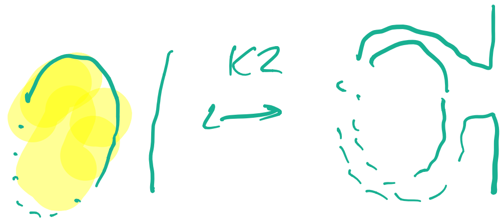
Non-semisimple quantum invariants

joint w/ Blanchet, Hikami, Geer,
Gainutdinov

please ask questions

$$\left\{ \begin{matrix} \text{framed} \\ \text{links} \\ \text{in } S^3 \end{matrix} \right\} \Big/ \text{K1-K2} \xrightarrow[1:1]{\text{surgery}} \left\{ \begin{matrix} \text{cmp} \\ \text{3-mnf} \end{matrix} \right\}$$

$$L \cup \bigcirc \overset{K1}{\longleftrightarrow} L$$



Non-semisimple quantum invariants

$$
\begin{tikzcd}
\left\{\begin{matrix}\text{framed}\\ \text{links}\\ \text{in } S^3\end{matrix}\right\} \Big/ \text{K1-K2} \arrow[r, "\text{surgery}", "1:1"'] \arrow[d, "U_q(sl_2)"', "\begin{matrix}\text{ribbon}\\ \text{Hopf}\\ \text{algebra}\end{matrix}"] & \left\{\begin{matrix}\text{cmp}\\ \text{3-mnf}\end{matrix}\right\} \arrow[d, "\begin{matrix}\text{`finiteness'}\\ q = e_p\\ \text{root of unity}\end{matrix}"] \\
\left\{\begin{matrix}\text{quantum}\\ \text{link}\\ \text{invariants}\end{matrix}\right\} \arrow[r, "\text{``integration''}"] & \left\{\begin{matrix}\text{quantum}\\ \text{3-mnf}\\ \text{invariants}\end{matrix}\right\}
\end{tikzcd}
$$

Examples:

links

1. RT
2. ADO

3-mnf

1. WRT
2. HKR
3. CGP

① WRT (Witten-Reshetikhin-Turaev) '91

$H = U_h(sl_2)$ gener. by $H, E, F \,/\, \mathbb{Q}[[h]]$

$[H,E]=2E \quad [H,F]=-2F, \quad [E,F]=\frac{K-K^{-1}}{v-v^{-1}}$ with

$q=e^h, \quad v=q^{h/2}, \quad K=v^H$

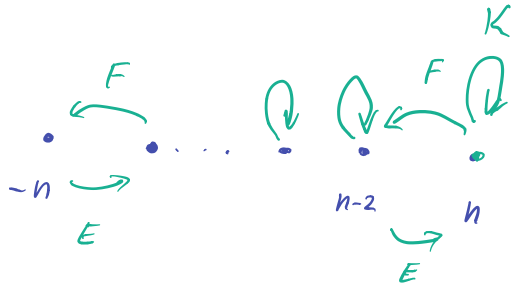
$$R = v^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} v^{\frac{n(n-1)}{2}} \frac{\{1\}^n}{[n]!} F^n \otimes E^n = \alpha \otimes \beta$$

$$[n] = \frac{\{n\}}{\{1\}}, \quad \{n\} = v^n - v^{-n}$$

For generic q

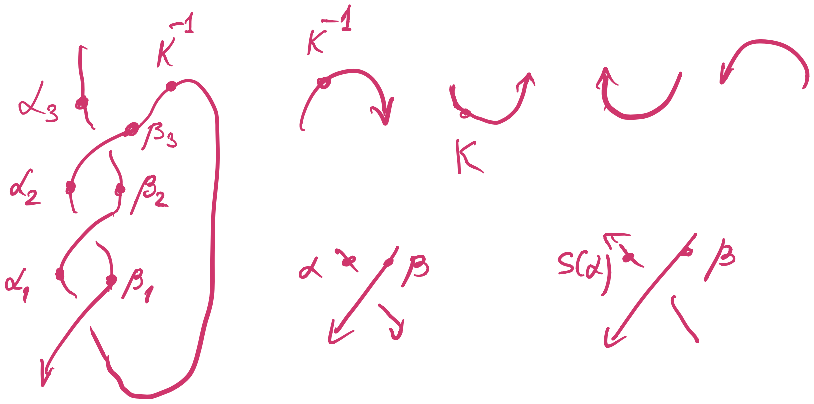
irreps

f.d. $V \in \operatorname{Rep} H \Rightarrow V = \bigoplus_n V_n$



$(n+1)$-dimens.

