The Teichmüller TQFT Volume Conjecture for Twist Knots

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(joint work with François Guéritaud and Eiichi Piguet-Nakazawa)

arXiv:1903.09480
How you can follow/use this talk:

**Live**: You can download the slides on the K-OS website (helps for following recurring notations).

Recurring example (the figure-eight knot): Slides 6, 8, 10, 18.

**From the future**: downloading the slides can also help! (eventual mistakes will have been hopefully corrected at this point).

**From the future and you are interested in our paper**: the pictures and main example in these slides can be a good complement to the technical details in arXiv:1903.09480.
Our Goal

Proving the Teichmüller TQFT volume conjecture for twist knots.

Context: quantum topology, volume conjectures.

Topology: triangulating the twist knot complements

Geometry: the triangulations contain the hyperbolicity

Algebra: computing the Teichmüller TQFT

Analysis: the hyperbolic volume appears asymptotically

(Optional: parts/sketches of proofs, at the audience’s preference)
'84: **Jones polynomial**, new knot invariant.

'90: **Witten** retrieves the Jones polynomial via **quantum physics**.

90s: **New topological invariants** (TQFTs of **Reshitikin-Turaev**, **Turaev-Viro**, . . .) are discovered via the intuition from physics.

Andersen-Kashaev '11: **Teichmüller TQFT** of a **triangulated** 3-manifold $M$, an "**infinite-dimensional** TQFT".

Its **partition function** $\{Z_b(M) \in \mathbb{C}\}_{b>0}$ yields an **invariant** of $M$.

**Volume Conjecture (Andersen-Kashaev '11)**

*If $M$ is a triangulated hyperbolic knot complement, then its hyperbolic volume $\text{Vol}(M)$ appears as an exponential decrease rate in $Z_b(M)$ for the limit $b \to 0^+$.*
**Introduction**

Topology (triangulations)
Geometry (angles)
Algebra (TQFT)
Analysis (asymptotics)
Conclusion

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**TOPOLOGY**

* Diagram of knot $K$

* Triangulation of $S^3 \setminus K$

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**QUANTUM INVARIANTS**

$J_K(N, q)$

$Z_b(S^3 \setminus K)$

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**SEMI-CLASSICAL LIMIT**

$q = e^{2i\pi/N}, \ N \to \infty$

$\sim e^N \frac{\text{vol}(S^3 \setminus K)}{2\pi}$

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**HYPERBOLIC GEOMETRY**

* Sums
* Integrals

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**VOLUME CONJECTURES**

* $b \to 0^+$

$\sim e^\frac{1}{b^2} \frac{-\text{vol}(S^3 \setminus K)}{2\pi}$

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**HYPERTOBOLIC GEOMETRY**

* Twist Knots $K_n$

* Colored Jones polynomials

* Teichmüller TQFT

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* TH 1
* TH 2
* TH 3

* TH 4

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The Teichmüller TQFT Volume Conjecture for Twist Knots
Our tetrahedra have ordered vertices (⇒ oriented edges too). \( \sim \) two possible signs \( \epsilon(T) \in \{\pm\} \).

A triangulation \( X = (T_1, \ldots, T_N, \sim) \) of a 3-manifold \( M \) is the datum of \( N \) tetrahedra and a gluing relation \( \sim \) pairing their faces while respecting the vertex order.

We consider ideal triangulations of open 3-manifolds, i.e. where the tetrahedra have their vertices removed.

\[
S^3 \setminus \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\} = T^+ \cup T^-
\]

\[
X^3 = \{T^+, T^-\}, \quad X^2 = \{A, B, C, D\}, \quad X^1 = \{\rightarrow, \rightarrow\}, \quad X^0 = \{\}.
\]

Face maps \( x_0, \ldots, x_3 : X^3 \to X^2 \), for example \( x_0(T^+) = B \).
Thurston: from a **diagram** of a knot $K$, one can construct an **ideal triangulation** $X$ of the knot complement $M = S^3 \setminus K$.

The $n$-th twist knot $K_n$ and the triangulation $X_n$ ($n$ odd, $p = \frac{n-3}{2}$)

**Theorem (TH 1, B.A.-P.N. '18)**

For all $n \geq 2$, we construct an **ideal triangulation** $X_n$ of the complement of the **twist knot** $K_n$, with $\left\lfloor \frac{n+4}{2} \right\rfloor$ tetrahedra.
Sketch of proof of TH1: First draw a tetrahedron around each **crossing** of $K$, whose diagram lives in the **equatorial plane** of $S^3$.

Then **collapse** the tetrahedra into segments ($K \leadsto \cdot$).

