Virtual Artin groups

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K-OS

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The term "virtual" will have different meanings : a virtual Artin group is a natural extension of the virtual braid group VB_n while a virtual property is referred to combinatorial properties of finite index subgroups.

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In summary: this will be a virtual talk on virtual properties of virtual objects...

Definition. The *virtual braid group* on n strands, denoted VB_n, is defined by the presentation with generators $\sigma_1, \ldots, \sigma_{n-1}, \tau_1, \ldots, \tau_{n-1}$, and relations:

(v1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$, $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for $|i-j| = 1$,

(v2) $\tau_i \tau_j = \tau_j \tau_i$ for $|i-j| \geq 2$, $\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j$ for $|i-j| = 1$, $\tau_i^2 = 1$ for $1 \leq i \leq n-1$,

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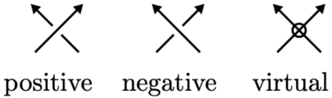
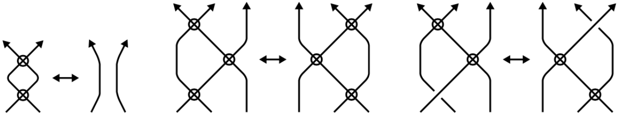
Remark. The relations (v1) correspond to usual braid relations in the braid group B_n, and they "correspond" to locality in space and Reidemeister-III moves. The relations (v2) correspond to Coxeter relations in the symmetric group S_n, and they "correspond" to virtual Reidemeister II and III moves. The relations (v3), called *mixed relations*, and they correspond to mixed Reidemeister moves.

A diagrammatic representation





Definition. There are two surjective homomorphisms $\pi_K : \mathrm{VB}_n \to S_n$ and $\pi_P : \mathrm{VB}_n \to S_n$ defined by:

$$\pi_K(\sigma_i) = 1 \text{ and } \pi_K(\tau_i) = (i, i+1) \text{ for all } 1 \leq i \leq n-1,$$
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Proposition

The homomorphism $\iota : S_n \to \mathrm{VB}_n$ is injective and we have the following semi-direct product decompositions:

$$\mathrm{VB}_n = KB_n \rtimes S_n \text{ and } \mathrm{VB}_n = VP_n \rtimes S_n\,.$$

Set:

$$\lambda_{i,i+1} = \tau_i \, \sigma_i \,, \qquad \lambda_{i+1,i} = \sigma_i \, \tau_i \,. \tag{1}$$

For $1 \leq i < j - 1 \leq n$:

$$\lambda_{i,j} = \tau_{j-1}\tau_{j-2}\cdots\tau_{i+1}\lambda_{i,i+1}\tau_{i+1}\cdots\tau_{j-2}\tau_{j-1}, \qquad \lambda_{j,i} = \tau_{j-1}\tau_{j-2}\cdots\tau_{i+1}\lambda_{i+1,i}\tau_{i+1}\cdots\tau_{j-2}\tau_{j-1}. \tag{2}$$

Set:

$$\begin{aligned} \lambda_{i,i+1} &= \tau_i \, \sigma_i \, , \\ \lambda_{i+1,i} &= \sigma_i \, \tau_i \, . \end{aligned} \quad (1)$$

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Theorem (Bardakov 2004)

The group VP_n admits a presentation with generators $\{\lambda_{i,j} \mid 1 \leq i \neq j \leq n\}$ and relations:

- $\lambda_{i,j}\lambda_{k,\ell} = \lambda_{k,\ell}\lambda_{i,j}$ for i, j, k, ℓ pairwise distinct,
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The group VP_n coincides with the Quasi triangular group QTr_n related to Yang-Baxter equations.

Theorem (Bartholdi-Enriquez-Etingof-Rains 2006)

The group VP_n admits a finite classifying space.