universal link invariant



$$J_{3_1} = \alpha_3 \beta_2 \alpha_1 K^{-1} \beta_3 \alpha_2 \beta_1 \in Z(H)$$

$$C = \{1\}^2 FE + vK + vK^{-1}$$

$$C \circlearrowright V_n \qquad v^{n+1} + v^{-n-1}$$

$$\mathbb{Z}[v^{\pm 1}] \ni J_{3_1}(V_n) \qquad (n+1)\text{-colored Jones polynomial}$$

$$WRT(M) = \sum_{n=0}^{p-2} q\dim(V_n) J_K(V_n) \Big|_{\substack{q=e_p \\ v=e_{2p}}}$$

$$q\dim(V_n) = tr_{V_n}(K^{-1}) = v^{-n} + v^{-n+2} + \ldots + v^{n} =$$
$$= [n+1]$$

② HKR (Hennings - Kauffman - Radford) '98

H is finite dim. ribbon Hopf

Ex: $\mathcal{U}^{rest}_{q=e_p}$ generated by $E, F, K \Big/ \begin{matrix} E^p = F^p = 0 \\ K^{2p} = 1 \end{matrix}$

all reps have integral weights

$$\{ F^i K^b E^j \mid 0 \le i, j \le p-1, 0 \le b \le 2p \}$$

$$K w_\lambda = v^\lambda w_\lambda \quad \Rightarrow \quad v^{2p\lambda} = 1 \Rightarrow \lambda \in \mathbb{Z}$$

Any f. dim Hopf algebra H has an integral $\mu \in H^*$

$$(\mu \otimes id)\Delta x = \mu(x) 1$$

If H is ribbon $\Rightarrow \exists\ J_K \in Z(H)$

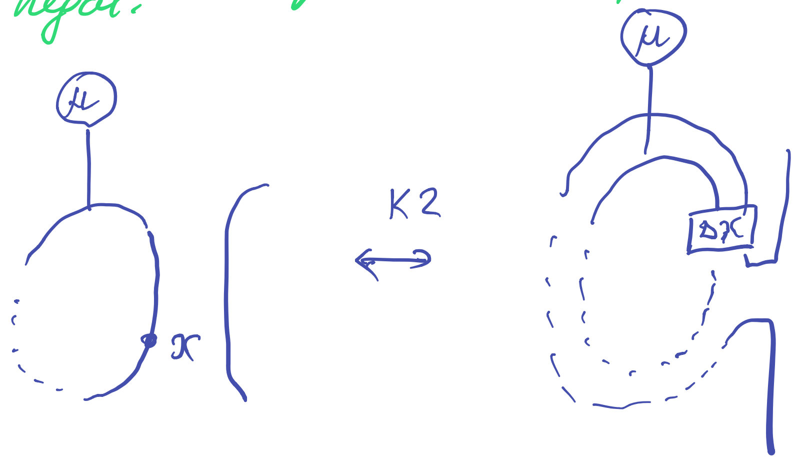
topolog. invariant

$$M = S^3(K_0) \Rightarrow HKR(M) := \mu(J_K)$$

$|L| = m$

$$M = S^3(L) \qquad HKR(M) := \frac{\mu^{\otimes m}(J_L)}{\mu(J_{U_+})^{\sigma_+}\,\mu(J_{U_-})^{\sigma_-}}$$

