Epimorphisms between knot groups and the $SL(2,\mathbb{C})$-character variety.

[K-OS] Seminar, 28 April 2022

Joint work with Teruaki Kitano, Steven Sivek and Raphael Zentner

All knots are assumed tame and contained in the 3-sphere S^3.

If $K \subset S^3$ is a knot, then $N(K)$ denotes a closed regular neighbourhood and $E(K) = \overline{S^3 \setminus N(K)}$ the knot exterior.

Fixing an orientation of S^3 restricts to a preferred orientation on knot exteriors.

Let $\pi_1(K)$ be the knot group $\pi_1(E(K))$.

$\mu_K, \lambda_K \in \pi_1(K)$ is a preferred meridian-longitude pair, represented by simple closed curves on $\partial E(K)$.

λ_K is realized as the boundary of a Seifert surface of K.

Partial order

A knot K_1 *dominates* another knot K_2, if there is an epimorphism $\varphi : \pi_1(K_1) \twoheadrightarrow \pi_1(K_2)$. We writte $K_1 \geq K_2$.

The relation $\geq$ provides a partial order on the set of **prime** knots in S^3.

The transitivity and reflexivity of $\geq$ are clear.

The antisymmetry, follows from the fact that knot groups are hopfian and that prime knots are determined by their groups.

It is no longer true for a connected sum of knots.

Every knot dominates the unknot U.

A connected sum $K_1 \sharp K_2$ of knots dominates each summand.

$K_1 \geq K_2 \Rightarrow \Delta_{K_2}(t) | \Delta_{K_1}(t)$.

Examples of domination

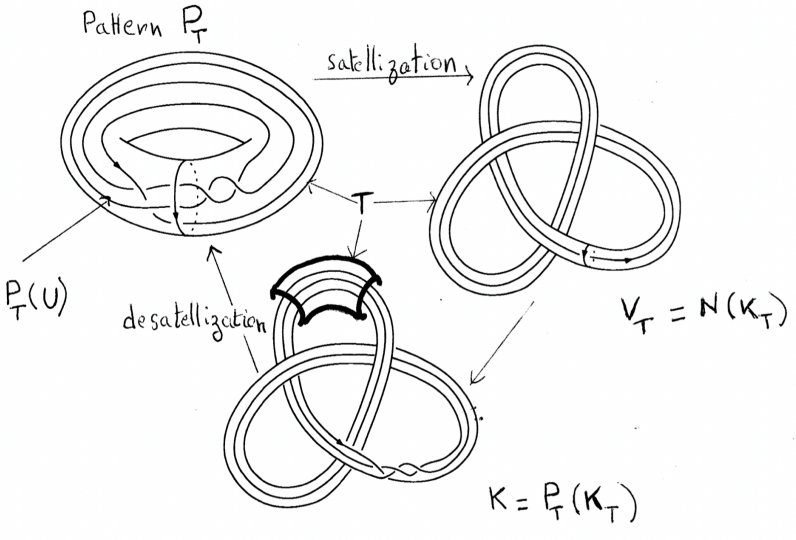
-Degree one map $f : (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2))$ then $f_\star : \pi_1(K_1) \to \pi_1(K_2)$ epimorphism.

-Desatellization by pinching the exterior of a companion onto a solid torus, which is still a degree-one map.

-Rotational symmetry : the quotient map induces an epimorphism between the knot groups.

All these epimorphisms have the property to be induced by non zero degree maps and thus to preserve meridians.

These extra conditions generate finer orders on the set of (prime) knots.


Pattern P_T
satellization
T
$P_T(U)$
desatellization
$V_T = N(K_T)$
$K = P_T(K_T)$

$$K \geq P_T(U)$$

Finer orders

$K_1 \geq_\mu K_2$ (meridionally dominates) if there is an epimorphism $\varphi : \pi_1(K_1) \twoheadrightarrow \pi_1(K_2)$. which preserves meridians.

