The Teichmüller TQFT Volume Conjecture for Twist Knots

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24th September 2020

(joint work with François Guéritaud and Eiichi Piguet-Nakazawa)

arXiv:1903.09480

How you can follow/use this talk:

Live: You can download the slides on the K-OS website (helps for following recurring notations).

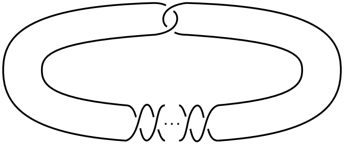
Recurring example (the figure-eight knot): Slides 6, 8, 10, 18.

From the future: downloading the slides can also help! (eventual mistakes will have been hopefully corrected at this point).

From the future and you are interested in our paper: the pictures and main example in these slides can be a good complement to the technical details in arXiv:1903.09480.

Our Goal

Proving the **Teichmüller TQFT volume conjecture** *for* **twist knots**.



0. **Context**: **quantum topology**, **volume conjectures**.
1. **Topology**: **triangulating** the **twist knot** complements
2. **Geometry**: the triangulations contain the **hyperbolicity**
3. **Algebra**: computing the **Teichmüller TQFT**
4. **Analysis**: the **hyperbolic volume** appears **asymptotically**

(Optional: parts/sketches of proofs, at the audience's preference)

'84: **Jones polynomial**, new knot invariant.

'90: **Witten** retrieves the Jones polynomial via **quantum physics**.

90s: **New topological invariants** (TQFTs of **Reshitikin-Turaev**, **Turaev-Viro**, ...) are discovered via the intuition from physics.

Andersen-Kashaev '11: **Teichmüller TQFT** of a **triangulated** 3-manifold M, an "**infinite-dimensional** TQFT".

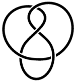
Its **partition function** $\{Z_{\mathbf{b}}(M) \in \mathbb{C}\}_{\mathbf{b}>0}$ yields an **invariant** of M.

Volume Conjecture (Andersen-Kashaev '11)

If M is a triangulated hyperbolic knot complement, then its **hyperbolic volume** $\mathrm{Vol}(M)$ *appears as an* **exponential decrease rate** *in $Z_{\mathbf{b}}(M)$ for the limit* $\mathbf{b} \to 0^+$.

TOPOLOGY

TWIST KNOTS K_n



QUANTUM INVARIANTS

SEMI-CLASSICAL LIMIT →

VOLUME CONJECTURES

HYPERBOLIC GEOMETRY

$J_K(N,q)$ $\xrightarrow{q=e^{2i\pi/N},\ N\to\infty}$ $\sim e^{N\frac{\mathrm{vol}(S^3\setminus K)}{2\pi}}$

Diagram of knot K	Colored Jones polynomials	→ Sums ↓ Integrals →
Triangulation of $S^3 \setminus K$	**Teichmüller TQFT**	→ Integrals →

Saddle point method

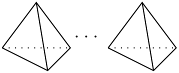
Hyperbolic Volume

TH 1 TH 2

$Z_{\mathbf{b}}(S^3 \setminus K)$ $\xrightarrow[\mathbf{b}\to 0^+]{}$ $\sim e^{\frac{1}{\mathbf{b}^2}\frac{-\mathrm{vol}(S^3\setminus K)}{2\pi}}$

TH 3

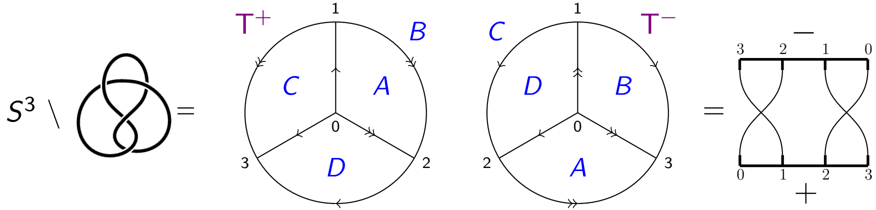
TH 4



Our **tetrahedra** have **ordered vertices** ($\Rightarrow$ **oriented edges** too).
$\rightsquigarrow$ two possible **signs** $\epsilon(T) \in \{\pm\}$.

