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Heisenberg Homology on surfaces configurations.

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$$\Sigma = \Sigma_{g,1}, \quad g \geq 1$$

$$C_n = \left\{ \begin{array}{l} \text{unordered} \\ \text{configurations} \\ \text{of } n \text{ points} \end{array} \right\}, \quad n \geq 2$$

Introduction: Lawrence (1990), Krammer, Bigelow

LKB representations:

$$B_k \longrightarrow GL(H_n(C_n(D^2_n)), L)$$

$\mathbb{Z}^2$-cover

$\Rightarrow$ Linearity over $\mathbb{R}$ (Bigelow 2001, Krammer 2002)

Goal: similar representations on $H_n(C_n(\Sigma), \mathcal{H})$

Heisenberg group.

I Braid group of surfaces

$B_n(\Sigma) = \pi_1(C_n(\Sigma))$

Presentation in closed case:

Scott (1970), Gonzalez-Meneses (2001)

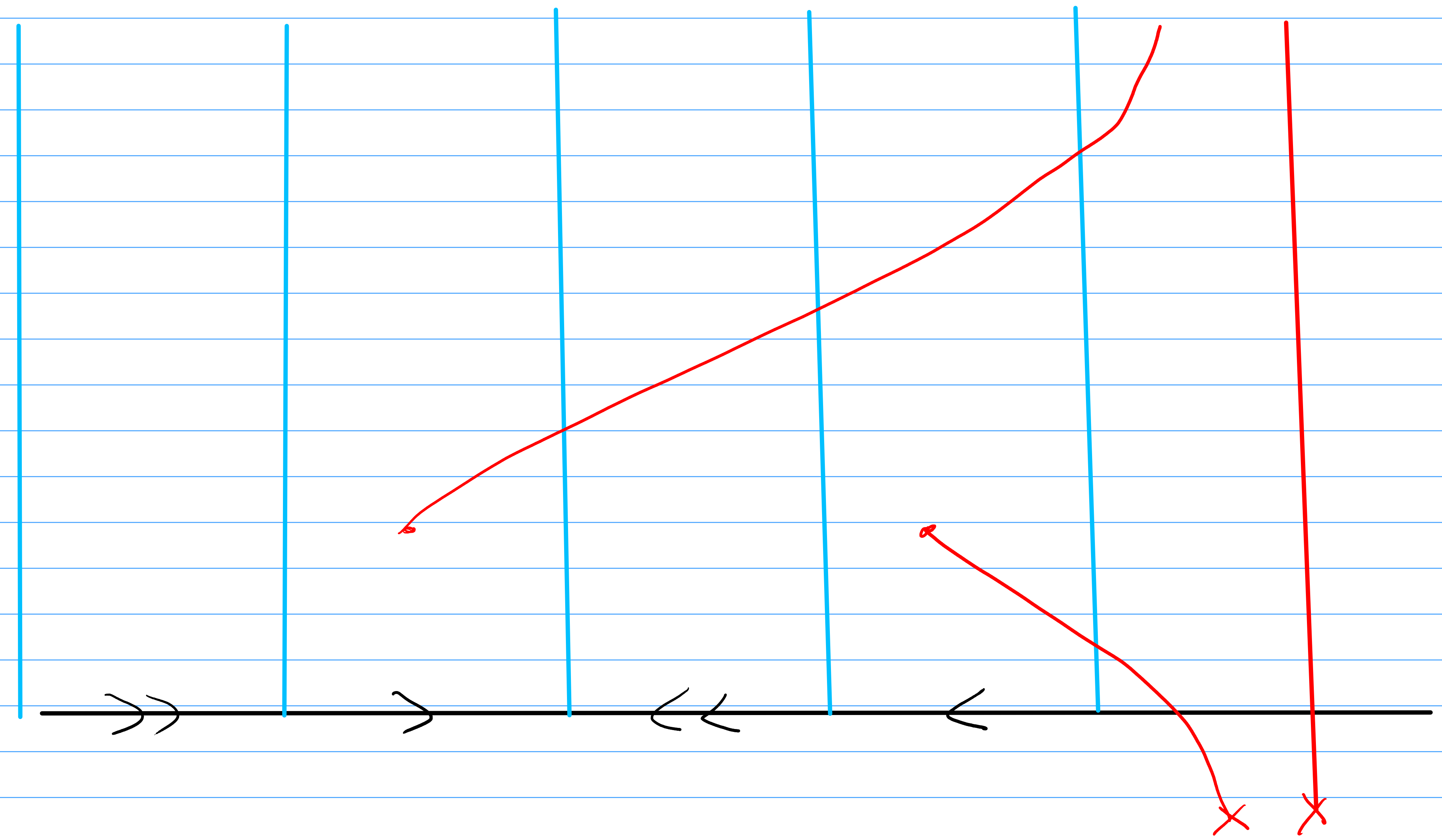
Boundary case: Bellingeri (2002)