Theorem (Dyes-Nycas 2015)

The group VB_n has trivial center.

$$\begin{aligned} \chi_{i,i+1} &= \sigma_i \ , \\ \chi_{i+1,i} &= \tau_i \ \sigma_i \ \tau_i \ . \end{aligned} \tag{3}$$

For $1 \leq i < j-1 \leq n$:

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Theorem (Godelle-Paris 2012; B.-Cisneros-Paris 2016)

The word problem is solvable in VB_n.

Other (different) proof: Chterental (2015)

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Theorem (B. - Paris 2020)

Let $n > 5$, $m > 2$ and $n \geq m$. Complete classification of all the homomorphisms from VB_n to S_m, from S_n to VB_m, and from VB_n to VB_m.

Corollaries: $Out(VB_n)$ is isomorphic to the Klein group, VB_n is hopfian and cohopfian

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Remark. The relations (v1) are the relations of Artin's presentation for the braid group B_n, and they determine an embedding of B_n into VB_n. The relations (v2) are the relations of the Coxeter type presentation of the symmetric group S_n, and they determine an embedding of S_n into VB_n.

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The relations (v3) mimic the action of the symmetric group on its associated root system.

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$$A[\Gamma] = \langle S \mid \underbrace{sts\dots}_{m_{st}} = \underbrace{tst\dots}_{m_{st}} \text{ for } s,t \in S\,,\ s \neq t\,,\ m_{s,t} \neq \infty \rangle\,.$$

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The *Coxeter group* associated with Γ, denoted $W[\Gamma]$, is the quotient of $A[\Gamma]$ by the relations $s^2 = 1$, $s \in S$.

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The *virtual Artin group* associated with Γ, denoted $\mathrm{VA}[\Gamma]$, is the group defined by the presentation with generating set $\mathcal{S} \sqcup \mathcal{T}$ and relations (for $s, t \in S$, $s \neq t$ and $m_{s,t} \neq \infty$):

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Remark. (v1) relations define the Artin group $A[\Gamma]$, and we have a natural homomorphism $\iota_A : A[\Gamma] \to \mathrm{VA}[\Gamma]$ which sends s to σ_s for all $s \in S$. (v2) relations define the Coxeter group $W[\Gamma]$, and we have a natural homomorphism $\iota_W : W[\Gamma] \to \mathrm{VA}[\Gamma]$ which sends s to τ_s for all $s \in S$. (v3) relations mimic the action of $W[\Gamma]$ on its root system.

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We set $\mathrm{KVA}[\Gamma] = \mathrm{Ker}(\pi_K)$ and $\mathrm{PVA}[\Gamma] = \mathrm{Ker}(\pi_P)$. Recall the homomorphism $\iota_W : W[\Gamma] \to \mathrm{VA}[\Gamma]$ which sends s to τ_s for all $s \in S$. Observe that:

Definition. We see in the presentations of $\mathrm{VA}[\Gamma]$ and $W[\Gamma]$ that there are surjective homomorphisms $\pi_K : \mathrm{VA}[\Gamma] \to W[\Gamma]$ and $\pi_P : \mathrm{VA}[\Gamma] \to W[\Gamma]$ defined by:

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So:

Proposition

Let Γ be a Coxeter graph. The homomorphism $\iota_W : W[\Gamma] \to \mathrm{VA}[\Gamma]$ is injective and we have the following semi-direct product decompositions:

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One can obtain explicit group presentations for $\mathrm{KVA}[\Gamma]$ and $\mathrm{PVA}[\Gamma]$ using underlying root systems.