A **triangulation** $X = (T_1, \ldots, T_N, \sim)$ of a 3-manifold M is the datum of N **tetrahedra** and a **gluing relation** $\sim$ pairing their faces while **respecting the vertex order**.

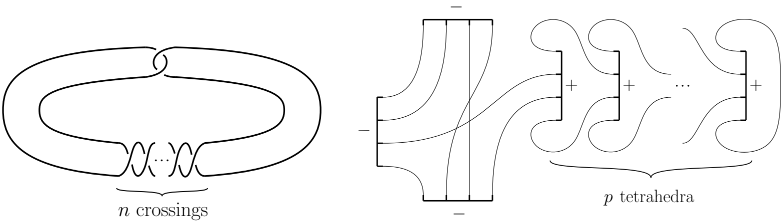
We consider **ideal triangulations** of **open** 3-manifolds, i.e. where the tetrahedra have their **vertices removed**.



$$X^3 = \{T^+, T^-\}, \quad X^2 = \{A, B, C, D\}, \quad X^1 = \{\rightarrow, \twoheadrightarrow\}, \quad X^0 = \{\cdot\}$$

face maps $x_0, \ldots, x_3 \colon X^3 \to X^2$, for example $x_0(T^+) = B$.

Thurston: from a **diagram** of a knot K, one can construct an **ideal triangulation** X of the knot complement $M = S^3 \setminus K$.

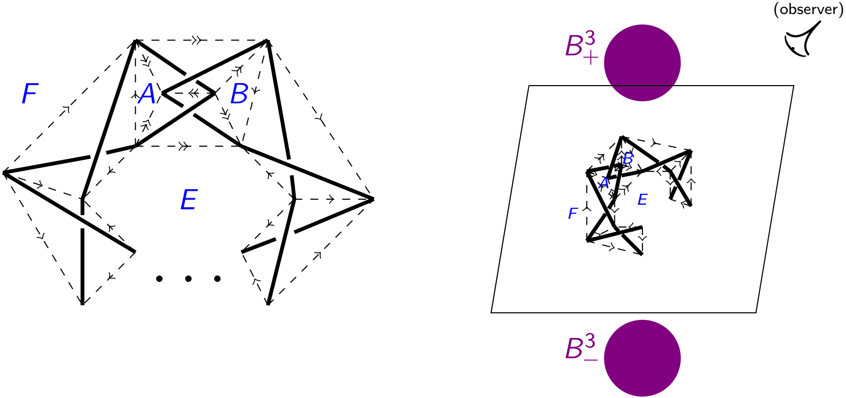


The n-th twist knot K_n and the triangulation X_n (n odd, $p = \frac{n-3}{2}$)

Theorem (TH 1, B.A.-P.N. '18)

For all $n \geqslant 2$, we construct an **ideal triangulation** *X_n of the complement of the* **twist knot** *K_n, with $\left\lfloor \frac{n+4}{2} \right\rfloor$ tetrahedra.*

Sketch of proof of TH1: First draw a tetrahedron around each **crossing** of K, whose diagram lives in the **equatorial plane** of S^3.

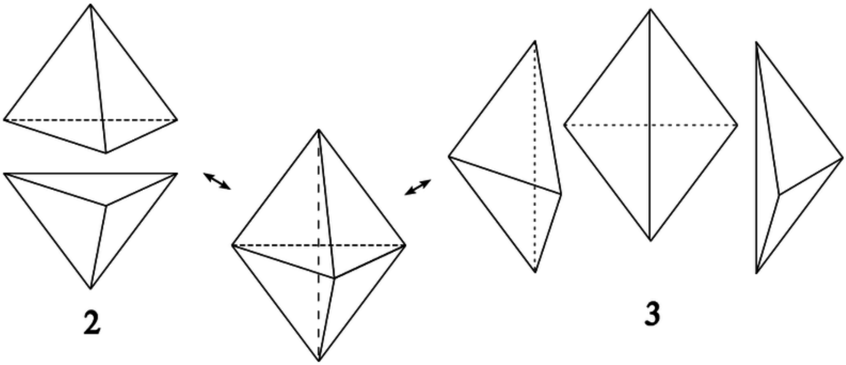


Then **collapse** the tetrahedra into segments ($K \rightsquigarrow \cdot$).
Hence the collapsed S^3 decomposes into **two polyhedra**.
Finally, **triangulate** the two polyhedra (several possible ways).