Generators: $\sigma_1, \dots \sigma_{n-1}$

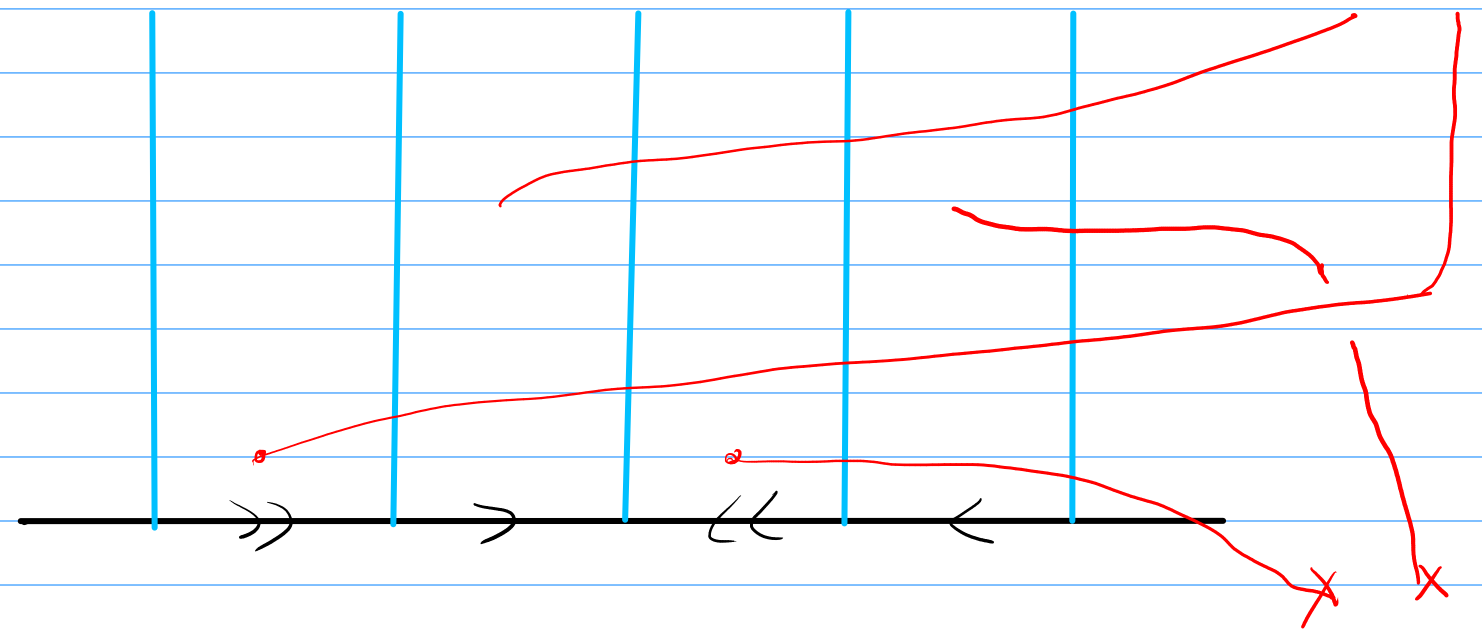
$\alpha_1, \beta_1, \quad \dots \quad, \alpha_g, \beta_g$