Definition. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Let $\Pi = \{\alpha_s \mid s \in S\}$ be an abstract set in one-to-one correspondence with S. The elements of Π are called *simple roots*. Let V be the real vector space having Π as a basis and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be the symmetric bilinear form, called the *canonical bilinear form*, defined by:

$$\langle \alpha_s, \alpha_t \rangle = \begin{cases} -2\cos(\pi/m_{s,t}) & \text{if } m_{s,t} \neq \infty\,, \\ -2 & \text{if } m_{s,t} = \infty\,. \end{cases}$$

There is a faithful linear representation $W[\Gamma] \hookrightarrow \mathrm{GL}(V)$, called the *canonical linear representation* of $W[\Gamma]$, preserving the bilinear form $\langle \cdot, \cdot \rangle$, and defined by:

$$s(v) = v - \langle v, \alpha_s \rangle \alpha_s\,, \quad \text{for } v \in V\,,\ s \in S\,.$$

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We will assume $W[\Gamma]$ to be embedded into $\mathrm{GL}(V)$ via this linear representation. We set $\Phi[\Gamma] = \{w(\alpha_s) \mid s \in S \text{ and } w \in W[\Gamma]\}$ which we call *root system*.

We denote by $\cdot$ the action of $W[\Gamma]$ on $\mathrm{KVA}[\Gamma]$. In other words, for $w \in W[\Gamma]$ and $g \in \mathrm{KVA}[\Gamma]$ we set $w \cdot g = \iota_W(w)\, g\, \iota_W(w)^{-1}$.

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Lemma (B–Paris–Thiel)

Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Let $u, v \in W[\Gamma]$ and $s, t \in S$. If $u(\alpha_s) = v(\alpha_t)$, then $u \cdot \sigma_s = v \cdot \sigma_t$ (in $\mathrm{KVA}[\Gamma]$).

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Definition. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Let $\beta \in \Phi[\Gamma]$. We choose $w \in W[\Gamma]$ and $s \in S$ such that $w(\alpha_s) = \beta$ and we set $\delta_\beta = w \cdot \sigma_s \in \mathrm{KVA}[\Gamma]$. By the above lemma this definition does not depend on the choice of w and s.

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Lemma (B–Paris–Thiel)

Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Then $\{\delta_\beta \mid \beta \in \Phi[\Gamma]\}$ generates $\mathrm{KVA}[\Gamma]$.

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Theorem (B–Paris–Thiel)

Let Γ a Coxeter graph. Then $\mathrm{KVA}[\Gamma]$ is an Artin group whose standard generating set is $\{\delta_\beta \mid \beta \in \Phi[\Gamma]\}$.

Definition. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. We define a new Coxeter matrix $\hat{M} = (\hat{m}_{\beta,\gamma})_{\beta,\gamma \in \Phi[\Gamma]}$ indexed by the elements of $\Phi[\Gamma]$ as follows.

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(a) We set $\hat{m}_{\beta,\beta} = 1$ for all $\beta \in \Phi[\Gamma]$.

(b) Let $\beta, \gamma \in \Phi[\Gamma]$, $\beta \neq \gamma$. If there exist $w \in W[\Gamma]$ and $s, t \in S$ such that $\beta = w(\alpha_s)$, $\gamma = w(\alpha_t)$ and $m_{s,t} \neq \infty$, then we set $\hat{m}_{\beta,\gamma} = m_{s,t}$. This definition does not depend on the choice of w, s and t. We set $\hat{m}_{\beta,\gamma} = \infty$ otherwise.

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We denote by $\hat{\Gamma}$ the Coxeter graph of $\hat{M}$. Moreover, for convenience, we will denote by $\{\hat{\delta}_\beta \mid \beta \in \Phi[\Gamma]\}$ the standard generating set of the corrsponding Artin group $A[\hat{\Gamma}]$.

Example

Suppose Γ is the graph A_{n-1}

Then $W[\Gamma] = S_n$, $A[\Gamma] = B_n$ and $\mathrm{VA}[\Gamma] = \mathrm{VB}_n$. Let $U = \mathbb{R}^n$ and $\{e_1, \dots, e_n\}$ its canonical basis. We consider the action by permutation of S_n on U. Let V be the hyperplane of U defined by the equation $x_1 + \cdots + x_n = 0$. Then V is invariant under the action of S_n and the induced representation $W[\Gamma] = S_n \to \mathrm{GL}(V)$ is the canonical representation. For $i, j \in \{1, \dots, n\}$, $i \neq j$, we set $\alpha_{i,j} = e_j - e_i$. Then $\Pi = \{\alpha_{1,2}, \dots, \alpha_{n-1,n}\}$ is the set of simple roots, and $\Phi[\Gamma] = \{\alpha_{i,j} \mid 1 \leq i \neq j \leq n\}$ is the root system of S_n. The set $\{\hat{\delta}_{\alpha_{i,j}} \mid 1 \leq i \neq j \leq n\}$ is the generating set of $A[\widehat{A_{n-1}}]$ The Coxeter matrix $\hat{M}$ is defined by:

