + \left\lfloor \frac{p^2}{2} \right\rfloor$$

Values of $t(K)$

The computation of $t(T_{p,q})$ relies on two bounds

- A cobordism $T_{p,p} \sqcup T_{p,q} \to T_{p,q+p}$ gives an upper bound
  - This relies on a computation of $\mathrm{Kh}(T_{p,p})$ due to Stošić
- A nonorientable cobordism $T_{p,q} \to T_{r,s}$ gives a matching lower bound



A nonorientable cobordism $T_{5,9} \to T_{1,0}$

Invariants from Lee reduced homology

- $C(K)/(x-1) \otimes_{\mathbb{Z}} \mathbb{Q}$ has two independent filtrations
- A one-parameter way to collapse these two a single $\mathbb{R}$-filtration

$$(q-3h)\alpha + h(2-\alpha)$$

- For each α, define $\Phi_K(\alpha)$ using filtration on homology.

Invariants from Lee reduced homology

Theorem

- Φ *is a concordance homomorphism valued in PL functions on* $[0, 2]$
- $\Phi_K(0) = \Phi_K(2) = 0$, $\Phi'_K(0) = s(K)$, *and* $\Phi'_K(2) = -t(K)$
- $|\Phi_K(\alpha)| \leq 2g_4(K)\min(\alpha, 2-\alpha)$

The slice genus bound

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The slice genus bound

- Suppose K bounds a ribbon surface Σ
- K_n is the result of inserting n positive full twists into each band of Σ
- $\Phi_K \geq \Phi_{K_1} \geq \cdots \geq \Phi_{K_n} \geq \cdots$
- $\lim_{n\to\infty} \Phi_{K_n}(\alpha) = -2g(\Sigma)\min(\alpha, 2-\alpha)$

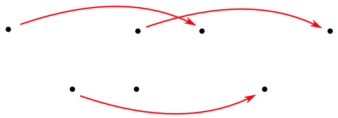
Algebraic structure

$\mathrm{CKh}^{\pm}(K) := C(K)/(x-1) \otimes_{\mathbb{Z}} \mathbb{Q}$ has two independent filtrations, but is not a filtered complex...

- $d(\mathrm{CKh}^{\pm}(K)_{i,j}) \subset \mathrm{CKh}^{\pm}(K)_{i-3,j+1}$
- Terms in d changing first filtration by exactly -3 are the Lee differential
- Terms in d changing second filtration by exactly 1 are the $E(-1)$ differential
- Can also have deeper terms

Example: the $(2,1)$-cable of $T_{2,3}$

Part of the Khovanov homology of $(T_{2,3})_{2,1}$:



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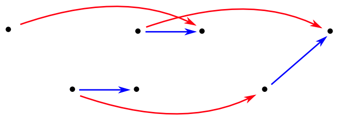

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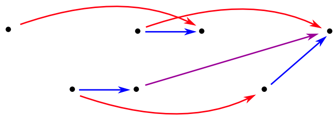

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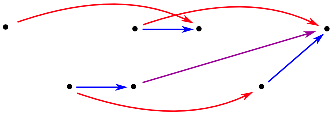
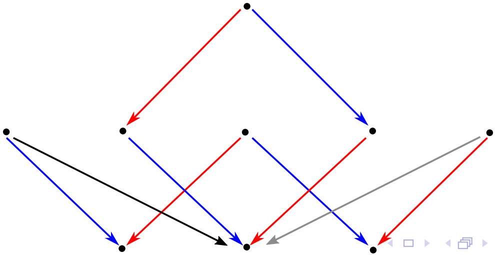
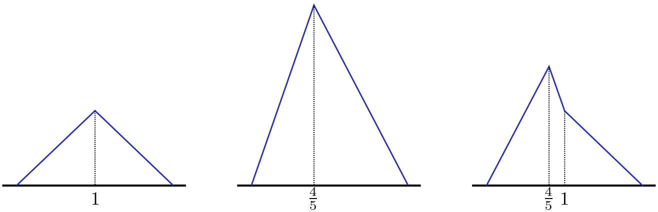


Could come from a piece of HOMFLY homology like this:



Simplest calculations

On small examples, can calculate $\mathrm{Kh}(K)$ by computer and manually fill in higher differentials.



$\Phi_K(\alpha)$ for $K = T_{2,3}$, $T_{3,4}$, and the $(2,1)$ cable of $T_{2,3}$

Three stranded pretzel knots

Theorem

For $K = P(-2k, a, b)$ with $a, b > 2k$ and odd, the first singularity of $\Phi_K(\alpha)$ occurs at $\frac{8k}{8k+2}$.

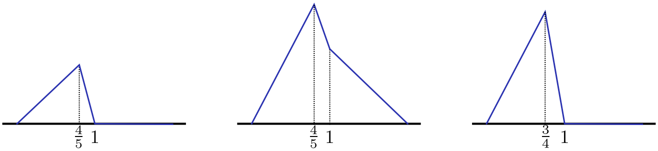
Uses computations of $\mathrm{Kh}(K)$ due to Manion.

Theorem

If a_k and b_k are two sequences of odd integers with $a_k, b_k > 2k$, then the pretzel knots $P(-2k, a_k, b_k)$ are a basis for a $\mathbb{Z}^\infty$ summand of the smooth concordance group.

Other $(2, n)$ cables

- The Khovanov homology of the cable knot $K_{2,2n+1}$ varies with n in a predictable way
- $\Phi_{K_{2,2n+1}} - \Phi_{T_{2,2n+1}}$ is independent of n for sufficiently large or small n



$\Phi_K(\alpha)$ for $K = (T_{2,3})_{2,2n+1} \# \overline{T_{2,2n+1}}$ in the ranges $2n+1 < 0$, $0 < 2n+1 < 6$, and $6 < 2n+1$.

Other torus knots

Conjecture

$\Phi'_{T_{p,p+1}}(\alpha)$ is discontinuous at $\alpha_k = \frac{2+2k}{2+3k}$ for $k = p-2, p-4, \ldots,$ and

$$\Delta\Phi'(\alpha_k) = -4 - 6k$$

Conjecture

$$\Phi_{T_{p,p+q}} = \Phi_{T_{p,q}} + \Phi_{T_{p,p+1}}$$

Further invariants

Conjecture

Each of the concordance invariants from knot Floer homology has an analogue defined in terms of $C(K)$.

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