$$\Sigma = \widehat{\mathbb{R}^2_+}/\sim$$

Graph of α_1 in $\Sigma \times I$



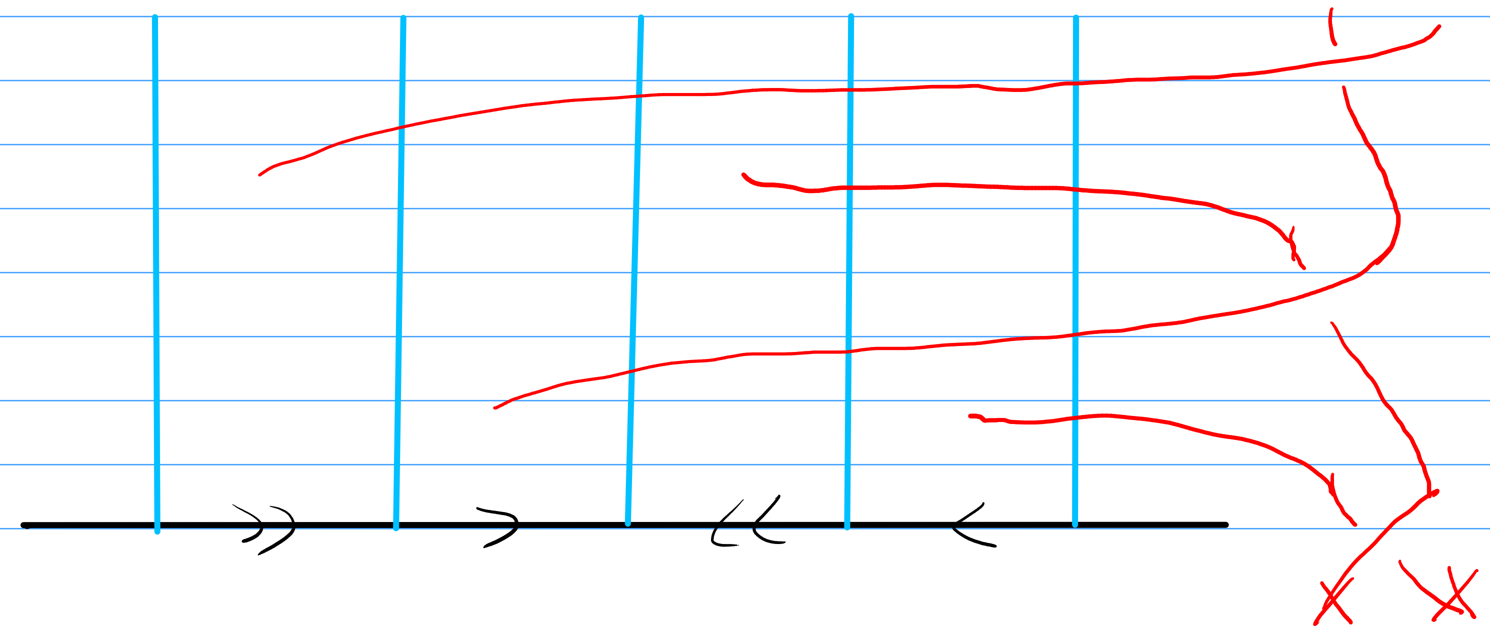
SCR relation



α_1

σ_1

β_1



σ_1

β_1

σ_1

α_1

σ_1

$$\alpha_1 \sigma_1 \beta_1 = \sigma_1 \beta_1 \sigma_1 \alpha_1 \sigma_1$$

(start from the right)

Bellingeri presentation:

(BR1) $[\sigma_i, \sigma_j] = 1$ for $|i-j| \geq 2$;
(BR2) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for $|i-j| = 1$;
(CR1) $[\alpha_r, \sigma_i] = [\beta_r, \sigma_i] = 1$ for $i > 1$;
(CR2) $[\alpha_r, \sigma_1 \alpha_r \sigma_1] = [\beta_r, \sigma_1 \beta_r \sigma_1] = 1$;
(CR3) $[\alpha_r, \sigma_1^{-1} \alpha_s \sigma_1] = [\alpha_r, \sigma_1^{-1} \beta_s \sigma_1] =$
$[\beta_r, \sigma_1^{-1} \alpha_s \sigma_1] = [\beta_r, \sigma_1^{-1} \beta_s \sigma_1] = 1$ $(r < s)$;
(SCR) $\sigma_1 \beta_r \sigma_1 \alpha_r \sigma_1 = \alpha_r \sigma_1 \beta_r$.

II Heisenberg quotient

$\mathcal{H} = \mathcal{H}(\Sigma) \overset{!}{=} \mathbb{Z} \times H$

as a set $\quad H = H_1(\Sigma)$

$(k, x)(\ell, y) = (k + \ell + x \cdot y,\ x + y)$

Central extension:

$0 \to \mathbb{Z} \to \mathcal{H} \to H \to 0$

defined with the cocycle

$\omega(x, y) = x \cdot y$

THM: $g \geq 1$, $n \geq 2$

There exists a quotient map

$\varphi : B_n(\Sigma) \longrightarrow \mathcal{H}(\Sigma)$

with $\ker(\varphi) = \langle [\sigma_1, x], x \in B_n(\Sigma) \rangle^N$

"Make σ_1 central"

Case $n \geq 3$: $\ker \varphi = \Gamma_3(B_n(\Sigma))$

(Bellingeri-Gervais-Guaschi in closed case)

Case $n = 2$ $\quad \Gamma_3(\Sigma) \subsetneq \ker \varphi \subsetneq \Gamma_2(\Sigma)$

<u>Proof</u> $\quad u = (1, 0)$

$a_i = [\alpha_i]$, $\quad b_i = [\beta_i] \in H$

$\tilde a_i = (0, a_i)$, $\quad \tilde b_i = (0, b_i) \in \mathcal{H}$

Presentation for $\mathcal{H}$:

$\langle u, \tilde a_i, \tilde b_i ;\ u$ central

$\quad \tilde a_i \tilde b_j = \tilde b_j \tilde a_i \quad i \neq j$

$\quad \tilde a_i \tilde b_i = u^2 \tilde b_i \tilde a_i \rangle$

$\varphi \quad B_n(\Sigma) \longrightarrow \mathcal{H}$

$\sigma_i \longmapsto u$

$\alpha_i \longmapsto \tilde a_i$

$\beta_i \longmapsto \tilde b_i$

Relations for $B_n(\Sigma)$ / σ_1 central:

(BR1) $[\sigma_i, \sigma_j] \cancel{= 1}$ ~~for~~ $|i-j| \geq 2$;

(BR2) $\sigma_i\sigma_j\sigma_i = \cancel{\sigma_j\sigma_i\sigma_j}$ for $|i-j| = 1$; $\quad \sigma_i = \sigma_1$

(CR1) $[\alpha_r, \sigma_i] = [\beta_r, \sigma_i] = 1$ for $i > 1$;

(CR2) $[\alpha_r, \sigma_1\alpha_r\sigma_1] = [\beta_r, \sigma_1\beta_r\sigma_1] = 1$; $\quad$ OK

(CR3) $[\alpha_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\alpha_r, \sigma_1^{-1}\beta_s\sigma_1] =$
$[\beta_r, \sigma_1^{-1}\alpha_s\sigma_1] = [\beta_r, \sigma_1^{-1}\beta_s\sigma_1] = 1$ $(r < s)$;

(SCR) $\sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r.$ $\quad \sigma_1^2 \beta_r \alpha_r = \alpha_r \beta_r$

σ_1 central $\quad$ ||

III Heisenberg homology

$\tilde{\mathcal{C}}_n(\Sigma)$ regular cover associated with φ.