(2, 3)-**Pachner moves** are moves between **ideal triangulations**.

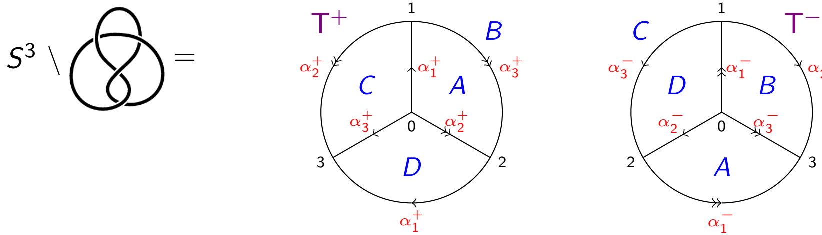
Matveev-Piergallini: X and X' triangulate the **same** M if and only if they are related by a **finite sequence** of Pachner moves.

$\Rightarrow$ Useful for constructing **topological invariants** for M.



source of the picture: Wikipedia

$\mathcal{A}_X$ is the space of **angle structures** on $X = (T_1, \dots, T_N, \sim)$, i.e. of $3N$-tuples $\alpha \in (0,\pi)^{3N}$ of **dihedral angles** on edges, such that the angle sum is π at each vertex and 2π around each edge.



$$\mathcal{A}_X = \left\{ \alpha = \begin{pmatrix} \alpha_1^+ \\ \alpha_2^+ \\ \alpha_3^+ \\ \alpha_1^- \\ \alpha_2^- \\ \alpha_3^- \end{pmatrix} \in (0,\pi)^6 \;\middle|\; \begin{array}{c} \alpha_1^+ + \alpha_2^+ + \alpha_3^+ = \pi \\ \alpha_1^- + \alpha_2^- + \alpha_3^- = \pi \\ (\rightarrow)\ 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- = 2\pi \\ (\twoheadrightarrow)\ 2\alpha_2^+ + \alpha_3^+ + 2\alpha_1^- + \alpha_3^- = 2\pi \end{array} \right\} \ni \begin{pmatrix} \frac{\pi}{3} \\ \vdots \\ \frac{\pi}{3} \end{pmatrix}$$

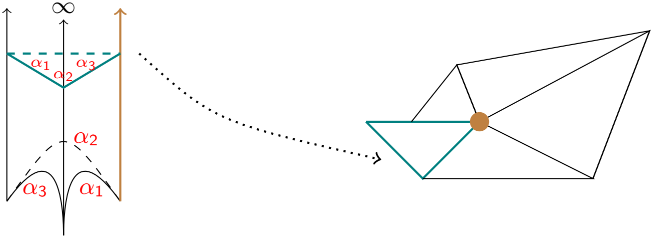
α fixed $\rightsquigarrow$ angle maps $\alpha_1, \alpha_2, \alpha_3 \colon X^3 \to \mathbb{R}$, for example $\alpha_2(T^+) = \alpha_2^+$.

The 3-dimensional **hyperbolic space** is $\mathbb{H}^3 = \mathbb{R}^2 \times \mathbb{R}_{>0}$ with

$$(ds)^2 = \frac{(dx)^2 + (dy)^2 + (dz)^2}{z^2},$$

a metric which has constant curvature -1.