- $\hat{m}_{\alpha_{i,j},\alpha_{i,j}} = 1$ for $1 \leq i \neq j \leq n$.
- $\hat{m}_{\alpha_{i,j},\alpha_{k,\ell}} = 2$ for i, j, k, ℓ pairwise distinct.
- $\hat{m}_{\alpha_{i,j},\alpha_{j,k}} = 3$ for i, j, k pairwise distinct.
- $\hat{m}_{\alpha_{i,j},\alpha_{k,\ell}} = \infty$ otherwise.

Theorem (B–Paris–Thiel)

Let Γ be a Coxeter graph. Then the map

$$\begin{array}{ccc} \{\hat{\delta}_\beta \mid \beta \in \Phi[\Gamma]\} & \to & \{\delta_\beta \mid \beta \in \Phi[\Gamma]\} \\ \hat{\delta}_\beta & \mapsto & \delta_\beta \end{array}$$

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Recall the homomorphism $\iota_A : A[\Gamma] \to \mathrm{VA}[\Gamma]$ which sends s to σ_s for all $s \in S$.

Corollary (B–Paris–Thiel)

Let Γ be a Coxeter graph. Then $\iota_A : A[\Gamma] \to \mathrm{VA}[\Gamma]$ is injective.

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Proof. Let $M = (m_{s,t})_{s,t \in S}$ be the matrix associate to Γ. For $X \subset S$, we set $M_X = (m_{s,t})_{s,t \in X}$ and we denote by Γ_X the Coxeter graph of M_X. By Van der Lek [1983]:

The homomorphism $A[\Gamma_X] \to A[\Gamma]$ induced by the inclusion is injective.

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Let $\Pi = \{\alpha_s \mid s \in S\}$ be the set of simple roots. Observe that the map

$$S \to \Pi : s \mapsto \alpha_s$$

induces an isomorphism from Γ to $\hat{\Gamma}_\Pi$, and therefore $A[\Gamma] \simeq A[\hat{\Gamma}_\Pi]$. By composing with $A[\hat{\Gamma}_\Pi] \hookrightarrow A[\hat{\Gamma}]$, we get an injective homomorphism from $A[\Gamma]$ into $A[\hat{\Gamma}] = \mathrm{KVA}[\Gamma]$ which sends s to δ_{α_s} for all $s \in S$. Since $\delta_{\alpha_s} = \sigma_s$ for all s, this homomorphism is ι_A. □

Theorem (B–Paris–Thiel)

Let Γ be a Coxeter graph. Then the center of $\mathrm{KVA}[\Gamma]$ (and therefore of $VA[\Gamma]$) is trivial.

Remark. Determining the center of $A[\Gamma]$ for any Coxeter graph is an open problem.

Definition. Let Γ be a (finite) Coxeter graph. We say that Γ (or $A[\Gamma]$ or $W[\Gamma]$ or $\mathrm{VA}[\Gamma]$) is of *spherical type* if $W[\Gamma]$ is finite.

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Definition. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. A subset $X \subset S$ is called *free of infinity* if $m_{s,t} \neq \infty$ for all $s, t \in X$.

General principle

Let Γ be a Coxeter graph. If we understand $A[\Gamma_X]$ for all free of infinity subset X of S, then we can understand $A[\Gamma]$.