$\Rightarrow H_*(\tilde{\mathcal{C}}_n(\Sigma)) = H_*(\mathcal{C}_n(\Sigma), \mathbb{Z}[\mathcal{H}])$ is a $\mathbb{Z}[\mathcal{H}]$-module

More generally

$H_*(\mathcal{C}_n(\Sigma), V_\rho)$

$V \otimes S(\tilde{\mathcal{C}}_n(\Sigma))$ ↑ right representation of $\mathcal{H}$

$\rho \curvearrowright \mathbb{Z}[\mathcal{H}] \curvearrowleft$ deck

→ Tautological f. d. rep.

$\dim = g + 2$

$$(k, x) \longmapsto \begin{pmatrix} 1 & p & \frac{1}{2}(k+pq) \\ 0 & I_g & q \\ 0 & 0 & 1 \end{pmatrix}$$

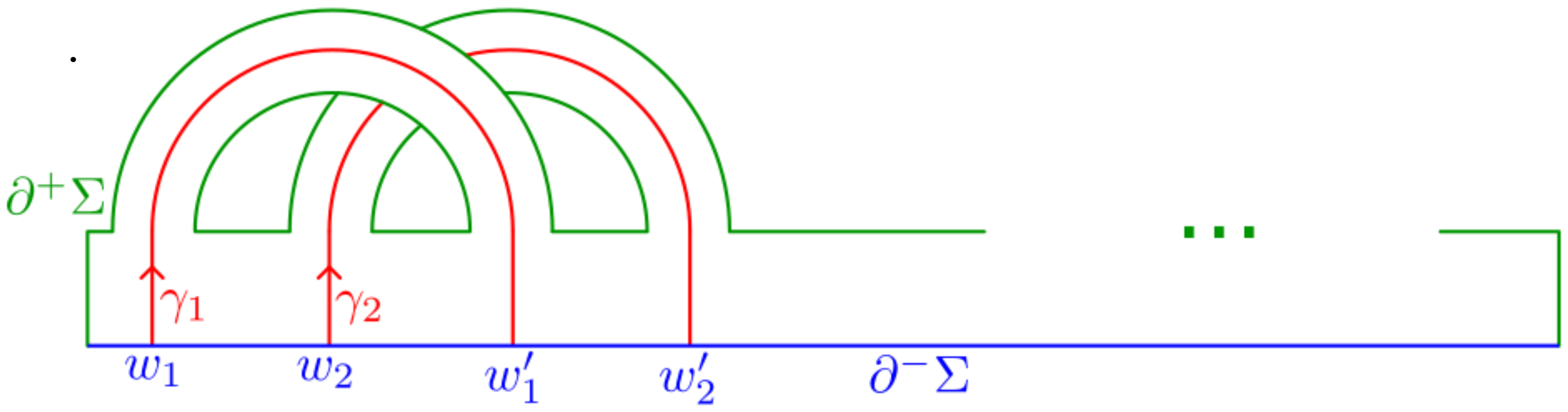
$x = \sum p_i a_i + q_i b_i$

→ Schrödinger representation on $L^2(\mathbb{R}^g)$

$$\left[\Pi_\hbar\left(k, x = \sum_{i=1}^{g} p_i a_i + q_i b_i\right) \cdot \psi\right](s) = \underbrace{e^{i\hbar\frac{k+p\cdot q}{2}} e^{ip\cdot s}}\psi(s + \hbar q)$$

$\hbar \in \mathbb{R} - \{0\}$, right action

Model for Σ :



$$\Gamma = \bigsqcup_{i=1}^{2g} \gamma_i$$

$$\mathcal{C}_n(\Gamma) = \bigsqcup_{k \in K} E_k$$

Type ∂ $\to$ ∂ boun ar ∞

$$K = \{ k = (k_1, \dots, k_{2g}), \ \textstyle\sum k_i = n \}$$

$$\# K = \binom{2g+n-1}{n}$$

We compute

at least 1 pt in $\partial^-\Sigma$

any $\mathcal{H}$-mod

$$H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-\Sigma); V) =$$

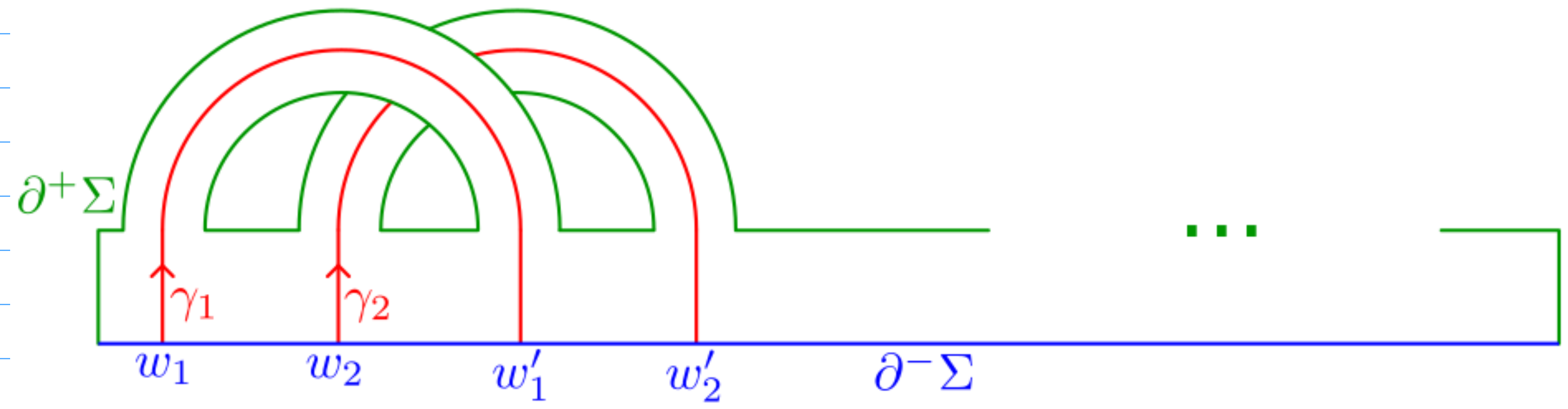
$$\lim_{\underset{\varepsilon}{\longleftarrow}} H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-\Sigma) \cup \mathcal{C}_n^{\varepsilon}(\Sigma); V)$$