$K_1 \geq_1 K_2$ (1-dominates) if there is a degree 1 map $f : E(K_1) \to E(K_2)$.

$K_1 \geq_1 K_2 \Rightarrow K_1 \geq_\mu K_2 \Rightarrow K_1 \geq K_2$, but there is no reverse implication (M. Suzuki).

There exists plenty of non-meridional epimorphisms between knot groups.

If $\pi_1(K) =<< \alpha >>$, α not an automorphic image of the meridian, then there is a knot K' and an epimorphism $\varphi : \pi_1(K') \twoheadrightarrow \pi_1(K)$ such that $\varphi(\mu_{K'}) = \alpha$ (F. Gonzalez-Acuña)

It is conjectured that knot groups contain infinitely many inequivalent such killer elements (D.Silver - W. Whitten - S. Williams)

Thm (I. Agol - Y. Liu)

A knot K dominates at most finitely many knots in S^3.

Problem (J. Simon 1975)

If $g(K)$ denotes the genus of K, does $K_1 \geq K_2$ imply $g(K_1) \geq g(K_2)$?

Let $Vol(K) =$ sum of volumes of hyperbolic pieces in the geometric decomposition of $E(K)$.

Question

Does $K_1 \geq K_2$ imply $Vol(K_1) \geq Vol(K_2)$?

The answer to these two questions is yes if $K_1 \geq_1 K_2$, but still unknown if $K_1 \geq_\mu K_2$.

Character variety of a knot

Let $\mathfrak{R}(K) = \mathrm{Hom}(\pi_1(K), SL(2;\mathbb{C}))$ be the space of representations from $\pi_1(K)$ in $SL(2;\mathbb{C})$.

The character variety of K is the GIT quotient $\mathfrak{X}(K) = \mathfrak{R}(K)//SL(2;\mathbb{C})$.

$\mathfrak{X}(K)$ is a complex affine algebraic set.

$\mathfrak{R}^{irr}(K) \subset \mathfrak{R}(K)$ denote the subspace of irreducible representations.

$\mathfrak{X}^{irr}(K) \subset \mathfrak{X}(K)$ the subspace of characters of irreducible representations.

$\mathfrak{X}^{irr}(K)$ is isomorphic to the quotient of $\mathfrak{R}^{irr}(K)$ by the action by conjugation of $SL(2;\mathbb{C})$.

Character variety

Let $K, K_0 \subset S^3$ be two knots.

An epimorphism $\varphi : \pi_1(K) \twoheadrightarrow \pi_1(K_0)$ induces an injective map

$\varphi^* : \mathfrak{X}(K_0) \hookrightarrow \mathfrak{X}(K)$ defined by $\varphi^*(\chi_\rho) = \chi_{\rho \circ \varphi}$.

This map is algebraic and closed in the Zariski topology.

If $\chi_\rho = \varphi^*(\chi_{\rho'}) = \chi_{\rho' \circ \varphi}$ for some irreducible representation $\rho' \in \mathfrak{R}(K_0)$.

Then ρ and $\rho' \circ \varphi$ irreducible $\Rightarrow$ after conjugating, if necessary, $\rho = \rho' \circ \varphi$.

Irreducible representations of knot groups

If K is hyperbolic, Thurston's hyperbolic Dehn surgery theorem $\Rightarrow$ there is a *canonical component* $X_0 \subset \mathfrak{X}(K)$, which is a curve, and contains the character of a faithful discrete representation of $\pi_1(K)$.

For the general case, one uses Kronheimer-Mrowka's theorem : for any $n \neq 0$, $\pi_1(K(\frac{1}{n}))$, admits an irreducible representation in $SU(2)$, where the homology sphere $K(\frac{1}{n})$ is the result of the Dehn surgery of slope $\frac{1}{n}$ on K :

Proposition

Let K be a non-trivial knot. Then $\mathfrak{X}(K)$ contains a curve X which contains infinitely many irreducible characters $\chi_\rho \in \mathfrak{X}_{SU(2)}(K) \subset \mathfrak{X}(K)$ for which $\rho(\mu_K \lambda_K^n) = Id$ for some $n \in \mathbb{Z}^*$.