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Theorem (B–Paris–Thiel)

Let Γ be a Coxeter graph of spherical type or of affine type, let $M = (m_{,t})_{s,t \in S}$ be its Coxeter matrix, and let $\mathcal{X}$ be a free of infinity subset of $\Phi[\Gamma]$. Then $\mathcal{X}$ is finite and $\hat{\Gamma}_{\mathcal{X}}$ is of spherical type or of affine type. Moreover, if $\hat{\Gamma}_{\mathcal{X}}$ is of spherical type, then $|\mathcal{X}| \leq |S|$.

Spherical type and affine type: the word problem

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Theorem (B–Paris–Thiel)

Let Γ be a Coxeter graph of spherical type or of affine type. Then $\mathrm{VA}[\Gamma]$ has a solution to the word problem.

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The set of reflections in $W[\Gamma]$ is $R = \{wsw^{-1} \mid w \in W \text{ and } s \in S\}$ and $W[\Gamma]$ acts freely and properly discontinuously on $I \setminus \cup_{r \in R} H_r$, where, for $r \in R$, H_r denotes the hyperplane of V fixed by r. Set:

$$\mathcal{M}[\Gamma] = (I \times I) \setminus \left(\bigcup_{r \in R} (H_r \times H_r) \right) .$$

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This is a connected manifold of dimension $2\,|S|$ on which $W[\Gamma]$ acts freely and properly discontinuously.

We set:

$$\mathcal{N}[\Gamma] = \mathcal{M}[\Gamma]/W[\Gamma] .$$

Theorem (Van der Lek [1983])

Let Γ be a finite Coxeter graph. The fundamental group of $\mathcal{N}[\Gamma]$ is isomorphic to $A[\Gamma]$.

Definition. A CW-complex X is a *classifying space* for a (discrete) group G if $\pi_1(X) = G$ and the universal covering of X is contractible.

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Theorem (Deligne [1972], Paolini–Salvetti [2021])

Let Γ be a Coxeter graph of spherical type or of affine type. Then $A[\Gamma]$ satisfies the $K(\pi, 1)$ conjecture.

Theorem (B–Paris–Thiel)

Let Γ be a Coxeter graph of spherical type. Then $A[\hat{\Gamma}] = \mathrm{KVA}[\Gamma]$ satisfies the $K(\pi, 1)$ conjecture.

Remark. The above makes sense only when Γ is of spherical type since this is the only case where $\hat{\Gamma}$ is finite.

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Notation. For a group G we denote by $\mathrm{cd}(G)$ the cohomological dimension of G and by $\mathrm{vcd}(G)$ the virtual cohomological dimension of G.

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Corollary (B–Paris–Thiel)

Let Γ be a Coxeter graph of spherical type and let n be the number of vertices of Γ. Then $\mathrm{cd}(\mathrm{KVA}[\Gamma]) = \mathrm{vcd}(\mathrm{VA}[\Gamma]) = n$. In particular, $\mathrm{KVA}[\Gamma]$ is torsion free and $\mathrm{VA}[\Gamma]$ is virtually torsion free.

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Theorem (B–Paris–Thiel)

Let Γ be a connected Coxeter graph of affine type and let n be the number of vertices of Γ. Then $\mathrm{cd}(\mathrm{KVA}[\Gamma]) \leq n$ and $\mathrm{vcd}(\mathrm{VA}[\Gamma]) \leq 2n-1$. In particular, $\mathrm{KVA}[\Gamma]$ is torsion free and $\mathrm{VA}[\Gamma]$ is virtually torsion free.

We denote by $\cdot$ the action of $W[\Gamma]$ on $\mathrm{PVA}[\Gamma]$. In other words, for $w \in W[\Gamma]$ and $g \in \mathrm{PVA}[\Gamma]$ we set $w \cdot g = \iota_W(w)\, g\, \iota_W(w)^{-1}$.