$\sigma_\pm$ # posit. / negat. eigenvalues of $lk(L)$



$$\Delta x = x_{(1)} \otimes x_{(2)}$$

$$\mu(x)\mathbb{1} = (\mu \otimes id)\Delta x$$

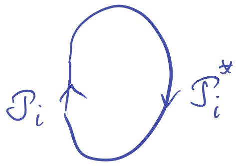
Thm (Kerler '96, Chen-Kappum-Srinivasan '07) $M \in \mathbb{Q}HS$

$$HKR(M) = \begin{cases} |H_1(M)| \; WRT(M) & b_1(M) = 0 \\ 0 & b_1(M) > 0 \end{cases}$$

$H \circlearrowright H = \bigoplus_i P_i$ as regular rep

P_i are indecomp. projectives $\quad \cdots \to P_1 \to \mathbb{K}$

$$\boxed{\mathrm{qdim}(P_i) = 0}$$



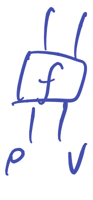
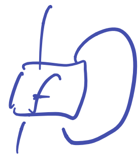
RT-functor

$$\begin{array}{c} \mathbb{K} \\ \uparrow \\ P_i \otimes P_i^* \\ \uparrow \\ \mathbb{K} \end{array} \quad = P_1 \oplus \cdots$$

$$\mu = \sum_i a_i \, \mathrm{qchar}_{P_i} \in H^*$$

Modified traces (Geer–Patureau–Turaev)

$$\{ t_P : \mathrm{End}\, P \to \mathbb{K} \}_{P \in H\text{-pmod}}$$

- $t_P(fg) = t_P(gf)$
- $t_{P \otimes V}\Big($  $\Big) = t_P\Big($  $\Big)$ partial trace property

$\forall \; V \in H\text{-mod}$

Thm (B-Blanchet-Gainutdinov) '18

Let H be a f. dim pivotal unimodular Hopf algebra

$$t_H(x) = \mu(gx) = \mu_g(x) \quad \forall x \in H$$

$$\mu_g(xy) = \mu_g(yx)$$

g pivotal

B-Blanchet-Geer

(2') $M = S^3(L)$, $K \subset M$

$$BBGe(M, K_a) = (\mu \otimes \mu_g P_a) J_{L \cup K}$$

top. invariant (M, K)

$$\operatorname{Im} P_a = \mathcal{P}_a, \quad P_a^2 = P_a$$

- $BBGe(M, U_a) \neq 0$ if $b_1(M) > 0$
- extends to TQFTs
- generalizes to all quantum groups

De Renzi-Geer-Patureau

- reps of $MCG(\Sigma_g)$

special cases of Meusburger

③ CGP (Costantino - Geer - Patureau) '11

$\overline{\mathcal{U}}^H$ unrolled quantum $\mathfrak{sl}(2)$ at ζ_p

$$H, E, F \Big/ E^p = F^p = 0 \qquad \cancel{K^{2p} = 1}$$

Rep $\overline{\mathcal{U}}^H$ has p-dim. projective module of highest weight $\lambda \in \mathbb{C}$

Rep $\overline{\mathcal{U}}^H$ admit $\lambda \in \mathbb{C}/2\mathbb{Z}$ grading

$$V_\lambda, \; V_{\lambda+2}, \; \dots \; V_{\lambda+2p-2}$$

$\lambda \in 0,1$

$$M = S^3(K) \qquad \lambda \in H^1(M, \mathbb{C}/2\mathbb{Z})$$

$$CGP(M,\lambda) = \sum_{k=0}^{p-1} \operatorname{mdim}(V_{\lambda+2k}) \underbrace{J_K(V_{\lambda+2k})}_{ADO_K(\lambda, \zeta_p)}$$

top. invariant (M,λ)

$$\operatorname{mdim}(V_\lambda) = \frac{\{\lambda+1\}}{\{p\lambda\}}$$

ADO (Akutsu - Deguchi - Ohtsuki) invariants of links colored by projectives

$q^\lambda = t$

$\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$

Properties of CGP

- extend to a non-semisimple TQFTs (Blanchet-CGP)

Dehn twist has ∞ order

- classify lens spaces

- $p=2$ CGP = Reidemeister torsion

(categorified by HFH)

- conjecturally categorifies

(Gukov - Putrov - Pei - Vafa)

Q1: What is the relationship between CGP and WRT?