at least 2 pts are ε-close

THM : $H_n^{BM}(\mathcal{C}_n, \mathcal{C}_n^-; V) = \bigoplus_{k \in K} V$

and all other groups vanish

Case $V = \mathbb{Z}[\mathcal{H}]$: free of rk $\# K$



About the proof.

Use $h : [0,1]\times\Sigma \longrightarrow \Sigma$:

$$(t,x) \longmapsto h_t(x)$$

→ deformation retraction on $\Gamma\cup\partial^-\Sigma$.

→ h_t is 1-Lipschitz, $0\leq t<1$,
embedding at
for appropriate distance.

Arguments:

→ Deformation retraction.

→ Excision.

Key points

→ $$C_n^{\varepsilon}(\underset{\substack{\| \\ \Sigma_t}}{h_t(\Sigma)}) \hookrightarrow C_n^{\varepsilon}(\Sigma) \quad , 0\leq t<1$$

→ $$C_n(h_1\circ h_t^{-1}) : C_n(\Sigma_t) \longrightarrow C_n(\overset{\substack{\Sigma_1 \\ \|}}{\Gamma\cup\partial^-\Sigma})$$

has bad locus.

IV Action of Mapping Classes.

$f \in \mathcal{M} = \mathcal{M}(\Sigma, \partial\Sigma)$

$\mathrm{Gr}(f)$ induces an action on $\mathcal{H}$

$f_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H} \cong \mathbb{Z} \times H$

$\overset{\cap}{\mathrm{Aut}^+(\mathcal{H})}$ $\quad (k, x) \longmapsto (k + \underbrace{\psi_f(x)}_{}, f_*(x))$

$\psi_f \in H^* = H^1(\Sigma)$

Q1 : $\{ f , f_{\mathcal{H}} = \mathrm{Id}_{\mathcal{H}} \}$?

Q2 : $\{ f , f_{\mathcal{H}} \in \mathrm{Inn}(\mathcal{H}) \}$

Let $\Psi : \mathcal{M} \longrightarrow \mathrm{Aut}^+(\mathcal{H})$

$f \longmapsto f_{\mathcal{H}}$

$\mathcal{X}(\Sigma) = \{ \text{non trivial vector fields} \} / \text{homotopy}$

$\curvearrowleft H^1(\Sigma)$

$\mathrm{Chill}(\Sigma) = \{ f , f \text{ acts trivially on } \mathcal{X}(\Sigma) \}$

$\hat{\mathcal{T}}(\Sigma) = \{ f , f \text{ acts trivially on } H \}$

Prop 1 : $\mathrm{Ker}\, \Psi = \mathrm{Chill}(\Sigma)$

Prop 2 : $\Psi^{-1}(\mathrm{Inn}(\mathcal{H})) = \mathcal{T}(\Sigma)$

Proof of Prop 1

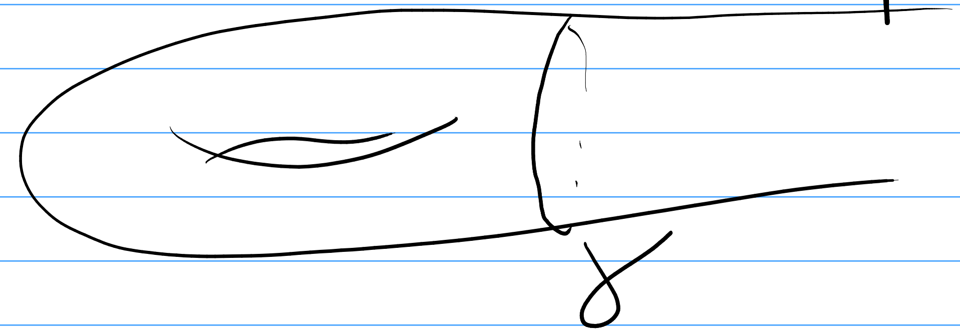
$f_{\mathcal{H}} : (k, x) \to (k + \psi_f(x), f_*(x))$