Minimal knots

Proposition

Let $K \subset S^3$ be a hyperbolic knot. If $\mathfrak{X}(K)$ has only two irreducible components. Then $K \geq K_0 \Rightarrow K = K_0$ or K_0 is unknotted.

Proof : Let $\varphi : \pi_1(K) \twoheadrightarrow \pi_1(K_0)$ be an epimorphism.

Let $X_0(K)$ be the canonical component of $\mathfrak{X}(K)$, which has dimension 1.

Hypothesis $\Rightarrow X_0(K)$ unique component of irreducible characters

$\Rightarrow \varphi^*(\mathfrak{X}^{irr}(K_0)) = X_0(K)$ if K_0 is knotted since $\dim \mathfrak{X}^{irr}(K_0) \geq 1$.

$\Rightarrow \exists\ \chi_{\rho'} \in \mathfrak{X}^{irr}(K_0)$ such that $\varphi^*(\chi_{\rho'}) = \chi_\rho$, with $\chi_\rho \in X_0(K)$

and ρ a faithful discrete representation in $SL(2, \mathbb{C})$.

Hence, after conjugation, $\rho = \rho' \circ \varphi \Rightarrow \ker \varphi \subset \ker \rho = \{1\}$.

$\Rightarrow \varphi$ is an isomorphism and $K = K_0$.

By T. Mattman, the $(-2, 3, n)$-pretzel knot, with n not divisible by 3, satisfies the hypothesis of the proposition when $n \geq 7$ or $n \leq -1$.

The key point of the proof is that the induced map $\phi^* : \mathfrak{X}(K_0) \to \mathfrak{X}(K)$ is surjective.

The remaining of the talk will be devoted to the following question :

Question (Rigidity)

Let K, K_0 be prime knots and let $\varphi : \pi_1(K) \twoheadrightarrow \pi_1(K_0)$ be an epimorphism such that the induced map $\phi^* : \mathfrak{X}(K_0) \to \mathfrak{X}(K)$ is surjective. Does it imply that $K = K_0$?

At least we would like to know what properties share K and K_0.

Do they have the same genus ? The same Alexander polynomial ? The same volume ?

Simple domain

Proposition

Let $\varphi : \pi_1(K) \twoheadrightarrow \pi_1(K_0)$ be an epimorphism such that the induced map $\phi^* : X(K_0) \to X(K)$ is surjective. If K is a hyperbolic or a torus knot, then $K = K_0$.

Proof of the previous Proposition $\Rightarrow$ true if K is a hyperbolic knot.

K (p, q)-torus knot $\Rightarrow$ K_0 (r, s)-torus knot with $r|p$ and $s|q$.

$\mathfrak{X}^{irr}(K)$ is a collection of $\frac{(p-1)(q-1)}{2}$ lines $\Rightarrow (p-1)(q-1) = (r-1)(s-1)$

$\Rightarrow (p, q) = (r, s)$, and $K = K_0$.

Hyperbolic target

Thm (B-Kitano-Sevek-Zentner)

Let $K, K_0 \subset S^3$ be two knots and let $\varphi : \pi_1(K) \twoheadrightarrow \pi_1(K_0)$ be an epimorphism, such that $\varphi^* : \mathfrak{X}(K_0) \to \mathfrak{X}(K)$ is surjective. If K_0 is a **hyperbolic** *knot, then :*

$K = K_0$ or K is a satellite knot whose every simple companion has winding number 0 and a knotted pattern.

In the later case, φ is induced by a degree-one map $f : E(K) \to E(K_0)$ which pinches the exteriors of the winding number 0 companions into solid tori.

Corollary

(i) If K is a fibred knot then $K \cong K_0$.

