The Turaev-Viro Volume Conjecture

Giulio Belletti

Scuola Normale Superiore, Pisa

14 May 2020

Partly joint work with R. Detcherry, E. Kalfagianni, T. Yang

Topological quantum field theories

Topological quantum field theories $\rightsquigarrow$ Invariants of 3-manifolds

Topological quantum field theories $\rightsquigarrow$ Invariants of 3-manifolds, representations of the mapping class group

Topological quantum field theories $\rightsquigarrow$ Invariants of 3-manifolds, representations of the mapping class group, quantization of moduli spaces of surfaces...

Topological quantum field theories $\rightsquigarrow$ Invariants of 3-manifolds, representations of the mapping class group, quantization of moduli spaces of surfaces...

What information do the TQFT invariants hold?

The invariants of Turaev-Viro take a compact 3-manifold, The invariants of Turaev-Viro take a compact 3-manifold, a natural r (the level)

The invariants of Turaev-Viro take a compact 3-manifold, a natural r (the level) and a $2r$-th root of unity q (think $q = e^{i\hbar}$),

The invariants of Turaev-Viro take a compact 3-manifold, a natural r (the level) and a $2r$-th root of unity q (think $q = e^{i\hbar}$), and give a real number: the TV invariant $TV_r(M, q)$.

The invariants of Turaev-Viro take a compact 3-manifold, a natural r (the level) and a $2r$-th root of unity q (think $q = e^{i\hbar}$), and give a real number: the TV invariant $TV_r(M, q)$.

Main problem: what is the asymptotic behavior of $TV_r(M, q)$?

The invariants of Turaev-Viro take a compact 3-manifold, a natural r (the level) and a $2r$-th root of unity q (think $q = e^{i\hbar}$), and give a real number: the TV invariant $TV_r(M, q)$.

Main problem: what is the asymptotic behavior of $TV_r(M, q)$?

Motivation: $r \to +\infty$ and $q \to 1$ would correspond to $\hbar \to 0$ or the semiclassical limit from physics.

The Witten asymptotics expansion conjecture

When we choose $q = e^{\pi i/r}$, growth is at most polynomial.

The Witten asymptotics expansion conjecture

When we choose $q = e^{\pi i/r}$, growth is at most polynomial.

Witten's conjecture:

$$TV_r(M) \sim_{r\to\infty} \left| \int_A e^{2\pi i CS(A)} r^{(h^1_A - h^0_A)} e^{-2\pi i \left(\frac{2I_A + h^0_A}{8}\right)} \tau_M(A)^{\frac{1}{2}} \right|^2$$

The Witten asymptotics expansion conjecture

When we choose $q = e^{\pi i/r}$, growth is at most polynomial.

Witten's conjecture:

$$TV_r(M) \sim_{r\to\infty} \left| \int_A e^{2\pi i CS(A)} r^{(h_A^1 - h_A^0)} e^{-2\pi i \left(\frac{2I_A + h_A^0}{8}\right)} \tau_M(A)^{\frac{1}{2}} \right|^2$$

Main features:

1. polynomial growth;

The Witten asymptotics expansion conjecture

When we choose $q = e^{\pi i/r}$, growth is at most polynomial.

Witten's conjecture:

$$TV_r(M) \sim_{r\to\infty} \left| \int_A e^{2\pi i CS(A)} r^{(h_A^1 - h_A^0)} e^{-2\pi i \left(\frac{2 I_A + h_A^0}{8}\right)} \tau_M(A)^{\frac{1}{2}} \right|^2$$

Main features:

1. polynomial growth;
2. topological information (no geometry).

The volume conjecture

If instead $q = e^{2\pi i/r}$ we see a very different behavior:

The volume conjecture

If instead $q = e^{2\pi i/r}$ we see a very different behavior:

Conjecture (Turaev-Viro Volume conjecture (Chen-Yang))

If M is a hyperbolic 3-manifold

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| TV_r \left(M, e^{\frac{2\pi i}{r}} \right) \right| = \mathrm{Vol}(M).$$

The volume conjecture

If instead $q = e^{2\pi i/r}$ we see a very different behavior:

Conjecture (Turaev-Viro Volume conjecture (Chen-Yang))

If M is a hyperbolic 3-manifold

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| TV_r \left(M, e^{\frac{2\pi i}{r}} \right) \right| = \mathrm{Vol}(M).$$

The volume conjecture

If instead $q = e^{2\pi i/r}$ we see a very different behavior:

Conjecture (Turaev-Viro Volume conjecture (Chen-Yang))

If M is a hyperbolic 3-manifold

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| TV_r \left(M, e^{\frac{2\pi i}{r}} \right) \right| = \mathrm{Vol}(M).$$

Alternatively,

$$TV_r(M) \sim_{r\to\infty} P(r) e^{\frac{r}{2\pi}\mathrm{Vol}(M)}$$

The volume conjecture

If instead $q = e^{2\pi i/r}$ we see a very different behavior:

Conjecture (Turaev-Viro Volume conjecture (Chen-Yang))

If M is a hyperbolic 3-manifold

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| TV_r \left(M, e^{\frac{2\pi i}{r}} \right) \right| = \mathrm{Vol}(M).$$