Hence the collapsed $S^3$ decomposes into **two polyhedra**.

Finally, **triangulate** the two polyhedra (several possible ways).
(2, 3)-Pachner moves are moves between ideal triangulations.

Matveev-Piergallini: X and X’ triangulate the same M if and only if they are related by a finite sequence of Pachner moves.

⇒ Useful for constructing topological invariants for M.
$\mathcal{A}_X$ is the space of **angle structures** on $X = (T_1, \ldots, T_N, \sim)$, i.e. of $3N$-tuples $\alpha \in (0, \pi)^{3N}$ of **dihedral angles** on edges, such that the angle sum is $\pi$ at each vertex and $2\pi$ around each edge.

$\mathcal{A}_X = \left\{ \alpha = \begin{pmatrix} \alpha_1^+ \\ \alpha_2^+ \\ \alpha_3^+ \\ \alpha_1^- \\ \alpha_2^- \\ \alpha_3^- \end{pmatrix} \in (0, \pi)^6 \right\}$

$$\begin{align*}
\alpha_1^+ + \alpha_2^+ + \alpha_3^+ &= \pi \\
\alpha_1^- + \alpha_2^- + \alpha_3^- &= \pi \\
(\rightarrow) \quad 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- &= 2\pi \\
(\rightarrow) \quad 2\alpha_2^+ + \alpha_3^+ + 2\alpha_1^- + \alpha_3^- &= 2\pi
\end{align*}$$

$\alpha$ fixed $\sim$ angle maps $\alpha_1, \alpha_2, \alpha_3 : X^3 \to \mathbb{R}$, for example $\alpha_2(\mathcal{T}^+) = \alpha_2^+$.
The 3-dimensional **hyperbolic space** is $\mathbb{H}^3 = \mathbb{R}^2 \times \mathbb{R}_{>0}$ with

$$(ds)^2 = \frac{(dx)^2 + (dy)^2 + (dz)^2}{z^2},$$

a metric which has constant curvature $-1$.

A knot is **hyperbolic** if its complement $M$ can be endowed with a complete hyperbolic metric of finite **volume** $\text{Vol}(M)$.

$\sim$ a specific $\alpha \in A_X$ on $X = (T_1, \ldots, T_N, \sim)$ triangulation of $M$.

$\alpha_1 + \alpha_2 + \alpha_3 = \pi$

$T \hookrightarrow \mathbb{H}^3$

$\sum_{\text{edge}} \alpha_j = 2\pi$ ( + others)

gluing gives a **manifold**
For all $n \geq 2$, the twist knot $K_n$ is hyperbolic.

**Theorem (TH2, B.A.-G.-P.N. ’20)**

For all $n \geq 2$, the triangulation $X_n$ of $S^3 \setminus K_n$ is geometric, i.e. it admits an angle structure $\alpha^0 \in A_{X_n}$ corresponding to the complete hyperbolic structure on the complement of $K_n$.

$X$ geometric $\iff$ $\exists$ solution to the nonlinear gluing equations of $X$ (difficult!)

Casson-Rivin, Futer-Guéritaud: approach via $A_X$, the solutions to the linear part: maximising the volume functional.
Dilogarithm function: \[ \text{Li}_2(z) = -\int_0^z \log(1 - u) \frac{du}{u} \quad \text{for } z \in \mathbb{C} \setminus [1, \infty). \]

Volume functional \( \text{Vol}: \mathcal{A}_X \to \mathbb{R}_{\geq 0} \) (strictly concave) is:

\[
\text{Vol}(\alpha) := \sum_{T \in \mathcal{X}^3} \Im \text{Li}_2(z(T)) + \arg(1 - z(T)) \log |z(T)|,
\]
where \( z(T) = \left( \frac{\sin \alpha_3(T)}{\sin \alpha_2(T)} \right)^{e(T)} e^{i\alpha_1(T)} \in \mathbb{R} + i\mathbb{R}_{>0} \) encodes the angles of \( T \).

Theorem (TH2, B.A.-G.-P.N. ’20)

For all \( n \geq 2 \), the triangulation \( X_n \) of \( S^3 \setminus K_n \) is geometric, i.e. it admits an angle structure \( \alpha^0 \in \mathcal{A}_X \) corresponding to the complete hyperbolic structure on the complement of \( K_n \).

Sketch of proof of TH2:
- Check that the open polyhedron \( \mathcal{A}_X \) is non-empty.
- General fact: the complete structure \( \alpha^0 \) exists \( \iff \max_{\mathcal{A}_X} \text{Vol} \) is reached in \( \mathcal{A}_X \).
- Prove that \( \max_{\mathcal{A}_X} \text{Vol} \) cannot be on \( \partial \mathcal{A}_X \) (case-by-case).
$S(\mathbb{R}^n) = \text{rapidly decreasing functions } f : \mathbb{R}^n \to \mathbb{C}$.