A knot is **hyperbolic** if its complement M can be endowed with a complete hyperbolic metric of finite **volume** $\mathrm{Vol}(M)$.
$\rightsquigarrow$ a specific $\alpha \in \mathcal{A}_X$ on $X = (T_1, \ldots, T_N, \sim)$ triangulation of M.



$\alpha_1 + \alpha_2 + \alpha_3 = \pi$
$T \hookrightarrow \mathbb{H}^3$

$\sum_{\text{edge}} \alpha_j = 2\pi$ (+ others)
gluing gives a **manifold**

For all $n \geqslant 2$, the twist knot K_n is **hyperbolic**.

Theorem (TH2, B.A.-G.-P.N. '20)

For all $n \geqslant 2$, the triangulation X_n of $S^3 \setminus K_n$ is **geometric**, *i.e. it admits an angle structure $\alpha^0 \in \mathcal{A}_{X_n}$ corresponding to the* **complete** *hyperbolic structure on the complement of K_n.*

X **geometric** $\Leftrightarrow \exists$ solution to the **nonlinear** gluing equations of X (difficult!)

Casson-Rivin, Futer-Guéritaud: approach via $\mathcal{A}_X$, the solutions to the **linear** part: maximising the **volume fonctional**.

Dilogarithm function: $\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u}$ for $z \in \mathbb{C} \setminus [1, \infty)$.

Volume functional $\mathrm{Vol}\colon \mathcal{A}_X \to \mathbb{R}_{\geqslant 0}$ (**strictly concave**) is:

$$\mathrm{Vol}(\alpha) := \sum_{T \in X^3} \Im \mathrm{Li}_2(z(T)) + \arg(1 - z(T)) \log |z(T)|,$$

where $z(T) = \left(\frac{\sin \alpha_3(T)}{\sin \alpha_2(T)}\right)^{\epsilon(T)} e^{i\alpha_1(T)} \in \mathbb{R} + i\mathbb{R}_{>0}$ encodes the angles of T.

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Sketch of proof of TH2:

- Check that the open polyhedron $\mathcal{A}_X$ is **non-empty**.
- General fact: the **complete** structure α^0 exists $\Leftrightarrow \max_{\mathcal{A}_X} \mathrm{Vol}$ is reached in $\mathcal{A}_X$.
- Prove that $\max_{\mathcal{A}_X} \mathrm{Vol}$ cannot be on $\partial \mathcal{A}_X$ (case-by-case).

$S(\mathbb{R}^n)$ = rapidly decreasing functions $f\colon \mathbb{R}^n \to \mathbb{C}$.

$S'(\mathbb{R}^n)$ = dual of $S(\mathbb{R}^n)$, **tempered distributions**.

Example: $X^2 = \{A, B\}$, **Dirac delta function**
$\delta(A) \in S'(\mathbb{R}^{X^2}) \cong S'(\mathbb{R}^2)$ acts by: $\forall f \in S(\mathbb{R}^2)$,

$$\delta(A)\cdot f = \iint_{(A,B)\in\mathbb{R}^2} dA\,dB\;\delta(A)\,f(A,B) = \int_{B\in\mathbb{R}} dB\, f(0,B) \;\in \mathbb{C}.$$

⚠ **Product** of Dirac deltas is sometimes but not always **defined**.

$\delta(A)\delta(A)$ is not defined (because of **linear dependance**).

$\delta(A+B)\delta(A-B) = \frac{1}{2}\delta(A)\delta(B) = (f \mapsto \frac{1}{2}f(0,0))$ is well-defined.