Alternatively,

$$TV_r(M) \sim_{r\to\infty} P(r) e^{\frac{r}{2\pi}\mathrm{Vol}(M)}$$

Theorem (B., Detcherry, Kalfagianni, Yang '18)

The volume conjecture holds for the complements of Fundamental Shadow Links (FSL)

The volume conjecture

If instead $q = e^{2\pi i/r}$ we see a very different behavior:

Conjecture (Turaev-Viro Volume conjecture (Chen-Yang))

If M is a hyperbolic 3-manifold

$$\lim_{r\to\infty} \frac{2\pi}{r} \log\left| TV_r\left(M, e^{\frac{2\pi i}{r}}\right)\right| = \mathrm{Vol}(M).$$

Alternatively,

$$TV_r(M) \sim_{r\to\infty} P(r) e^{\frac{r}{2\pi}\mathrm{Vol}(M)}$$

Theorem (B., Detcherry, Kalfagianni, Yang '18 + B., '20)

The volume conjecture holds for the complements of Fundamental Shadow Links (FSL) and another related family of cusped manifolds.

Outline

Turaev-Viro invariants

- Take a compact 3-manifold and a triangulation τ;

Turaev-Viro invariants

- Take a compact 3-manifold and a triangulation τ;
- Take c a coloring of each edge of τ with a natural number between 0 and $r-2$;

Turaev-Viro invariants

- Take a compact 3-manifold and a triangulation τ;
- Take c a coloring of each edge of τ with a natural number between 0 and $r-2$;
- Choose appropriately weight functions $|e|_c$ and $|T|_c$;

Turaev-Viro invariants

- Take a compact 3-manifold and a triangulation τ;
- Take c a coloring of each edge of τ with a natural number between 0 and $r-2$;
- Choose appropriately weight functions $|e|_c$ and $|T|_c$;

  $|e|_c$ takes a coloring c and an edge $e \in \tau$ and gives a complex number; $|T|_c$ takes a coloring c and a tetrahedron $T \in \tau$ and gives a complex number.

Turaev-Viro invariants

- Take a compact 3-manifold and a triangulation τ;
- Take c a coloring of each edge of τ with a natural number between 0 and $r-2$;
- Choose appropriately weight functions $|e|_c$ and $|T|_c$;

  $|e|_c$ takes a coloring c and an edge $e \in \tau$ and gives a complex number; $|T|_c$ takes a coloring c and a tetrahedron $T \in \tau$ and gives a complex number. Define

$$TV_r(M) := \left(\frac{\sqrt{2}\sin\left(\frac{2\pi}{r}\right)}{\sqrt{r}} \right)^{2|V|} \sum_c \prod_{e \in E} |e|_c \prod_{T \in \tau} |T|_c .$$

Turaev-Viro invariants

- Take a compact 3-manifold and a triangulation τ;
- Take c a coloring of each edge of τ with a natural number between 0 and $r-2$;
- Choose appropriately weight functions $|e|_c$ and $|T|_c$;

  $|e|_c$ takes a coloring c and an edge $e \in \tau$ and gives a complex number; $|T|_c$ takes a coloring c and a tetrahedron $T \in \tau$ and gives a complex number. Define

$$TV_r(M) := \left(\frac{\sqrt{2} \sin\left(\frac{2\pi}{r}\right)}{\sqrt{r}} \right)^{2|V|} \sum_c \prod_{e \in E} |e|_c \prod_{T \in \tau} |T|_c .$$

To finish, we need that $|e|$ and $|T|$ satisfy some equations.

The weight functions

The weight functions

The weight functions we are interested in come from the representation theory of $U_q(\mathfrak{sl}_2)$.

The weight functions

The weight functions we are interested in come from the representation theory of $U_q(\mathfrak{sl}_2)$.

$$|e|_{col} := (-1)^{col(e)} 2i \sin(2\pi(col(e)+1)/r)$$

The weight functions

The weight functions we are interested in come from the representation theory of $U_q(\mathfrak{sl}_2)$.

$$|e|_{col} := (-1)^{col(e)} 2i \sin(2\pi(col(e)+1)/r)$$

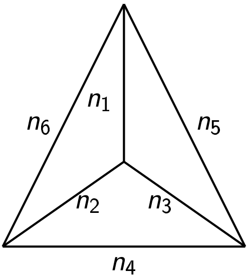


Figure: A coloring of a tetrahedron

The weight functions

The weight functions we are interested in come from the representation theory of $U_q(\mathfrak{sl}_2)$.

$$|e|_{col} := (-1)^{col(e)} 2i \sin(2\pi(col(e)+1)/r)$$

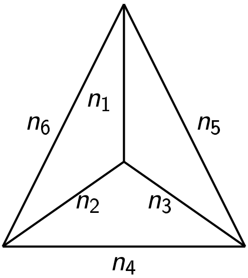


Figure: A coloring of a tetrahedron

$$|T|_{col} := \begin{vmatrix} n_4 & n_5 & n_6 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

This is called a *quantum 6j-symbol*.

The weight functions we are interested in come from the representation theory of $U_q(\mathfrak{sl}_2)$.

$$|e|_{col} := (-1)^{col(e)} 2i \sin(2\pi(col(e)+1)/r)$$

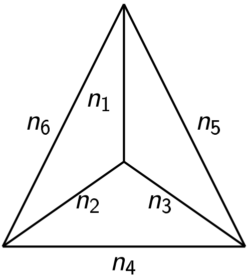


Figure: A coloring of a tetrahedron

$$|T|_{col} := \begin{vmatrix} n_4 & n_5 & n_6 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

This is called a *quantum 6j-symbol*.