$S'(\mathbb{R}^n) = \text{dual of } S(\mathbb{R}^n), \text{ tempered distributions.}$

Example: $X^2 = \{A, B\}$, Dirac delta function

$\delta(A) \in S'(\mathbb{R}^{X^2}) \cong S'(\mathbb{R}^2)$ acts by: $\forall f \in S(\mathbb{R}^2),$

$$\delta(A) \cdot f = \int\int_{(A,B)\in\mathbb{R}^2} dAdB \delta(A) f(A, B) = \int_{B\in\mathbb{R}} dB f(0, B) \in \mathbb{C}.$$  

⚠️ **Product** of Dirac deltas is sometimes but not always defined.

$\delta(A)\delta(A)$ is not defined (because of linear dependance).

$\delta(A+B)\delta(A-B) = \frac{1}{2} \delta(A)\delta(B) = (f \mapsto \frac{1}{2} f(0, 0))$ is well-defined.
Partition function for the triangulation $X$ (and $\alpha \in \mathcal{A}_X$, $b > 0$):

$$Z_b(X, \alpha) = \int_{\bar{x} \in \mathbb{R}^X} \prod_{T_1, \ldots, T_N} p_b(T)(\alpha)(\bar{x}) \in \mathbb{C}.$$

Tetrahedral operator: $p_b(T)(\alpha)(\bar{x}) \in S'(\mathbb{R}^X)$ is equal to

$$\frac{\delta (x_0(T) - x_1(T) + x_2(T)) e^{(2\pi i \epsilon(T)x_0(T) + (b+b^{-1})\alpha_3(T))(x_3(T)-x_2(T))}}{\Phi_b \left( (x_3(T) - x_2(T)) - \frac{i(b+b^{-1})}{2\pi} \epsilon(T)(\alpha_2(T) + \alpha_3(T)) \right)^{\epsilon(T)}}.$$

Faddeev’s quantum dilogarithm:

$$\Phi_b(x) := \exp \left( \int_{z \in \mathbb{R}^+} \frac{e^{-2izx}}{4 \sinh(zb) \sinh(zb^{-1})} \frac{dz}{z} \right).$$

Proposition (Andersen-Kashaev ’11)

$|Z_b(X, \alpha)|$ is invariant under angled Pachner moves on $(X, \alpha)$. 

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Partition function for the triangulation $X$ (and $\alpha \in A_x$, $b > 0$):

$$Z_b(X, \alpha) = \int_{\mathbb{R}^{2x}} d\mathbf{x} \prod_{T_1, \ldots, T_N} p_b(T)(\alpha)(\mathbf{x}) \in \mathbb{C}. $$

Tetrahedral operator: $p_b(T)(\alpha)(\mathbf{x}) \in S'((\mathbb{R}^{x2})$ is equal to

$$\frac{\delta \left( x_0(T) - x_1(T) + x_2(T) \right) e^{\left(2\pi i \epsilon(T)x_0(T)+(b+b^{-1})\alpha_3(T)\right)(x_3(T)-x_2(T))}}{\Phi_b \left( \left( x_3(T) - x_2(T) \right) - \frac{i(b+b^{-1})}{2\pi} \epsilon(T)(\alpha_2(T) + \alpha_3(T)) \right) ^{\epsilon(T)}}. $$

Volume Conjecture (Andersen-Kashaev ’11)

Let $X$ be a triangulation of a hyperbolic knot complement $M$.

(1) $\exists \lambda_x$ linear combination of dihedral angles, $\exists$ smooth function $J_X: \mathbb{R}_{>0} \times \mathbb{R} \rightarrow \mathbb{C}$ such that $\forall$ angle structures $\alpha$, $\forall$ $b > 0$, $\forall$

$$|Z_b(X, \alpha)| = \left| \int_{\mathbb{R}} J_X(b, x)e^{-(b+b^{-1})x\lambda_x(\alpha)} dx \right|. $$
Volume Conjecture (Andersen-Kashaev ’11)

Let \( X \) be a triangulation of a hyperbolic knot complement \( M \).