$f_{\mathcal{H}} = Id_{\mathcal{H}} \Rightarrow f_* = Id.$

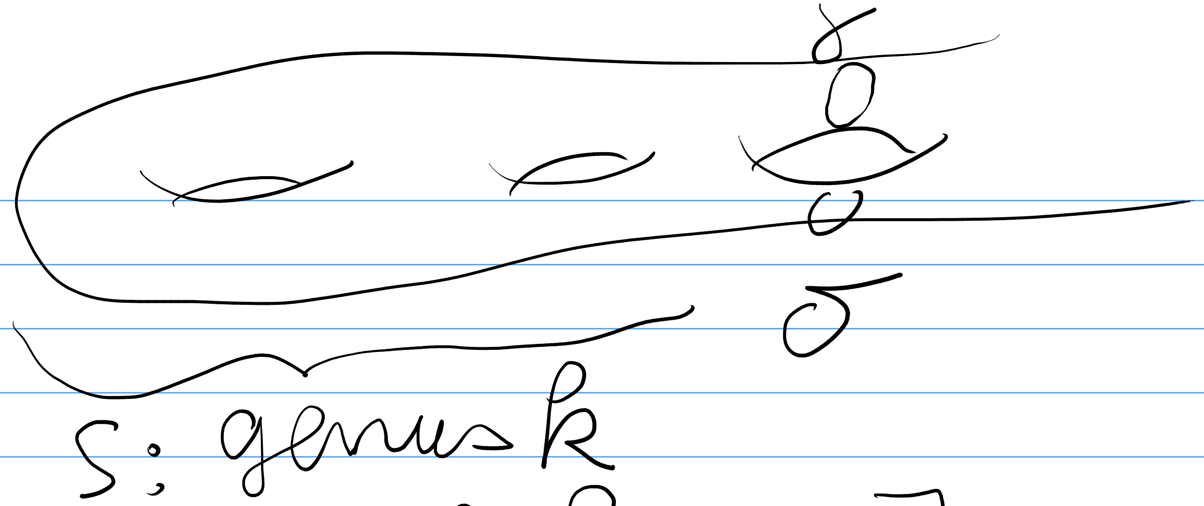
$Ker(\Psi) \subset \mathcal{T}(\Sigma)$

$\psi_{|\mathcal{T}(\Sigma)} \quad \mathcal{T}(\Sigma) \longrightarrow H^1(\Sigma)$

is an homomorphism

Case 

$\psi_{T_\gamma} = 0$

Case 

$P\psi_{|T_\gamma \circ T_\delta^{-1}} = 2k[\gamma]$

↑ oriented as ∂S

↑ Poincaré dual

If $X \in \mathcal{X}(\Sigma)$

$\psi_f = f^* X - X \in H^1(\Sigma)$

for $f \in \mathcal{T}(\Sigma)$

Proof of prop 2

Inner action in $\mathcal{H}$

$$(k,x)(l,y)(-k,-x) =$$

$$(k+l+x.y,\ x+y)(-k,-x) =$$

$$(l + \underline{2x.y},\ y)$$

$$f_{\mathcal{H}} \in \mathrm{Inn}(\mathcal{H}) \Rightarrow f \in \mathcal{T}(\Sigma)$$

For $f \in \mathcal{T}(\Sigma)$, $f_{\mathcal{H}}$ is inner action of $(0, \frac{1}{2} P.\psi_f)$

1. We have an action

$$\mathrm{Chill}(\Sigma) \longrightarrow GL(H_n^{BM}(C_n, C_n^-; V))$$

2. Lift of ψ to the pull-back central extension

$\mathbb{Z}$ central kernel

$$\begin{tikzcd} \widetilde{\mathcal{T}}(\Sigma) \arrow[r, "\widetilde{\psi}'"] \arrow[d] & \mathcal{H} \arrow[d] \\ \mathcal{T}(\Sigma) \arrow[r, "\psi'"] & H \end{tikzcd}$$

$$\psi' = P \circ \psi$$

We have an action

$$\widetilde{\mathcal{T}}(\Sigma) \longrightarrow GL(H_n^{BM}(C_n, C_n^-; V))$$

→ The action of $f \in$ Chill is the natural action of $\mathcal{C}_n(f)$ on homology.

→ The action of $\tilde{f} \in \tilde{\mathcal{J}}(\Sigma)$:

The chain complex is

$$V \otimes S_*\left(\widetilde{\mathcal{C}_n(\Sigma)}^{\varphi}\right)$$

$$\rho \nwarrow \mathbb{Z}[\mathcal{H}] \nearrow \text{deck}$$

The action of $\tilde{f}$ is induced by $\rho(\tilde{\Psi}_f) \underset{\mathbb{Z}}{\otimes} S_*\left(\widetilde{\mathcal{C}_n(f)}\right)$.

3. Case $f \in \mathcal{M}(\Sigma)$

$$\begin{tikzcd}
B_n(\Sigma) \arrow[r, "\varphi"] \arrow[d, "\mathcal{C}_n(f)_\#"'] & \mathcal{H} \arrow[d, "f_{\mathcal{H}}"] \\
B_n(\Sigma) \arrow[r, "\varphi"] & \mathcal{H}
\end{tikzcd}$$

Local system $V \otimes S_*\left(\widetilde{\mathcal{C}_n(\Sigma)}^{\varphi}\right)$

$$\rho \nwarrow \mathbb{Z}[\mathcal{H}] \nearrow \text{deck}$$

Pull-back

$$V \otimes S_*\left(\widetilde{\mathcal{C}_n}^{f_{\mathcal{H}} \circ \varphi}\right)$$

$$\rho \nwarrow \mathbb{Z}[\mathcal{H}] \nearrow$$

Functoriality for homology with local systems provides a twisted action:

$$ {}_{f\mathcal{H}}H_*(\mathcal{C},\mathcal{C}^-;\mathcal{H}) \longrightarrow H_*(\mathcal{C},\mathcal{C}^-;\mathcal{H}) $$

with composition formula

$$ \mathcal{C}_n(g\circ f)_* = \mathcal{C}_n(g)_* \circ \underbrace{g\cdot \mathcal{C}_n(f)_*} $$

