Twisted L^2-torsion & character variety

joint with Sean Raimbault

Intro

M hyperbolic 3-mfd with finite volume. $M = \mathbb{H}^3/\Gamma$, $\rho_0 : \pi_1 M \to \Gamma \subset SL_2\mathbb{C}$ holonomy

By Mostow-Prasad rigidity, complete hyperbolic metric on M is unique

$\Rightarrow$ vol M is a topological invariant

Q: Relate it with "actual" topological invariants?

A: L^2-torsion

$$\tau^{(2)}(M) = \exp \frac{-\mathrm{Vol}\, M}{6\pi}$$

[Lott - Mathai M closed; Lück - Schick vol $M < \infty$]

Remark: Proof goes through a computation of the L^2-torsion by means of analytic tools (a "Cheeger-Müller THM")

Today: Given $\rho : \pi_1 M \to SL_2\mathbb{C}$, study a twisted L^2-torsion $\tau^{(2)}(M,\rho)$.

In particular, previous result generalizes as

$$\tau^{(2)}(M,\rho_0) = \exp\left(\frac{-11\,\mathrm{Vol}\, M}{12\pi}\right)$$

Waßerman

for ρ_0 holonomy

Q (Lück, in general setting: continuous? A (abelian, lin))

Theorem A (B - Raimbault '20)

The function

$$\tau^{(2)}(M,\cdot) : \{\rho : \pi_1 M \to SL_2\mathbb{C}\}/SL_2\mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$$

Character variety = $X(M)$

$$[\rho] \longmapsto \tau^{(2)}(M,\rho)$$

is real analytic on a neighborhood of the holonomy representation $[\rho_0]$

Motivation:

Just as $\rho_0 \longleftrightarrow$ unique complete hyperbolic metric on M,

representations ρ around ρ_0 in $X(M)$ $\longleftrightarrow$ non-complete hyperbolic metrics on M

Example: Dehn filling $p/q \in \mathbb{Q}$ on cusp

$M \hookrightarrow M_{p/q}$ closed.

for $p, q \gg 1$, $M_{p/q}$ hyperbolic

$\rho_{p/q} : \pi_1 M \to \pi_1 M_{p/q} \to SL_2\mathbb{C}$ holonomy of $M_{p/q}$

$X(M)$ $\quad \rho_0 \quad \rho_{p/q}$

metric on $M_{p/q}$ $\rightsquigarrow$ incomplete metric on M with $\mathrm{Vol}(M, \rho_{p/q}) = \mathrm{Vol}\, M_{p/q}$

Indeed, there is a volume function

$$\mathrm{Vol} : X(M) \longrightarrow \mathbb{R}$$
$$[\rho_0] \longmapsto \mathrm{Vol}\, M$$
$$[\rho_{p/q}] \mapsto \mathrm{Vol}\, M_{p/q}$$

and length function L with $L(\rho_0) = 0$ such that

$\underline{\mathrm{Vol} + L}$ analytic on $X(M)$

geometric

<u>Q</u>: Relate it with topological

Candidate: Twisted L^2-torsion

On the other hand,

Twisted L^2-torsion can be defined for any 3-manifold.

(nonhyperbolic $\Rightarrow$ Vol $= 0$)

<u>Theorem B</u> (B-R)

Let M irreducible with JSJ decomposition

$$M = M_1 \cup_{T_1} M_2 \cup_{T_2} \cdots \cup_{T_{n-1}} M_n$$

There is an open subset in $X(M)$ such that

$$\tau^{(2)}(M, \rho) = \prod_{\{i \mid M_i \text{ hyperbolic}\}} \tau^{(2)}(M_i, \rho_i)$$

In particular,

$\tau^{(2)}(M, \rho) = 1$ if M is a graph manifold

("good candidate" & "computable")