Turaev-Viro invariant of graphs

We can define, with little additional effort, a Turaev-Viro invariant for graphs in manifolds.

Turaev-Viro invariant of graphs

We can define, with little additional effort, a Turaev-Viro invariant for graphs in manifolds.

- Take a compact 3-manifold M, a graph $\Gamma \subseteq M$ and a triangulation τ such that $\Gamma \subseteq \tau^{(1)}$;

Turaev-Viro invariant of graphs

We can define, with little additional effort, a Turaev-Viro invariant for graphs in manifolds.

- Take a compact 3-manifold M, a graph $\Gamma \subseteq M$ and a triangulation τ such that $\Gamma \subseteq \tau^{(1)}$;
- Take col a coloring of Γ and col' a coloring of $\tau^{(1)}$ that extends it

Turaev-Viro invariant of graphs

We can define, with little additional effort, a Turaev-Viro invariant for graphs in manifolds.

- Take a compact 3-manifold M, a graph $\Gamma \subseteq M$ and a triangulation τ such that $\Gamma \subseteq \tau^{(1)}$;
- Take *col* a coloring of Γ and *col'* a coloring of $\tau^{(1)}$ that extends it
- Take the same weight functions $|e|_c$ and $|T|_c$;

Turaev-Viro invariant of graphs

We can define, with little additional effort, a Turaev-Viro invariant for graphs in manifolds.

- Take a compact 3-manifold M, a graph $\Gamma \subseteq M$ and a triangulation τ such that $\Gamma \subseteq \tau^{(1)}$;
- Take col a coloring of Γ and col' a coloring of $\tau^{(1)}$ that extends it
- Take the same weight functions $|e|_c$ and $|T|_c$;
- Define

$$TV_r(M, \Gamma, col) := \sum_{col'} \prod_{e \in E} |e|_{col'} \prod_{T \in \tau} |T|_{col'}.$$

Where's the difficulty?

Approach to prove the volume conjecture:

Where's the difficulty?

Approach to prove the volume conjecture:

$$TV_r(M) = \sum_{\text{some set}} \text{some big product.}$$

Where's the difficulty?

Approach to prove the volume conjecture:

$$TV_r(M) = \sum_{\text{some set}} \text{some big product}.$$

Approximate the big product with a smooth function to obtain

$$TV_r(M) \sim \sum_{\vec{i} \in \text{some set} \subseteq \mathbb{N}^n} f(\vec{i}, r)$$

Where's the difficulty?

Approach to prove the volume conjecture:

$$TV_r(M) = \sum_{\text{some set}} \text{some big product.}$$

Approximate the big product with a smooth function to obtain

$$TV_r(M) \sim \sum_{\vec{i} \in \text{some set} \subseteq \mathbb{N}^n} f(\vec{i}, r)$$

Most of the summands grow exponentially; however the signs can alternate and there are wild cancellations.

The approach via the Fourier transform

Poisson summation formula:

$$\sum_{\vec{i}\in\mathbb{Z}^n} f(\vec{i}, r) = \sum_{\vec{j}\in\mathbb{Z}^n} \hat{f}(\vec{j}, r)$$

The approach via the Fourier transform

Poisson summation formula:

$$\sum_{\vec{i}\in\mathbb{Z}^n} f(\vec{i}, r) = \sum_{\vec{j}\in\mathbb{Z}^n} \hat{f}(\vec{j}, r)$$

If you are lucky enough, one summand grows faster than any other

The approach via the Fourier transform

Poisson summation formula:

$$\sum_{\vec{i}\in\mathbb{Z}^n} f(\vec{i}, r) = \sum_{\vec{j}\in\mathbb{Z}^n} \hat{f}(\vec{j}, r)$$

If you are lucky enough, one summand grows faster than any other and it's easy to estimate

The approach via the Fourier transform

Poisson summation formula:

$$\sum_{\vec{i}\in\mathbb{Z}^n} f(\vec{i}, r) = \sum_{\vec{j}\in\mathbb{Z}^n} \hat{f}(\vec{j}, r)$$

If you are lucky enough, one summand grows faster than any other and it's easy to estimate and it grows at the correct rate.

The approach via the Fourier transform

Poisson summation formula:

$$\sum_{\vec{i} \in \mathbb{Z}^n} f(\vec{i}, r) = \sum_{\vec{j} \in \mathbb{Z}^n} \hat{f}(\vec{j}, r)$$

If you are lucky enough, one summand grows faster than any other and it's easy to estimate and it grows at the correct rate.

Theorem (Ohtsuki, 2018)

The Turaev-Viro volume conjecture holds for Dehn fillings of the Figure Eight knot.