(ii) the knot K has the same Alexander polynomial as K_0.

Satellite knot

If E(K) contains an incompressible, non $\partial-$parallel embedded torus T, K is a satellite knot.

T bounds a solid torus V_T which contains K such that K meets any meridian disk.

The core K_T of V_T is a companion of K.

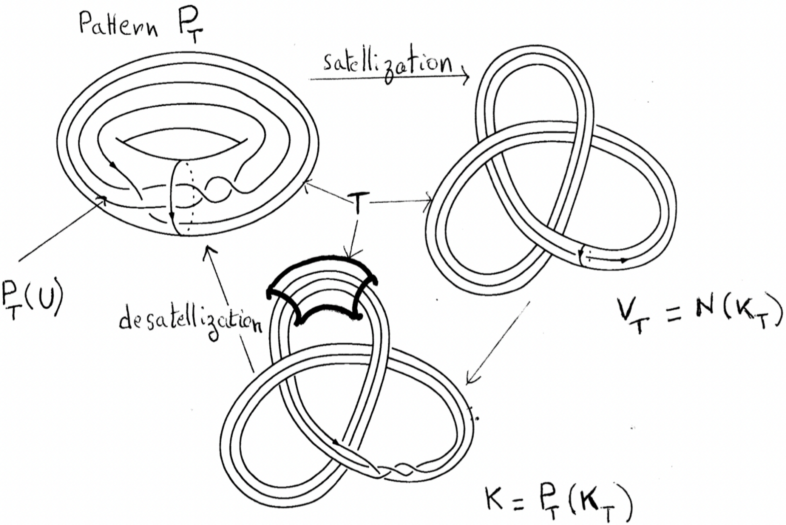
Winding number of K with respect to K_T = algebraic intersection number of K with any meridian disk of V_T.

Let μ_T and λ_T on T denote a meridian and longitude of K_T.

K has winding number $w \Leftrightarrow$ the class $[\mu_T] = [\mu_K]^{\pm w} \in H_1(E(K);\mathbb{Z}) \cong \mathbb{Z}$.

The pair $P_T = (V_T, K)$ is the pattern of the satellization.

The satellite knot is denoted $K = P_T(K_T)$


Pattern P_T
satellization
T
$P_T(U)$
desatellization
$V_T = N(K_T)$
$K = P_T(K_T)$

$$K \geq P_T(U)$$

Satellite knot

$K = P_T(K_T) \Rightarrow E(K) = E(K_T) \cup_T Y$ with :

- $\partial E(K_T) = T$.
- $Y = \overline{V_T \setminus N(K)}$ and $\partial Y = T \cup \partial E(K)$.

Pinching $E(K_T)$ to $S^1 \times D^2$ induces a degree-1 map :

$E(K) \to (S^1 \times D^2) \cup_T Y = E(K')$, with $K' = P_T(U)$.

If $P_T(U)$ is knotted, the pattern is said knotted, otherwise it is said unknotted.

This operation is called a desatellization of $K = P_T(K_T)$ to $K' = P_T(U)$.

In particular $K \geq P_T(U) \Rightarrow$ a satellite knot with knotted pattern is never minimal.

First step of the proof

Proposition (Satellite knots)

Under the hypotheses of the Thm, if K is a satellite knot, then one of the following cases occurs :

1. *K is a satellite of K_0 with winding number 1 and unknotted pattern.*

The epimorphism φ is induced by a degree-one map $f : E(K) \to E(K_0)$

which pinches the satellite part $Y = E(K) \setminus int(E(K_0))$ to a neighborhood

of $\partial E(K_0)$.

2. *Every Simple companion of K has winding number 0 and knotted pattern. Moreover the epimorphism φ is induced by a degree-one map*

$f : E(K) \to E(K_0)$ which pinches the exteriors of the winding number 0 companions into solid tori.