Partition function for the triangulation X (and $\alpha \in \mathcal{A}_X$, $\mathbf{b} > 0$):

$$Z_{\mathbf{b}}(X, \alpha) = \int_{\overline{x} \in \mathbb{R}^{X^2}} d\overline{x} \prod_{T_1, \ldots, T_N} p_{\mathbf{b}}(T)(\alpha)(\overline{x}) \qquad \in \mathbb{C}.$$

Tetrahedral operator: $p_{\mathbf{b}}(T)(\alpha)(\overline{x}) \in S'(\mathbb{R}^{X_2})$ is equal to

$$\frac{\delta\left(x_0(T) - x_1(T) + x_2(T)\right) e^{\left(2\pi i \epsilon(T) x_0(T) + (\mathbf{b}+\mathbf{b}^{-1})\alpha_3(T)\right)(x_3(T) - x_2(T))}}{\Phi_{\mathbf{b}}\left((x_3(T) - x_2(T)) - \frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}\epsilon(T)(\alpha_2(T) + \alpha_3(T))\right)^{\epsilon(T)}}.$$

Faddeev's quantum dilogarithm:

$$\Phi_{\mathbf{b}}(x) := \exp\left(\int_{z \in \mathbb{R} + i0^+} \frac{e^{-2izx}\, dz}{4\sinh(z\mathbf{b})\sinh(z\mathbf{b}^{-1})z}\right).$$

Proposition (Andersen-Kashaev '11)

$|Z_{\mathbf{b}}(X, \alpha)|$ *is* **invariant** *under angled Pachner moves on* (X, α).

Partition function for the triangulation X (and $\alpha \in \mathcal{A}_X$, $\mathbf{b} > 0$):

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Volume Conjecture (Andersen-Kashaev '11)

Let X be a triangulation of a **hyperbolic knot complement** *M.*

(1) *$\exists\ \lambda_X$ linear combination of dihedral angles, $\exists$ smooth function $J_X : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C}$ such that $\forall$ angle structures α, $\forall\ \mathbf{b} > 0$,*

$$|Z_{\mathbf{b}}(X, \alpha)| = \left| \int_{x \in \mathbb{R}} J_X(\mathbf{b}, x) e^{-(\mathbf{b}+\mathbf{b}^{-1}) x \lambda_X(\alpha)} dx \right|.$$

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(2) *The* **hyperbolic volume** $\mathrm{Vol}(M)$ *is obtained as the following* **semi-classical limit***:*

$$\lim_{\mathbf{b} \to 0^+} 2\pi \mathbf{b}^2 \ \log |J_X(\mathbf{b}, 0)| \ = \ -\mathrm{Vol}(M).$$

Theorem (TH3, B.A.-P.N. '18)

(1) is proven for **all twist knots***, via algebraic computations.*

Theorem (TH4, B.A.-G.-P.N. '20)

(2) is proven for **all twist knots***, via asymptotic analysis.*

Proof of TH3, easiest example: For $K = 4_1$, we find $Z_{\mathbf{b}}(X, \alpha) =$

$$\iiiint \frac{dAdBdCdD \;\; \delta(B-D+C)\,\delta(C-A+B) \;\; \Phi_{\mathbf{b}}\left(D-B+\frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}(\alpha_2^- + \alpha_3^-)\right)}{e^{\left(2\pi i B + (\mathbf{b}+\mathbf{b}^{-1})\alpha_3^+\right)(C-A)} e^{\left(-2\pi i C + (\mathbf{b}+\mathbf{b}^{-1})\alpha_3^-\right)(B-D)} \Phi_{\mathbf{b}}\left(A-C-\frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}(\alpha_2^+ + \alpha_3^+)\right)}.$$

Then we change the variables: $2x = B + C + \frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}(\alpha_1^+ - \alpha_1^-)$,
$2y = B - C + \frac{i(\mathbf{b}+\mathbf{b}^{-1})}{2\pi}(\alpha_1^+ + \alpha_1^- - 2\pi)$ and $A = D = B + C$.