Upper bound on the 6j-symbol

We need an upper bound on the growth of the 6j-symbol:

Theorem (B., Detcherry, Kalfagianni, Yang '18)

$$\frac{2\pi}{r}\log\left\|\begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix}\right\| \leqslant v_8 + O\left(\frac{\log(r)}{r}\right).$$

Upper bound on the 6j-symbol

We need an upper bound on the growth of the 6j-symbol:

Theorem (B., Detcherry, Kalfagianni, Yang '18)

$$\frac{2\pi}{r} \log \left\| \begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix} \right\| \leqslant v_8 + O\left(\frac{\log(r)}{r}\right).$$

$v_8 \cong 3.66$ *is the volume of the right-angled ideal hyperbolic octahedron.*

Upper bound on the 6*j*-symbol

We need an upper bound on the growth of the 6*j*-symbol:

Theorem (B., Detcherry, Kalfagianni, Yang '18)

$$\frac{2\pi}{r}\log\left\|\begin{vmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \end{vmatrix}\right\| \leqslant v_8 + O\left(\frac{\log(r)}{r}\right).$$

$v_8 \cong 3.66$ is the volume of the right-angled ideal hyperbolic octahedron.
It's sharp!

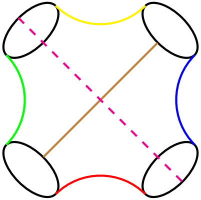
$$\frac{2\pi}{r}\log\left\|\begin{vmatrix} \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \\ \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \end{vmatrix}\right\| = v_8 + O\left(\frac{\log(r)}{r}\right).$$

Fundamental Shadow Links (FSL)

Basic building block:

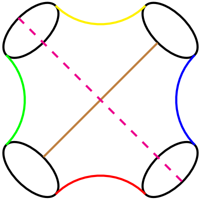
Fundamental Shadow Links (FSL)

Basic building block:



Fundamental Shadow Links (FSL)

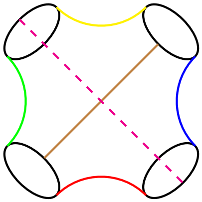
Basic building block:



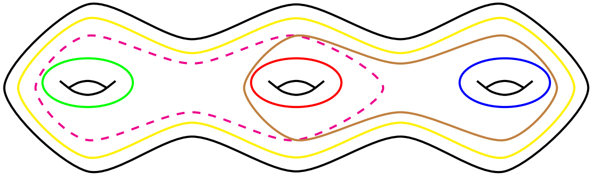
Glue g of them up

Fundamental Shadow Links (FSL)

Basic building block:

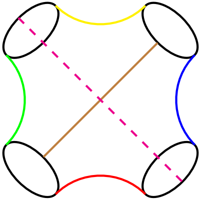


Glue g of them up:

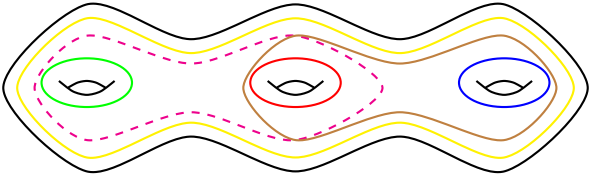


Fundamental Shadow Links (FSL)

Basic building block:



Glue g of them up:



and double: get a *fundamental shadow link* (FSL) in $\#^{g+1}(S^1 \times S^2)$.

Properties of FSLs

1. Always hyperbolic: decompose into $2g$ right-angle ideal octahedra;

Properties of FSLs

1. Always hyperbolic: decompose into $2g$ right-angle ideal octahedra;
2. Universal family: every 3-manifold is a Dehn filling of a FSL $\Leftrightarrow$ every 3-manifold contains a FSL;

Properties of FSLs

1. Always hyperbolic: decompose into $2g$ right-angle ideal octahedra;
2. Universal family: every 3-manifold is a Dehn filling of a FSL $\Leftrightarrow$ every 3-manifold contains a FSL;
3. Easy Turaev-Viro invariants

Properties of FSLs

1. Always hyperbolic: decompose into $2g$ right-angle ideal octahedra;
2. Universal family: every 3-manifold is a Dehn filling of a FSL $\Leftrightarrow$ every 3-manifold contains a FSL;
3. Easy Turaev-Viro invariants

$$TV_r(FSL) = \left(\frac{2\sin\left(\frac{2\pi}{r}\right)}{r} \right)^{-c} \sum_{colorings} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix}^2$$

The volume conjecture for FSLs

It is easy to show that the Volume Conjecture holds for FSLs:

The volume conjecture for FSLs

It is easy to show that the Volume Conjecture holds for FSLs:

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \sum_{colorings} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix}^2 \right|$$

The volume conjecture for FSLs

It is easy to show that the Volume Conjecture holds for FSLs:

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \sum_{colorings} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix}^2 \right| \leqslant$$

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left((\#\text{ of colorings}) \begin{vmatrix} col(m_1) & col(m_2) & col(m_3) \\ col(m_4) & col(m_5) & col(m_6) \end{vmatrix}^{2g} \right)$$

The volume conjecture for FSLs

It is easy to show that the Volume Conjecture holds for FSLs:

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \sum_{colorings} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix}^2 \right| \leqslant$$

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left((\#\text{ of colorings}) \begin{vmatrix} col(m_1) & col(m_2) & col(m_3) \\ col(m_4) & col(m_5) & col(m_6) \end{vmatrix}^{2g} \right) \leqslant 2g v_8$$