- Unified WRT invariants

Habiro's cyclotomic expansion

$$J_K^{cyc} = \sum_{k=0}^{\infty} a_k \sigma_k \in \hat{Z}(U_q sl_2), \quad a_k \in \mathbb{Z}[q^{\pm 1}]$$

$$\sigma_k = \prod_{i=1}^{k} C^2 - (v^i + v^{-i})^2 = \prod_{i=1}^{k} (t + t^{-1} - q^i - q^{-i})$$

$C = \{1\}^2 FE + vK + v^{-1}K^{-1}$ Casimir

$C \curvearrowright V_n$ by $v^{n+1} + v^{-n-1}$

$t = q^{n+1}$ for $(n+1)$-colored Jones

$J_U^{cyc} = 1$

$$WRT(S^3(K_f)) = \sum_{n=0}^{p-1} q^{f\frac{n^2-1}{4}} [n]^2 J_K^{cyc}(V_{n-1})$$

$$WRT(M) = \sum_{k=0}^{\infty} \sum_{n=0}^{p-1} q^{\frac{b(n^2-1)}{4}} [n]^2 a_k \sigma_k \Big|_{t=q^n}$$

- Laplace transform (Le, B-Le)

$$t^{\pm a} \mapsto q^{-\frac{a^2}{b}} \qquad b = \pm 1 \text{ or } \frac{s}{t}$$

Habiro $M \in \mathbb{Z}HS$

B-Bühler-Le $M \in \mathbb{Q}HS$

$$\widehat{\mathbb{Z}[q]} \ni J_M = \sum_{k=0}^{\infty} a_k (q^{k+1})_{k+1}$$

$$(q^l)_n = (1-q^l)(1-q^{l+1}) \dots (1-q^{l+n-1})$$

- $ev_{q=e_p}(J_M) = WRT(M)$
- evaluations at $\{p^k \mid p \text{ prime}, k \in \mathbb{N}\}$ determine $\{WRT \mid e_p, p \in \mathbb{Z}\}$
- J_M determines Ohtsuki series

Q2: Does Habiro's cyclotomic expansion determine ADO?

Example: $K = 4_1$

$$J^{cyc}_{4_1} = \sum_{k=0}^{\infty} \sigma_k \qquad \sigma_k = \prod_{i=1}^{k} (t + t^{-1} - q^{i} - q^{-i})$$

$$J_{4_1}\Big|_{q=1} = \sum_{k=0}^{\infty} (t + t^{-1} - 2)^k = \frac{1}{1 - t - t^{-1} + 2} = \frac{1}{\Delta_{4_1}(t)}$$

celebrated MMR conjecture

(Melvin - Morton - Rozansky)

$p=2$ $\quad q=-1$

$$\sigma_2 = (t+t^{-1}+2)(t+t^{-1}-2) = t^2+t^{-2}-2$$

$$\sigma_3 = \sigma_1\sigma_2 \qquad \sigma_4 = \sigma_1\sigma_2^2 \quad \ldots$$

$$J_{4_1}\Big|_{q=-1} = (1+t+t^{-1}+2)\sum_{k=0}^{\infty}(t^2+t^{-2}-2)^k =$$

$$= \frac{\Delta_{4_1}(-t)}{\Delta_{4_1}(t^2)} \quad ADO_{p=2}$$

In general, $q=e_p$ $\qquad \sigma_p = t^p+t^{-p}-2$

$$J^{cyc}_{4_1} = \frac{\sum_{k=0}^{p-1}\sigma_k}{\Delta_{4_1}(t^p)} \quad ADO_p$$

Thm (S. Willetts, B-Blanchet)