Second step of the proof

The case of satellite knots with winding number 1 is ruled out by :

Proposition (Winding number 1)

If K is a satellite of K_0 with winding number 1 and unknotted pattern, then $\pi_1(K) = \pi_1(K_0) \star_{\pi_1(T)} \pi_1(Y)$ admits an irreducible $SU(2)$-representation which is irreducible on each side of the satellization.

One only needs K_0 to be knotted, but not to be hyperbolic.

The proof is based on holonomy perturbation methods for $SU(2)$ representations, directly issued from Zentner's proof that the splicing of two knot exteriors in S^3 admits an irreducible $SU(2)$ representation.

In general it is not true that a satellite knot admits such a representation, which is irreducible on each side.

Peripheral group

Lemma (peripheral)

$\varphi(\pi_1(\partial E(K))) \cong \mathbb{Z} \oplus \mathbb{Z}$. *Moreover, after possibly a conjugation in* $\pi_1(K_0)$, *one has :* $\varphi(\mu_K) = \mu_{K_0}^{\pm 1}$ *and* $\varphi(\lambda_K) = \lambda_{K_0}^{q}$, *with* $|q| \geq 1$.

Proof Take a Kronheimer-Mrowka's irreducible representation

$\rho : \pi_1(K) \to SU(2) \subset SL_2(\mathbb{C})$ such that $\rho(\mu_K \lambda_K) = \mathrm{Id}$.

$\exists\ \rho' : \pi_1(K_0) \to SL(2, \mathbb{C})$ such that $\rho = \rho' \circ \varphi$.

Then $\varphi(\lambda_K) = 1 \Rightarrow \rho(\lambda_K) = \mathrm{Id} \Rightarrow \rho(\mu_K) = \mathrm{Id}$, which is not possible.

If $\varphi(\mu_K^p \lambda_K^q) = 1$, abelianizing $\Rightarrow p[\mu_K] = 0 \in H_1(E(K); \mathbb{Z}) \cong \mathbb{Z} \Rightarrow p = 0$.

$\pi_1(K)$ is torsion-free $\Rightarrow q = 1$, which is not possible since $\varphi(\lambda_K) \neq 1$.

Representations

Proposition (Key proposition)

Let $K = P_T(K_T)$ be a satellite knot and $E(K) = E(K_T) \cup_T Y$. Under the assumptions of the Thm :

(i) *The restriction $\varphi_{|\pi_1(T)}$ is injective $\Leftrightarrow \varphi(\pi_1(Y)) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

(ii) *The restriction $\varphi_{|\pi_1(T)}$ is not injective $\Leftrightarrow \varphi(\pi_1(K_T)) \subset \mathbb{Z}$.*

A straightforward corollary is :

Corollary

Under the assumptions of the Thm, given any irreducible representation $\rho : \pi_1(K) \to SL(2;\mathbb{C})$:

(i) *If $\varphi_{|\pi_1(T)}$ is injective, then $\rho(\pi_1(Y))$ is abelian.*

(ii) *If $\varphi_{|\pi_1(T)}$ is not injective, then $\rho(\pi_1(K_T))$ is abelian.*

The proof follows from the two lemmas :

Lemma (injective)

The restriction $\varphi_{|\pi_1(T)}$ *is injective* $\Rightarrow \varphi(\pi_1(Y)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Lemma (not injective)

The restriction $\varphi_{|\pi_1(T)}$ *is not injective* $\Rightarrow \varphi(\pi_1(K_T)) \subset \mathbb{Z}$.

We prove Lemma injective. The proof of Lemma not injective is of the same spirit, arguing by contradiction and using Lemma injective.

Proof Lemma injective

Let $\rho_0 : \pi_1(K_0) \to SL(2;\mathbb{C})$ be a discrete faithful representation.

Consider the irreducible representation $\rho_K = \rho_0 \circ \varphi : \pi_1(K) \to SL(2;\mathbb{C})$.