The volume conjecture for FSLs

It is easy to show that the Volume Conjecture holds for FSLs:

$$\lim_{r\to\infty}\frac{2\pi}{r}\log\left|\sum_{colorings}\prod_{blocks}\begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6)\end{vmatrix}^2\right| \leqslant$$

$$\lim_{r\to\infty}\frac{2\pi}{r}\log\left((\#\text{ of colorings})\begin{vmatrix} col(m_1) & col(m_2) & col(m_3) \\ col(m_4) & col(m_5) & col(m_6)\end{vmatrix}^{2g}\right) \leqslant 2gv_8$$

On the other hand,

The volume conjecture for FSLs

It is easy to show that the Volume Conjecture holds for FSLs:

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \sum_{colorings} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix}^2 \right| \leqslant$$

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left((\#\text{ of colorings}) \begin{vmatrix} col(m_1) & col(m_2) & col(m_3) \\ col(m_4) & col(m_5) & col(m_6) \end{vmatrix}^{2g} \right) \leqslant 2gv_8$$

On the other hand,

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \sum_{colorings} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix}^2 \right|$$

The volume conjecture for FSLs

It is easy to show that the Volume Conjecture holds for FSLs:

$$\lim_{r\to\infty}\frac{2\pi}{r}\log\left|\sum_{colorings}\prod_{blocks}\begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6)\end{vmatrix}^2\right| \leqslant$$

$$\lim_{r\to\infty}\frac{2\pi}{r}\log\left((\#\text{ of colorings})\begin{vmatrix} col(m_1) & col(m_2) & col(m_3) \\ col(m_4) & col(m_5) & col(m_6)\end{vmatrix}^{2g}\right) \leqslant 2gv_8$$

On the other hand,

$$\lim_{r\to\infty}\frac{2\pi}{r}\log\left|\sum_{colorings}\prod_{blocks}\begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6)\end{vmatrix}^2\right| \geqslant$$

$$\lim_{r\to\infty}\frac{2\pi}{r}\log\begin{vmatrix} \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2} \\ \frac{r\pm1}{2} & \frac{r\pm1}{2} & \frac{r\pm1}{2}\end{vmatrix}^{2g}$$

The volume conjecture for FSLs

It is easy to show that the Volume Conjecture holds for FSLs:

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \sum_{colorings} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix}^2 \right| \leqslant$$

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left((\#\text{ of colorings}) \begin{vmatrix} col(m_1) & col(m_2) & col(m_3) \\ col(m_4) & col(m_5) & col(m_6) \end{vmatrix}^{2g} \right) \leqslant 2gv_8$$

On the other hand,

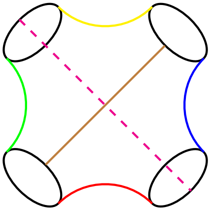
$$\lim_{r\to\infty} \frac{2\pi}{r} \log \left| \sum_{colorings} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix}^2 \right| \geqslant$$

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \begin{vmatrix} \frac{r\pm 1}{2} & \frac{r\pm 1}{2} & \frac{r\pm 1}{2} \\ \frac{r\pm 1}{2} & \frac{r\pm 1}{2} & \frac{r\pm 1}{2} \end{vmatrix}^{2g} = 2gv_8$$

A new family of manifolds

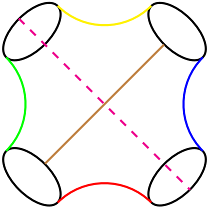
A new family of manifolds

Same basic building block:



A new family of manifolds

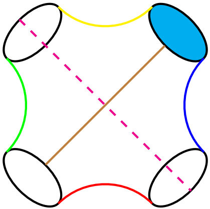
Same basic building block:



But now, we glue slightly differently.

A new family of manifolds

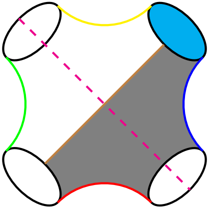
Same basic building block:



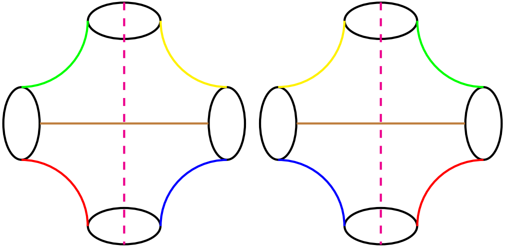
But now, we glue slightly differently. Each new block can only be glued either along a single triangular disk (in blue)

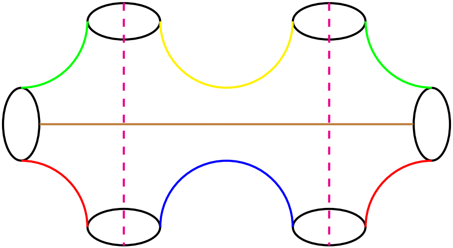
A new family of manifolds

Same basic building block:



But now, we glue slightly differently. Each new block can only be glued either along a single triangular disk (in blue) or a single hexagonal "face" delimited by 3 arcs (in gray).





After gluing g blocks, we get a big ball with some arcs in its boundary, some disks (from the blue disks of each block) and some faces (from the gray faces of each block).

After gluing g blocks, we get a big ball with some arcs in its boundary, some disks (from the blue disks of each block) and some faces (from the gray faces of each block).

Double the ball along the blue disks to obtain a genus $2g + 1$ handlebody with a link in its boundary

After gluing g blocks, we get a big ball with some arcs in its boundary, some disks (from the blue disks of each block) and some faces (from the gray faces of each block).