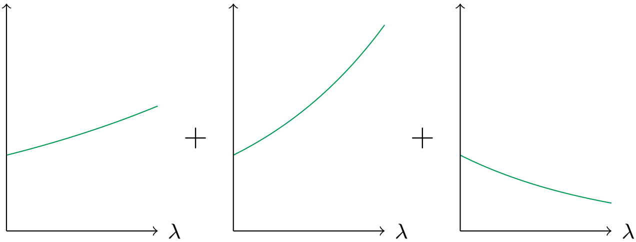
Thus, by taking the module, $|Z_{\mathbf{b}}(X, \alpha)| =$

$$\left| \iint \frac{dxdy \;\; \Phi_{\mathbf{b}}(x+y)}{e^{-8\pi i xy}\Phi_{\mathbf{b}}(x-y)} e^{-(\mathbf{b}+\mathbf{b}^{-1})\left((2\alpha_2^+ + \alpha_3^+)(x+y) + (2\alpha_2^- + \alpha_3^-)(x-y)\right)} \right|$$

Finally we obtain (1) via $(\rightarrow)$ $2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- = 2\pi$, with
$J_X(\mathbf{b}, x) = \int_{y\in\Gamma} dy\, e^{8\pi i xy} \frac{\Phi_{\mathbf{b}}(x+y)}{\Phi_{\mathbf{b}}(x-y)}$ and $\lambda_X(\alpha) = 4\alpha_2^+ + 2\alpha_3^+$.

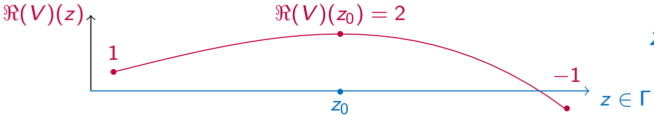
The **saddle point method** gives (under technical conditions) **asymptotics** of complex **integrals with parameters** of the form:

$$\left| \int_{\Gamma} \exp\left(\frac{1}{\mathbf{b}^2} V(z) \right) dz \right| \underset{\mathbf{b}\to 0^+}{\approx} \exp\left(\frac{1}{\mathbf{b}^2} \Re(V)(z_0) \right).$$



$$\left(\lambda \leftrightarrow \frac{1}{\mathbf{b}^2}\right)$$

$$\exp(\lambda \cdot 1) + \exp(\lambda \cdot 2) + \exp(\lambda \cdot (-1)) \underset{\lambda\to\infty}{\approx} \exp(\lambda \cdot 2)$$



z_0 = saddle point

Theorem (TH4, B.A.-G.-P.N. '20)

$$\lim_{\mathbf{b}\to 0^+} 2\pi\mathbf{b}^2 \ \log|J_{X_n}(\mathbf{b},0)| \ = \ -\mathrm{Vol}(S^3\setminus K_n).$$

Sketch of proof: (a) Semi-classical approximation:

$$|J_{X_n}(\mathbf{b},0)| \underset{\mathbf{b}\to 0^+}{\approx} \left|\int_\Gamma \exp\left(\frac{1}{\mathbf{b}^2}V(z)\right)dz\right|.$$

comes from $\log\Phi_{\mathbf{b}} \underset{\mathbf{b}\to 0^+}{\approx} \mathrm{Li}_2$ + technical error bounds

(b) Saddle point method:

$$\left|\int_\Gamma \exp\left(\frac{1}{\mathbf{b}^2}V(z)\right)dz\right| \underset{\mathbf{b}\to 0^+}{\approx} \exp\left(\frac{1}{\mathbf{b}^2}\Re(V)(z_0)\right).$$

we check that z_0 exists thanks to TH2 (geometricity).

(c) Finally, $\Re(V)(z_0) = -\frac{1}{2\pi}\mathrm{Vol}(S^3\setminus K_n)$, from $\mathrm{Li}_2 \leftrightarrow \mathrm{Vol}$.

Future possible directions:

Extend the **algorithm** Knot diagram $\longrightarrow$ Triangulation
(non-alternating knots, links, canonical choices)

Understand the **combinatorial simplifications** in $Z_{\mathbf{b}}(X, \alpha)$
($\leftrightarrow$ Neumann-Zagier datum?)

Adapt our method to **new formulations** of the Teichmüller TQFT
(e.g. for links)

Extend analytical techniques to get an **asymptotic expansion**
(Reidemeister torsion?)

Apply **geometric triangulations** to **other volume conjectures**
(colored Jones polynomials, Turaev-Viro invariants...)

Thank you for your attention!