1. \( \exists \lambda_X \) linear combination of dihedral angles, \( \exists \) smooth function \( J_X : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C} \) such that \( \forall \) angle structures \( \alpha \), \( \forall b > 0 \),

\[
|Z_b(X, \alpha)| = \left| \int_{x \in \mathbb{R}} J_X(b, x) e^{- (b + b^{-1}) \lambda_X(\alpha) x} \, dx \right|.
\]

2. The hyperbolic volume \( \text{Vol}(M) \) is obtained as the following semi-classical limit:

\[
\lim_{b \to 0^+} 2\pi b^2 \log |J_X(b, 0)| = -\text{Vol}(M).
\]

Theorem (TH3, B.A.-P.N. ’18)

(1) is proven for all twist knots, via algebraic computations.

Theorem (TH4, B.A.-G.-P.N. ’20)

(2) is proven for all twist knots, via asymptotic analysis.
Proof of TH3, easiest example: For $K = 4_1$, we find $Z_b(X, \alpha) =$

$$\int \int \int \int dAdBdCdD \ \delta(B - D + C) \delta(C - A + B) \ \Phi_b \left( D - B + \frac{i(b+b^{-1})}{2\pi} (\alpha_2^- + \alpha_3^-) \right)$$

$$\phi_b \left( A - C - \frac{i(b+b^{-1})}{2\pi} (\alpha_2^+ + \alpha_3^+) \right).$$

Then we change the variables: $2x = B + C + \frac{i(b+b^{-1})}{2\pi} (\alpha_1^+ - \alpha_1^-)$, $2y = B - C + \frac{i(b+b^{-1})}{2\pi} (\alpha_1^+ + \alpha_1^- - 2\pi)$ and $A = D = B + C$.

Thus, by taking the module, $|Z_b(X, \alpha)| =$

$$\left| \int \int dx dy \ \phi_b \left( x + y \right) e^{-(b+b^{-1})(2\alpha_2^+ + \alpha_3^+)(x+y)+(2\alpha_2^- + \alpha_3^-)(x-y)} \right|$$

Finally we obtain (1) via $(\rightarrow) \ 2\alpha_1^+ + \alpha_3^+ + 2\alpha_2^- + \alpha_3^- = 2\pi$, with

$$J_X(b, x) = \int_{y \in \Gamma} dy e^{8\pi i xy} \frac{\phi_b(x + y)}{\phi_b(x - y)} \ \text{and} \ \lambda_X(\alpha) = 4\alpha_2^+ + 2\alpha_3^+.$$

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The **saddle point method** gives (under technical conditions) asymptotics of complex integrals with parameters of the form:

\[
\left| \int_{\Gamma} \exp \left( \frac{1}{b^2} V(z) \right) \, dz \right| \approx \exp \left( \frac{1}{b^2} \mathcal{R}(V)(z_0) \right).
\]

\[
\lambda \exp(\lambda \cdot 1) + \exp(\lambda \cdot 2) + \exp(\lambda \cdot (-1)) \approx \exp(\lambda \cdot 2)
\]

\[
\mathcal{R}(V)(z)\quad \mathcal{R}(V)(z_0) = 2
\]

\[
z_0 = \text{saddle point}
\]
Theorem (TH4, B.A.-G.-P.N. ’20)

\[ \lim_{b \to 0^+} 2\pi b^2 \ log |J_{X_n}(b, 0)| = -\text{Vol}(S^3 \setminus K_n). \]

Sketch of proof: (a) Semi-classical approximation:

\[ |J_{X_n}(b, 0)| \approx \int_{\Gamma} \exp \left( \frac{1}{b^2} V(z) \right) dz. \]

comes from \( \log \Phi_b \approx \text{Li}_2 \) + technical error bounds

(b) Saddle point method:

\[ \left| \int_{\Gamma} \exp \left( \frac{1}{b^2} V(z) \right) dz \right| \approx \int_{0^+} \exp \left( \frac{1}{b^2} \Re(V)(z_0) \right). \]

we check that \( z_0 \) exists thanks to TH2 (geometricity).

(c) Finally, \( \Re(V)(z_0) = -\frac{1}{2\pi} \text{Vol}(S^3 \setminus K_n), \) from \( \text{Li}_2 \leftrightarrow \text{Vol}. \)
Future possible directions:

Extend the **algorithm** Knot diagram $\rightarrow$ Triangulation (non-alternating knots, links, canonical choices)

Understand the **combinatorial simplifications** in $\mathbb{Z}_b(X, \alpha)$ ($\leftrightarrow$ Neumann-Zagier datum?)

Adapt our method to **new formulations** of the Teichmüller TQFT (e.g. for links)

Extend analytical techniques to get an **asymptotic expansion** (Reidemeister torsion?)

Apply **geometric triangulations** to **other volume conjectures** (colored Jones polynomials, Turaev-Viro invariants...)
Thank you for your attention!