Double the ball along the blue disks to obtain a genus $2g+1$ handlebody with a link in its boundary and then double again to obtain a link in $\#^{2g+1}(S^1 \times S^2)$.

After gluing g blocks, we get a big ball with some arcs in its boundary, some disks (from the blue disks of each block) and some faces (from the gray faces of each block).

Double the ball along the blue disks to obtain a genus $2g+1$ handlebody with a link in its boundary and then double again to obtain a link in $\#^{2g+1}(S^1 \times S^2)$.

We call $\mathcal{M}$ the family of exteriors of these links.

Compare:

FSL:

Manifolds in $\mathcal{M}$:

Compare:

FSL:

- can glue each new block only along (triangular) disks;

Manifolds in $\mathcal{M}$:

Compare:

FSL:

- can glue each new block only along (triangular) disks;

Manifolds in $\mathcal{M}$:

- can glue each new block along a triangular disk or a hexagonal face;

Compare:

FSL:

- can glue each new block only along (triangular) disks;
- can glue it via several disks at a time;

Manifolds in $\mathcal{M}$:

- can glue each new block along a triangular disk or a hexagonal face;

Compare:

FSL:

- can glue each new block only along (triangular) disks;
- can glue it via several disks at a time;

Manifolds in $\mathcal{M}$:

- can glue each new block along a triangular disk or a hexagonal face;
- each new block is glued via a single region;

Compare:

FSL:

- can glue each new block only along (triangular) disks;
- can glue it via several disks at a time;
- can even glue a block to itself.

Manifolds in $\mathcal{M}$:

- can glue each new block along a triangular disk or a hexagonal face;
- each new block is glued via a single region;

Compare:

FSL:

- can glue each new block only along (triangular) disks;
- can glue it via several disks at a time;
- can even glue a block to itself.

Manifolds in $\mathcal{M}$:

- can glue each new block along a triangular disk or a hexagonal face;
- each new block is glued via a single region;
- cannot glue blocks to themselves.

Properties of $\mathcal{M}$

1. Always hyperbolic: decompose into $4g$ right-angle ideal octahedra;

Properties of $\mathcal{M}$

1. Always hyperbolic: decompose into $4g$ right-angle ideal octahedra;
2. Not all of the manifolds in $\mathcal{M}$ are complements of FSL

Properties of $\mathcal{M}$

1. Always hyperbolic: decompose into $4g$ right-angle ideal octahedra;
2. Not all of the manifolds in $\mathcal{M}$ are complements of FSL (tricky to see; done by counting thrice punctured spheres);

Properties of $\mathcal{M}$

1. Always hyperbolic: decompose into $4g$ right-angle ideal octahedra;
2. Not all of the manifolds in $\mathcal{M}$ are complements of FSL (tricky to see; done by counting thrice punctured spheres);
3. Their Turaev-Viro invariants are not so easy:

Properties of $\mathcal{M}$

1. Always hyperbolic: decompose into $4g$ right-angle ideal octahedra;
2. Not all of the manifolds in $\mathcal{M}$ are complements of FSL (tricky to see; done by counting thrice punctured spheres);
3. Their Turaev-Viro invariants are not so easy:

$$TV_r(M) = \sum \left| \qquad \right|^2$$

Properties of $\mathcal{M}$

1. Always hyperbolic: decompose into $4g$ right-angle ideal octahedra;
2. Not all of the manifolds in $\mathcal{M}$ are complements of FSL (tricky to see; done by counting thrice punctured spheres);
3. Their Turaev-Viro invariants are not so easy:

$$TV_r(M) = \sum \left| \sum \left(\prod |e|_{col} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix} \right) \right|^2$$

Properties of $\mathcal{M}$

1. Always hyperbolic: decompose into $4g$ right-angle ideal octahedra;
2. Not all of the manifolds in $\mathcal{M}$ are complements of FSL (tricky to see; done by counting thrice punctured spheres);
3. Their Turaev-Viro invariants are not so easy:

$$TV_r(M) = \sum \left| \sum \left(\prod |e|_{col} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix} \right) \right|^2$$

We could try to estimate each summand via a smooth function, apply a Fourier transform and then try the Poisson summation formula.

Properties of $\mathcal{M}$

1. Always hyperbolic: decompose into $4g$ right-angle ideal octahedra;
2. Not all of the manifolds in $\mathcal{M}$ are complements of FSL (tricky to see; done by counting thrice punctured spheres);
3. Their Turaev-Viro invariants are not so easy:

$$TV_r(M) = \sum \left| \sum \left(\prod |e|_{col} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix} \right) \right|^2$$

We could try to estimate each summand via a smooth function, apply a Fourier transform and then try the Poisson summation formula.

Instead, we directly apply *Barrett's Fourier transform* to the summands.

Barrett's Fourier transform

Take $\Gamma \subseteq M$ a graph colored with $\vec{i} := (i_1, \ldots, i_k)$.

Barrett's Fourier transform

Take $\Gamma \subseteq M$ a graph colored with $\vec{i} := (i_1, \ldots, i_k)$.