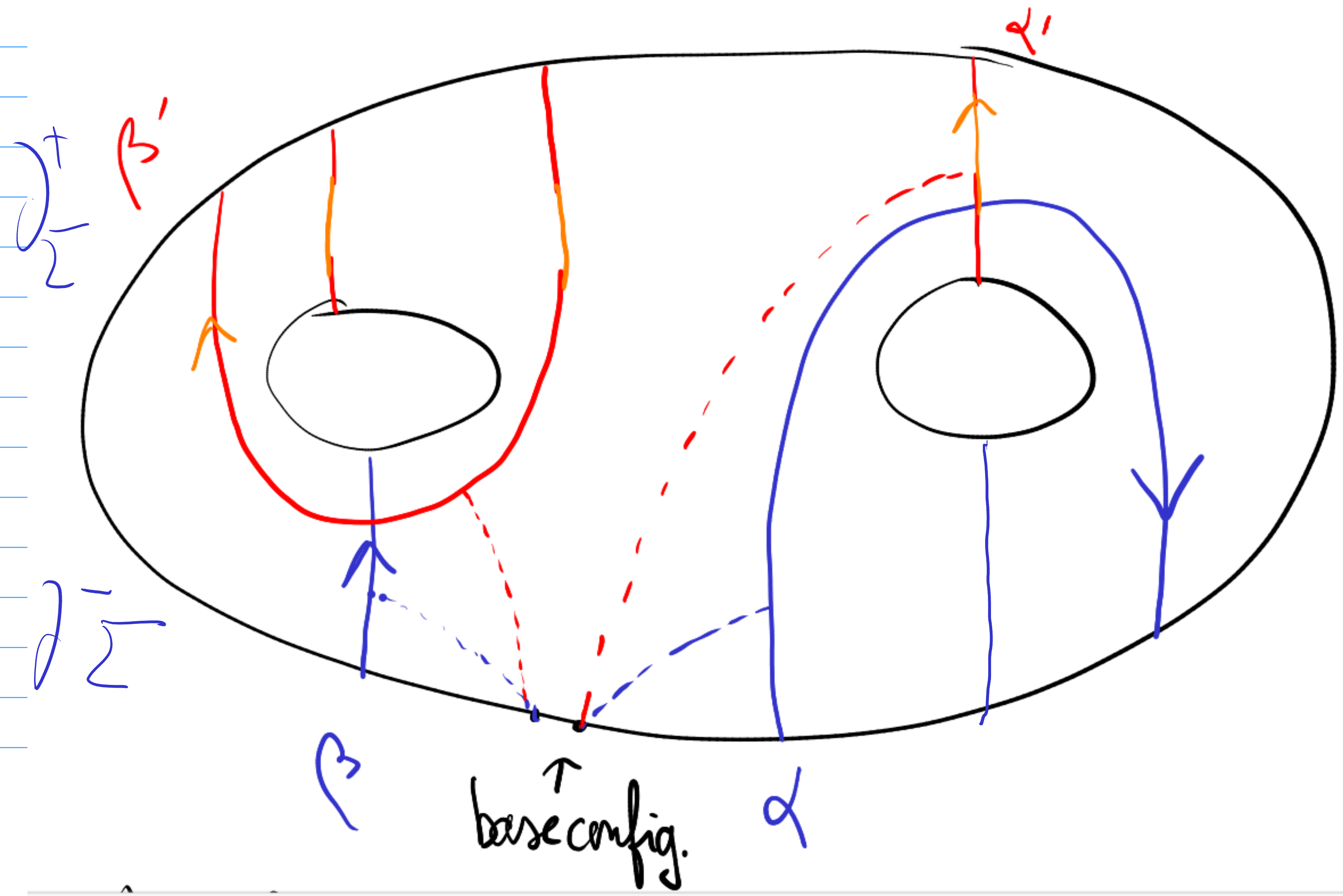
$$ {}_{(g\circ f)_{g\mathcal{H}}}H_*(\mathcal{C}_n,\mathcal{C}_n^-;V) \longrightarrow {}_{f_{g\mathcal{H}}}H_*(\mathcal{C}_n,\mathcal{C}_n^-;V) $$

We get a "functor on the left translation groupoid"

V Computation, case $n = 2$

Genus one case:

$k = ((2,0),(0,2),(1,1))$



Write basis $w(\alpha), w(\beta), v(\alpha,\beta)$

Dual basis $\tilde{w}(\alpha'), \tilde{w}(\beta'), \tilde{v}(\alpha',\beta')$

$\uparrow$

represented by 2 parallel copies

Intersection form

$$H_2^{BM}(\mathcal{C}_2, \mathcal{C}_2^-; \mathcal{H}) \otimes H_2(\mathcal{C}_2, \mathcal{C}_2^+; \mathcal{H}) \to \mathbb{Z}[\mathcal{H}]$$