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Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Let $u, v \in W[\Gamma]$ and $s, t \in S$. If $u(\alpha_s) = v(\alpha_t)$, then $u \cdot (\tau_s \sigma_s) = v \cdot (\tau_t \sigma_t)$ (in $\mathrm{PVA}[\Gamma]$).

We denote by $\cdot$ the action of $W[\Gamma]$ on $\mathrm{PVA}[\Gamma]$. In other words, for $w \in W[\Gamma]$ and $g \in \mathrm{PVA}[\Gamma]$ we set $w \cdot g = \iota_W(w)\, g\, \iota_W(w)^{-1}$.

Lemma (B–Paris–Thiel)

Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Let $u, v \in W[\Gamma]$ and $s, t \in S$. If $u(\alpha_s) = v(\alpha_t)$, then $u \cdot (\tau_s \sigma_s) = v \cdot (\tau_t \sigma_t)$ (in $\mathrm{PVA}[\Gamma]$).

Definition. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Let $\beta \in \Phi[\Gamma]$. We choose $w \in W[\Gamma]$ and $s \in S$ such that $w(\alpha_s) = \beta$ and we set $\zeta_\beta = w \cdot (\tau_s \sigma_s) \in \mathrm{KVA}[\Gamma]$.

Theorem (B–Paris–Thiel)

Let Γ a Coxeter graph. Then $\mathrm{PVA}[\Gamma]$ is generated $\{\zeta_\beta \mid \beta \in \Phi[\Gamma]\}$.

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Theorem (B–Paris–Thiel)

Let Γ a Coxeter graph. Then $\mathrm{PVA}[\Gamma]$ is generated $\{\zeta_\beta \mid \beta \in \Phi[\Gamma]\}$.

We can show a group presentation with generating set $\{\zeta_\beta \mid \beta \in \Phi[\Gamma]\}$ and where all relations are of the type:

$$\zeta_{\beta_m} \cdots \zeta_{\beta_2} \zeta_{\beta_1} = \zeta_{\beta_1} \zeta_{\beta_2} \cdots \zeta_{\beta_m} \qquad \beta_1, \dots, \beta_m \in \Phi[\Gamma]$$

Exemple. Suppose Γ is simply laced. Let $\beta, \gamma \in \Phi[\Gamma]$. $\mathrm{PVA}[\Gamma]$ is the group defined by the presentation with generating set $\{\zeta_\beta \mid \beta \in \Phi[\Gamma]\}$ and relations:

- $\zeta_\gamma \zeta_\beta = \zeta_\beta \zeta_\gamma$ for $\beta, \gamma \in \Phi[\Gamma]$, $\beta \neq \gamma$ and $\langle \beta, \gamma \rangle = 0$,
- $\zeta_\gamma \zeta_{\beta+\gamma} \zeta_\beta = \zeta_\beta \zeta_{\beta+\gamma} \zeta_\gamma$ for $\beta, \gamma \in \Phi[\Gamma]$, $\beta \neq \gamma$ and $\langle \beta, \gamma \rangle = -1$.

- $\mathrm{PVA}[\Gamma]$ is torsion free for Γ of spherical type?
- Are there diagrammatic representations for $\mathrm{PVA}[\Gamma]$ (e.g. like Gauss diagrams for VP_n)?

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Proposition (B–Paris–Thiel)

Let Γ be a Coxeter graph of spherical type. Let $PA[\Gamma]$ the colored (pure) subgroup of $A[\Gamma]$. Then $\dfrac{\mathrm{VA}[\Gamma]}{[\mathrm{PVA}[\Gamma], \mathrm{PVA}[\Gamma]]}$ is a crystallographic group and the embedding $\iota_A : A[\Gamma] \to \mathrm{VA}[\Gamma]$ induces an embedding $\hat{\iota}_A : \dfrac{A[\Gamma]}{[PA[\Gamma], PA[\Gamma]]} \to \dfrac{\mathrm{VA}[\Gamma]}{[\mathrm{PVA}[\Gamma], \mathrm{PVA}[\Gamma]]}$.

Thank you for your attention!