Barrett's Fourier transform of $TV_r\left(M, \Gamma, \vec{i}\right)$ takes as input some colors $\vec{j} := (j_1, \ldots, j_k)$:

Barrett's Fourier transform

Take $\Gamma \subseteq M$ a graph colored with $\vec{i} := (i_1, \ldots, i_k)$.

Barrett's Fourier transform of $TV_r\left(M, \Gamma, \vec{i}\right)$ takes as input some colors $\vec{j} := (j_1, \ldots, j_k)$:

$$\widehat{TV}_r\left(M, \Gamma, \vec{j}\right) := N(r)^g \sum_{(i_1,\ldots,i_k)} \left(\prod_{n=1}^{k} H(i_n, j_n)\right) TV_r\left(M, \Gamma, \vec{i}\right)$$

Barrett's Fourier transform

Take $\Gamma \subseteq M$ a graph colored with $\vec{i} := (i_1, \ldots, i_k)$.

Barrett's Fourier transform of $TV_r\left(M, \Gamma, \vec{i}\right)$ takes as input some colors $\vec{j} := (j_1, \ldots, j_k)$:

$$\widehat{TV}_r\left(M, \Gamma, \vec{j}\right) := N(r)^g \sum_{(i_1,\ldots,i_k)} \left(\prod_{n=1}^{k} H(i_n, j_n)\right) TV_r\left(M, \Gamma, \vec{i}\right)$$

with

$$H(i,j) := \left((-1)^{i+j} \frac{\sin\left(\frac{2\pi}{r}(i+1)(j+1)\right)}{\sin\left(\frac{2\pi}{r}\right)}\right)$$

Properties of Barrett's Fourier transform

- $\widehat{\widehat{TV}}_r(M,\Gamma) = N(r)^e TV(M,\Gamma)$;

Properties of Barrett's Fourier transform

- $\widehat{\widehat{TV}}_r(M, \Gamma) = N(r)^e TV(M, \Gamma)$;
- if $\Gamma \subseteq S^3$ is planar, then

$$\widehat{TV}_r(S^3, \Gamma, col) = TV_r(S^3, \Gamma^*, col);$$

Properties of Barrett's Fourier transform

- $\widehat{\widehat{TV}}_r(M,\Gamma) = N(r)^e TV(M,\Gamma)$;
- if $\Gamma \subseteq S^3$ is planar, then

$$\widehat{TV}_r(S^3,\Gamma,col) = TV_r(S^3,\Gamma^*,col);$$

- A sort of Poisson summation formula:

$$\sum_{col} |TV_r(M,\Gamma,col)| \sim \sum_{col'} |\widehat{TV}_r(M,\Gamma,col')|.$$

Properties of Barrett's Fourier transform

- $\widehat{\widehat{TV}}_r(M,\Gamma) = N(r)^e TV(M,\Gamma)$;
- if $\Gamma \subseteq S^3$ is planar, then

$$\widehat{TV}_r(S^3,\Gamma,col) = TV_r(S^3,\Gamma^*,col);$$

- A sort of Poisson summation formula:

$$\sum_{col} |TV_r(M,\Gamma,col)| \sim \sum_{col'} |\widehat{TV}_r(M,\Gamma,col')|.$$

The last property is not quite a Poisson summation formula since there are absolute values and holds asymptotically.

The Volume Conjecture for $\mathcal{M}$

$$TV_r(M) = \sum \left| \sum \left(\prod |e|_{col} \prod_{blocks} \begin{vmatrix} col(i_1) & col(i_2) & col(i_3) \\ col(i_4) & col(i_5) & col(i_6) \end{vmatrix} \right) \right|^2$$

The Volume Conjecture for $\mathcal{M}$

$$TV_r(M) = \sum_{col} |TV(S^3, \Gamma, col)|$$

The Volume Conjecture for $\mathcal{M}$

$$TV_r(M) = \sum_{col} |TV(S^3, \Gamma, col)| \sim \sum_{col'} |\widehat{TV}(S^3, \Gamma, col')|$$

The Volume Conjecture for $\mathcal{M}$

$$TV_r(M) = \sum_{col} |TV(S^3, \Gamma, col)| \sim \sum_{col'} |\widehat{TV}(S^3, \Gamma, col')|$$

$$= \sum_{col'} |TV(S^3, \Gamma^*, col')|$$

The Volume Conjecture for $\mathcal{M}$

$$TV_r(M) = \sum_{col} |TV(S^3, \Gamma, col)| \sim \sum_{col'} |\widehat{TV}(S^3, \Gamma, col')|$$

$$= \sum_{col'} |TV(S^3, \Gamma^*, col')| = TV_r(FSL)$$

The Volume Conjecture for $\mathcal{M}$

$$TV_r(M) = \sum_{col} |TV(S^3, \Gamma, col)| \sim \sum_{col'} |\widehat{TV}(S^3, \Gamma, col')|$$

$$= \sum_{col'} |TV(S^3, \Gamma^*, col')| = TV_r(FSL) \sim P(r) e^{\frac{r}{2\pi} 4 g v_8}$$

The mapping class group

$$MCG(\Sigma) := \pi_0(\mathrm{Diffeo}^+(\Sigma))$$

The mapping class group

$$MCG(\Sigma) := \pi_0(\mathrm{Diffeo}^+(\Sigma))$$

The representation theory of $MCG(\Sigma)$ is still poorly understood:

The mapping class group

$$MCG(\Sigma) := \pi_0(\text{Diffeo}^+(\Sigma))$$

The representation theory of $MCG(\Sigma)$ is still poorly understood:

Open question: Does $MCG(\Sigma)$ admit a faithful, finite dimensional representation (i.e. is $MCG(\Sigma)$ linear)?