Since ρ_0 is injective, $\varphi|_{\pi_1(T)}$ is injective $\Leftrightarrow$ $\rho_K|_{\pi_1(T)}$ is injective.

Therefore $\rho_K(\pi_1(K_T)) \not\subseteq \mathbb{Z}$ and $\rho_K(\pi_1(K_T))$ is not abelian.

If the restriction $\rho_{K|\pi_1(Y)}$ is not abelian, then the bending construction

along the torus T gives a curve of characters $\chi_{\rho_{K,t}} \in \mathfrak{X}^{irr}(K)$ $(t \in \mathbb{C})$

where $\chi_{\rho_{K,0}} = \chi_{\rho_K}$ and $\rho_{K,t}(\mu_K) = \rho_K(\mu_K)$.

Hence one can find a curve of characters $\chi_{\alpha_t} \in \mathfrak{X}^{irr}(K_0)$ such that

$\chi_{\rho_{K,t}} = \varphi^*(\chi_{\alpha_t}) = \chi_{\alpha_t \circ \varphi}$.

Since $\chi_{\alpha_0} = \chi_{\rho_0}$ this curve χ_{α_t} lies in the canonical component X_0 of $\mathfrak{X}(K_0)$.

But $\mathrm{tr}(\rho_K(\mu_K)) = \mathrm{tr}(\rho_{K,t}(\mu_K)) = \mathrm{tr}(\alpha_t(\mu_{K_0}^{\pm 1}))$ is constant for any t in contradiction with the property that the trace fonction $I_{\mu_{K_0}}$ at μ_{K_0} is not constant on the canonical component X_0 around χ_{ρ_0}.
Therefore the restriction $\rho_{K|\pi_1(Y)}$ is abelian.

$\Rightarrow \varphi(\pi_1(Y))$ abelian, since $\rho_K = \rho_0 \circ \varphi$ and ρ_0 isomorphism on its image.

$\Rightarrow \varphi : \pi_1(Y) \to \pi_1(K_0)$ factorises through $H_1(Y, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

$\Rightarrow \varphi(\pi_1(Y))$ is a torsion free quotient of $H_1(Y, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ which contains the subgroup $\varphi(\pi_1(\partial E(K))) \cong \mathbb{Z} \oplus \mathbb{Z}$.

$\Rightarrow \varphi$ induces an isomorphism from $H_1(Y, \mathbb{Z})$ onto $\varphi(\pi_1(Y))$.

Corollary (Pattern)

Let $K = P_T(K_T)$ be a satellite knot and $E(K) = E(K_T) \cup_T Y$. Under the assumptions of the Thm :

(i) $\varphi_{|\pi_1(T)}$ *is injective* $\Leftrightarrow$ *the pattern* $P_T(U)$ *is unknotted.*

(ii) $\varphi_{|\pi_1(T)}$ *is not injective* $\Leftrightarrow$ *the pattern* $P_T(U)$ *is knotted.*

(ii) $\varphi_{|\pi_1(T)}$ not injective $\Rightarrow \varphi(\pi_1(K_T)) \subset \mathbb{Z}$.

φ factorises through desatellization epimorphism $\pi : \pi_1(K) \twoheadrightarrow \pi_1(P_T(U))$

$P_T(U)$ unknotted $\Rightarrow \pi_1(P_T(U)) \cong \mathbb{Z} \Rightarrow \pi_1(K_0) = \varphi(\pi_1(K)) \cong \mathbb{Z} \Rightarrow \Leftarrow$.

Conversely $P_T(U)$ knotted $\Rightarrow \exists\, \beta : \pi_1(P_T(U)) \to SU(2)$ non abelian.

Hence $\rho = \beta \circ \pi : \pi_1(K) \to SU(2)$ is non abelian on $\pi_1(Y)$.

$\varphi_{|\pi_1(T)}$ injective $\Rightarrow$ any representation $\rho \in \mathfrak{X}^{irr}(K)$ is abelian on $\pi_1(Y)$.