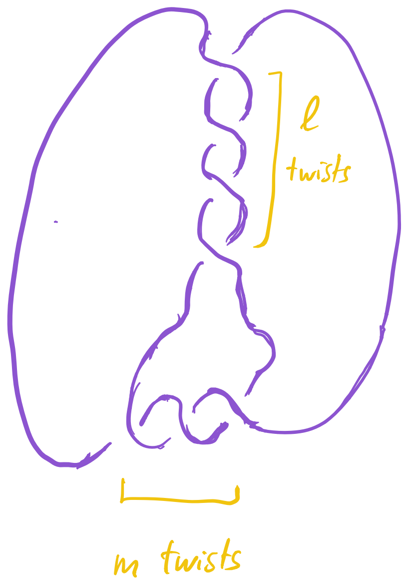
$$J_K^{cyc}\Big|_{q=e_p} = \frac{ADO_K(t, e_{2p})}{\Delta_K(t^p)} \in \hat{R}$$

$$\hat{R} = \lim_{\overleftarrow{n}} \frac{\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]}{I_n}$$

$$I_n = \{ (q^{n-i})_i \, (t)_{n-i} \mid 0 \le i \le n \}$$

$$(x)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$$

Consider a double-twist knot $K_{(\ell,m)}$



Example: $K_{(2,-1)}$

$K_{(1,1)} = 3_1$ $\quad K_{(-1,1)} = 4_1$

For $\ell, m > 0$

$$a_n = (-1)^n q^{-\frac{n(n-1)}{2}} \sum_{n = t_m \geq \dots \geq t_1 \geq 0} \prod_{i=1}^{m-1} q^{t_i(t_i+1)} \begin{bmatrix} t_{i+1} \\ t_i \end{bmatrix}$$

$$\sum_{n = s_\ell \geq s_{\ell-1} \geq \dots \geq s_1 \geq 0} \prod_{j=1}^{\ell-1} q^{s_j(s_j+1)} \begin{bmatrix} s_{j+1} \\ s_j \end{bmatrix}$$

$m=1$ Masbaum '03, Lovejoy-Osburn '17

Thm (B- Hikami)

$K = K_{(\ell,m)}$ double twist knot, p odd

$$ADO_K(t, e_{2p}) = \sum_{k=0}^{p-1} a_k \, \sigma_k \Big|_{q=e_p}$$

$$a_{n+kp}\Big|_{e_p} = a_n\Big|_{e_p} \; a_k\Big|_{q=1}$$

$$\Delta_K^{-1}(t) = \sum_{k=0}^{\infty} a_{kp}\Big|_{e_p} (t+t^{-1}-2)^k = \sum_{k=0}^{\infty} (-\ell m)^k (t+t^{-1}-2)^k$$

If M is 0-surgery on K,

$$CGP(M, \lambda) = \frac{WRT^{SO(3)}(M)}{\{p\lambda\}^2} + p \, a_{p-1}\Big|_{e_p}$$

- a_{p-1} generalizes Reidemeister torsion

$p=2$ $\qquad 1 + a_1 (t+t^{-1}+2) = \Delta_K(-t)$

$$a_1\Big|_{(-1)} = -\ell m$$

- $K = T(2,2a+1)$ torus knot Hikami '03

$$J_K(t,q) = q^a t^a \sum_{k_a \geq \dots \geq k_1 \geq 0} (tq)_{k_a} t^{k_a} \cdot \prod_{i=1}^{a-1} q^{k_i(k_i+1)} t^{2k_i} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}$$

$ev(q) = e_p$

$$J_K(t,e_p) = \frac{e_p^a\, t^{a(1-p)}}{\Delta_K(t^p)} \sum_{k_a \geq \dots \geq k_1 \geq 0}^{p-1} (te_p)_{k_a} t^{k_a} \cdot \prod_{i=1}^{a-1} e_p^{k_i(k_i+1)} t^{2k_i} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_{e_p}$$

$$\Delta_{T(2,2a+1)}(t) = t^{-a} \frac{(1+t^{2a+1})}{1+t}$$

$$\begin{bmatrix} n + bp \\ k + cp \end{bmatrix}_{e_p} = \begin{bmatrix} n \\ k \end{bmatrix}_{e_p} \begin{pmatrix} a \\ b \end{pmatrix}$$