The mapping class group

$$MCG(\Sigma) := \pi_0(\text{Diffeo}^+(\Sigma))$$

The representation theory of $MCG(\Sigma)$ is still poorly understood:

Open question: Does $MCG(\Sigma)$ admit a faithful, finite dimensional representation (i.e. is $MCG(\Sigma)$ linear)?

Theorem (Hadari, '15)

If $\phi \in MCG(\Sigma)$ has infinite order, there is a f.d. representation ρ of $MCG(\Sigma)$ such that $\rho(\phi)$ has infinite order.

The mapping class group

$$MCG(\Sigma) := \pi_0(\text{Diffeo}^+(\Sigma))$$

The representation theory of $MCG(\Sigma)$ is still poorly understood:

Open question: Does $MCG(\Sigma)$ admit a faithful, finite dimensional representation (i.e. is $MCG(\Sigma)$ linear)?

Theorem (Hadari, '15)

If $\phi \in MCG(\Sigma)$ has infinite order, there is a f.d. representation ρ of $MCG(\Sigma)$ such that $\rho(\phi)$ has infinite order.

Remember: the TQFT giving the Turaev-Viro invariants also gives a family of f.d. representations ρ_r of $MCG(\Sigma)$.

Pseudo-Anosov mappings and the AMU conjecture

Definition (Cop-out definition of pseudo-Anosov diffeomorphism)

Pseudo-Anosov mappings and the AMU conjecture

Definition (Cop-out definition of pseudo-Anosov diffeomorphism)

$f \in MCG(\Sigma)$ is pseudo-Anosov if M_f is hyperbolic.

Definition (Cop-out definition of pseudo-Anosov diffeomorphism)

$f \in MCG(\Sigma)$ is pseudo-Anosov if M_f is hyperbolic.
$f \in MCG(\Sigma)$ has a pseudo-Anosov part if M_f has a hyperbolic piece in its geometric decomposition $\Leftrightarrow ||M_f|| > 0$.

Pseudo-Anosov mappings and the AMU conjecture

Definition (Cop-out definition of pseudo-Anosov diffeomorphism)

$f \in MCG(\Sigma)$ is pseudo-Anosov if M_f is hyperbolic.
$f \in MCG(\Sigma)$ has a pseudo-Anosov part if M_f has a hyperbolic piece in its geometric decomposition $\Leftrightarrow ||M_f|| > 0$.

Conjecture (The AMU conjecture (Andersen-Masbaum-Ueno))

If $f \in \mathrm{MCG}(\Sigma)$ contains a pseudo-Anosov part, then for $r >> 0$ the quantum representation $\rho_r(\phi)$ has infinite order.

Theorem (Detcherry, Kalfagianni)

If

$$\liminf_{r \to +\infty} \frac{2\pi}{r} \log |TV(M_f)| > 0,$$

then f satisfies the AMU conjecture.

Theorem (Detcherry, Kalfagianni)

If

$$\liminf_{r\to+\infty} \frac{2\pi}{r} \log |TV(M_f)| > 0,$$

then f satisfies the AMU conjecture.

Idea: the dimension of ρ_r grows polynomially in r and $TV(M_f)$ can be computed by the eigenvalues of $\rho_r(f)$. But if $\rho_r(f)$ had finite order, its eigenvalues would be roots of 1.

Theorem (Detcherry, Kalfagianni)

If

$$\liminf_{r\to+\infty} \frac{2\pi}{r} \log |TV(M_f)| > 0,$$

then f satisfies the AMU conjecture.

Idea: the dimension of ρ_r grows polynomially in r and $TV(M_f)$ can be computed by the eigenvalues of $\rho_r(f)$. But if $\rho_r(f)$ had finite order, its eigenvalues would be roots of 1.

Theorem (Detcherry, Kalfagianni)

If $K \subseteq M$ is a knot,

$$\liminf_{r\to+\infty} \frac{2\pi}{r} \log |TV(M)| \leqslant \liminf_{r\to+\infty} \frac{2\pi}{r} \log |TV(M \setminus \nu(K))|$$

Theorem (B., Detcherry, Kalfagianni, Yang)

If M is a FSL there is a knot $K \subset M$ such that

Theorem (B., Detcherry, Kalfagianni, Yang)

If M is a FSL there is a knot $K \subset M$ such that

1. *$M \setminus K$ is fibered;*

Theorem (B., Detcherry, Kalfagianni, Yang)

If M is a FSL there is a knot $K \subset M$ such that

1. $M \setminus K$ *is fibered;*
2. *any monodromy of* $M \setminus K$ *has a pseudo-Anosov part and satisfies the AMU conjecture.*

Theorem (B., Detcherry, Kalfagianni, Yang)

If M is a FSL there is a knot $K \subset M$ such that

1. *$M \setminus K$ is fibered;*
2. *any monodromy of $M \setminus K$ has a pseudo-Anosov part and satisfies the AMU conjecture.*

Remark: it's unclear what mapping classes can arise from this.