II Twisted L^2-torsion

Our results

	Combinatorial	Twist	Analytic
complex	$C^*(\tilde{M}, \mathbb{C})$ cochains		$\Omega^*(\tilde{M}, \mathbb{C}^2)$ de Rham differential forms
Prehilbert structure	basis given by cells $(of\, M)$ $\rightsquigarrow$ hermitian product		norm: $\|\alpha\|^2 = \int_{\tilde{M}} \alpha \wedge \boxed{*} \alpha$ → Hodge star (α compact support)
L^2-completion	$C_{(2)}(\tilde{M}, \mathbb{C}^2_\rho)$		$L^{(2)}\Omega^*(\tilde{M}, \mathbb{C}^2_\rho)$
Laplacian	$C^i_{(2)}(\tilde{M}) \xrightarrow{d_i} C^{i+1}_{(2)}(\tilde{M})$, d_i^* adjoint; $\Delta_i(\rho)$: $C^i_{(2)}(\tilde{M})$ $= d_i^* d_i + d_{i-1} d_{i-1}^*$ self adjoint		idem
Von Neumann Trace, π-action	Defined for both		operators on Hilbert spaces, π-actions ($\mathcal{N}\pi$-modules)
Determinant, $L^{(2)}$ torsion	"$\log \det \Delta_i(\rho) =$ if $\ker \Delta_i = \{0\}$, ℓ^2-acyclicity		$\mathrm{Tr} \log \Delta_i(\rho)$, "formally" $\tau^{(2)}(M, \rho) = \prod_{i=0} (\det \Delta_i(\rho))^{\frac{(-1)^i i}{2}}$ determinant class ≥ 0, > 0

Remarks:

- In the general context of twisted L^2-invariants, addressing continuity/regularity problem was asked by Lück
- In the "simplest case" where the twist is $\phi : \pi_1 M \to \mathbb{Z}$, it was answered positively by Liu
- ρ-twisted L^2-torsions were considered by Li–Zhang, but neither regularity, nor determinant class property was addressed.

III Sketch of proof of Thm 1

$\tau^{(2)}(M,\rho)$ is real analytic around ρ_0 holonomy representation

- Step 1

For $\rho = \rho_0$, the analytic Laplacians $\Delta^i(\rho_0)$ have spectral gap ($\inf\{\text{e.v.}\} > 0$)

In particular, corresponding complexes are L^2-acyclic and of determinant class.

Proof of 1

It is a refinement of a result (known as "strong acyclicity") of Bergeron–Venkatesh.

It does not apply if ρ trivial

- Step 2: back into combinatorial

Construct a Whitney map

$$W^i : C^i(\tilde{M},\rho) \longrightarrow \Omega^i(\tilde{M},\rho)$$

take a partition of unity e_c of M with respect to the vertices c of a fixed triangulation of M (then lift to $\tilde{M}$), such that supp e_c included in $\bigcup_{c\in\sigma} \sigma$ simplices

then $W^i(f_\sigma) \longmapsto i! \sum_{c_i \in \sigma} w_{\sigma,c_i}$

where $\sigma = \{c_0, \dots, c_i\}$

$$w_{\sigma,c_i} = (-1)^i e_{c_i} \bigwedge_{j\neq i} d e_{c_j}$$

Then use W^i

$\Longrightarrow$ combinatorial Laplacians have spectral gap.

- Step 3: Propagate the spectral gap.

- $\rho \longmapsto \Delta^i(\rho)$ is analytic (can be written as a matrix with operator-valued coefficients that varies polynomially with ρ)
- $\exists$ neighborhood of ρ_0 s.t. $\Delta^i(\rho)$ has spectral gap as well
- $\log \Delta^i(\rho)$ is analytic
- $\underbrace{\operatorname{Tr} \log \Delta^i(\rho)}_{\log \operatorname{Det} \Delta^i(\rho)}$ —
- $\prod_i \operatorname{Det} \Delta^i(\rho)^{(-1)^i \frac{i}{2}}$ analytic ∎

Conclusion:

Let $\rho_0 \in U \subset X(n)$

with coordinates $\underline{u} = (u_1, \dots, u_n)$

we proved

$$\log T^2(M, \rho_u) = c\left(\mathrm{Vol}(\rho_u) + L(\rho_u)\right) + o\left(L(\rho_u)^2\right)$$

Q: Compute higher order terms?