$\Rightarrow$ Matrix coefficients

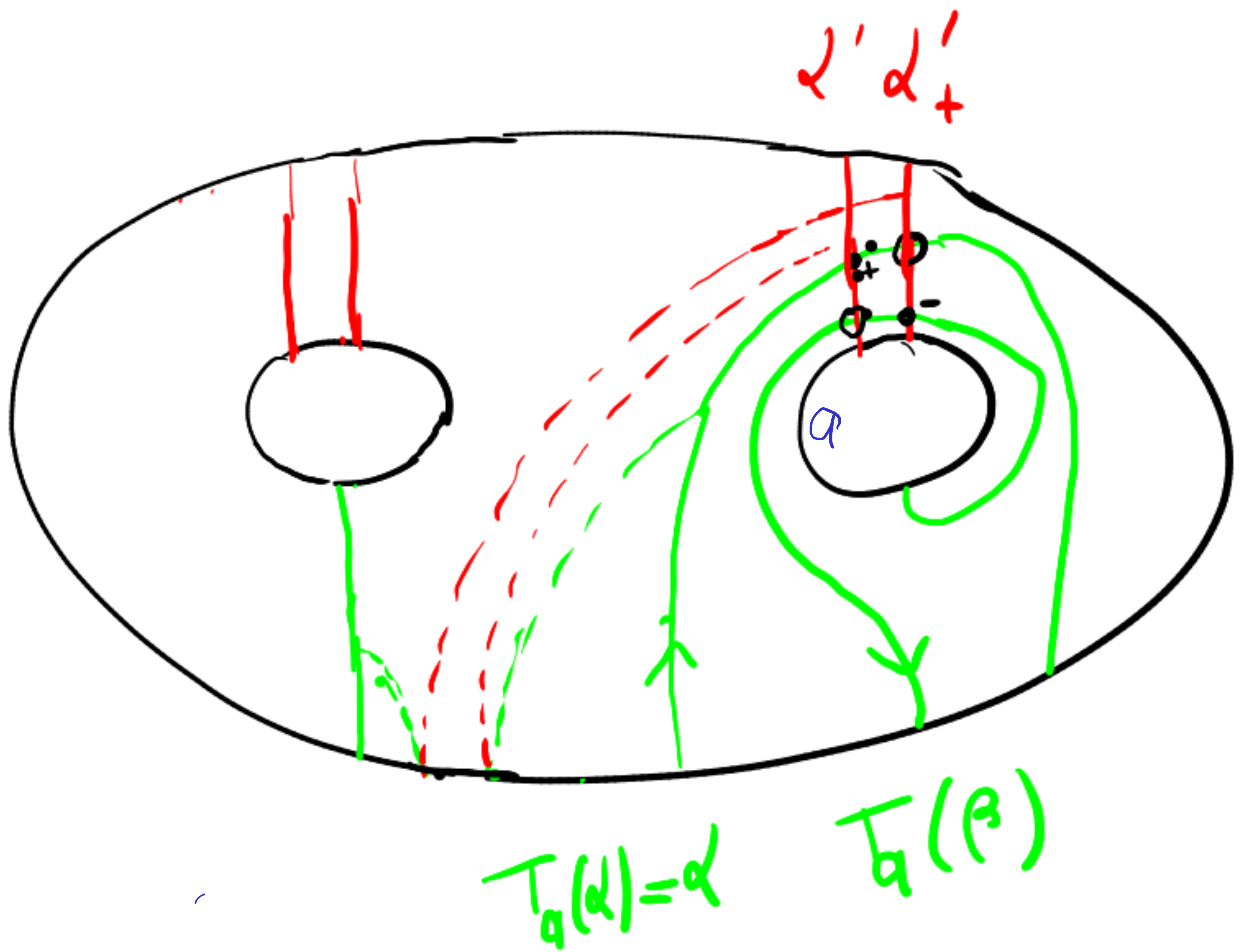


Figure for computing

$\langle T_a . v(\alpha,\beta), \overline{w}(\alpha'') \rangle =$

$-a^{-1}b + u^{-1}a^{-1}b$

Theorem 36. *With respect to the ordered basis $(w(\alpha), w(\beta), v(\alpha,\beta))$:*

(a) *The matrix for the isomorphism*

$$\mathcal{T}_a = \mathcal{C}_2(T_a)_* \colon {}_{(T_a)_\mathcal{H}}H_*^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}]) \longrightarrow H_*^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$$

is

$$M_a = \begin{pmatrix} 1 & u^2a^{-2}b^2 & (u^{-1}-1)a^{-1}b \\ 0 & 1 & 0 \\ 0 & -a^{-1}b & 1 \end{pmatrix}.$$

(b) *The matrix for the isomorphism*

$$\mathcal{T}_b = \mathcal{C}_2(T_b)_* \colon {}_{(T_b)_\mathcal{H}}H_*^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}]) \longrightarrow H_*^{BM}(\mathcal{C}_2(\Sigma), \partial^-; \mathbb{Z}[\mathcal{H}])$$

is

$$M_b = \begin{pmatrix} u^{-2}b^2 & 0 & 0 \\ -u^{-1} & 1 & 1-u^{-1} \\ -u^{-1}b & 0 & b \end{pmatrix}.$$

Sage computation:

Matrix for genus 1 separating curve.

(in any $\Sigma_{g,1}$.)