$\Rightarrow \varphi_{|\pi_1(T)}$ not injective.

Winding number

Corollary (Winding number)

Let $K = P_T(K_T)$ be a satellite knot and $E(K) = E(K_T) \cup_T Y$. Let w be the winding number of K with respect to K_T. Under the assumptions of the Thm :

(i) $\varphi_{|\pi_1(T)}$ *is injective* $\Leftrightarrow w \neq 0$.

(ii) $\varphi_{|\pi_1(T)}$ *is not injective* $\Leftrightarrow w = 0$.

(i) $\varphi_{|\pi_1(T)}$ injective $\Rightarrow$ pattern unknotted $\Rightarrow K(\frac{1}{n}) = E(K_T) \cup_T Y(\frac{1}{n})$ with $Y(\frac{1}{n}) \subset S^3$ the exterior of a non trivial knot if $n \geq 2$ (Y. Mathieu).

If $w = 0$, $K(\frac{1}{n}) = E(K_T) \cup_T Y(\frac{1}{n})$ is a splicing of two knot exteriors.

By R. Zentner $\exists\, \rho : \pi_1(K(\frac{1}{n})) \to SU(2)$ which is non abelian on both sides of the splicing.

$\Rightarrow \rho$ induces a representation of $\pi_1(K)$ which is non abelian on $\pi_1(Y)$

$\Rightarrow \varphi_{|\pi_1(T)}$ not injective $\Rightarrow\Leftarrow$. Hence $w \neq 0$.

Conversely $w \neq 0 \Rightarrow$ any non abelian representation $\rho_T : \pi_1(K_T) \to SU(2)$ can be extended to $\pi_1(K)$ by an abelian representation on $\pi_1(Y)$.

$\Rightarrow \varphi_{|\pi_1(T)}$ injective.

Winding number $\neq 0$

Proposition

Under the assumptions of the Thm, if the satellite knot K admits a simple companion K_T with winding number $w \neq 0$. Then $w = 1$ and $K_T = K_0$.

Proof

$w \neq 0 \Leftrightarrow \varphi_{|\pi_1(T)}$ injective $\Leftrightarrow \varphi(\pi_1(Y)) \cong \mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(\partial E(K_0))$.

K_T simple is either a hyperbolic or a torus knot.

K_T torus knot $\Rightarrow \pi_1(K_T)$ has a non trivial center $\mathcal{Z} \subset \pi_1(T) \subset \pi_1(Y)$.

$\varphi(\pi_1(Y)) \cong \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \varphi(\mathcal{Z})$ is in the center of $\varphi(\pi_1(K)) = \pi_1(K_0)$.

$\pi_1(K_0)$ centerless $\Rightarrow \varphi(\mathcal{Z}) = \{1\}$ contradicting the injectivity of $\varphi_{|\pi_1(T)}$.

$\Rightarrow K_T$ is hyperbolic.

Let $\rho_T : \pi_1(K_T) \to SL(2;\mathbb{C})$ discrete and faithful.

Since $w \neq 0$, ρ_T extends to $\pi_1(K)$ by an abelian representation on $\pi_1(Y)$.

By hypothesis $\rho_T = \rho \circ \varphi$ with $\rho \in \mathfrak{X}^{irr}(\pi_1(K_0)) \Rightarrow \varphi_{|\pi_1(K_T)}$ is injective.

$\varphi(\pi_1(\partial E(K_T))) \subset \pi_1(\partial E(K_0))$ since $\varphi(\pi_1(Y)) \subset \pi_1(\partial E(K_0))$.

Waldhausen covering thm + Gonzalez-Acuna and Whitten thm $\Rightarrow$

$\varphi_|$ can be realised by a finite cyclic covering map $h : E(K_T) \to E(K_0)$ such that $h_\star = \varphi_|$.

Using the Smith conjecture, one can show that $w = \text{degree}(h) = 1$ and thus $K_T = K_0$.