Heegaard Floer Theory and non-cuspidal curves

Maciej Borodzik
joint with Beibei Liu and Ian Zemke

Institute of Mathematics, University of Warsaw

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Curves in $\mathbb{C}P^2$

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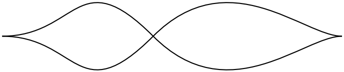

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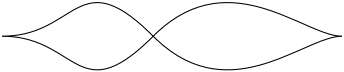

Definition

The *genus* of C is the geometric genus. Minimal genus of a smooth closed surface Σ that surjects onto C.

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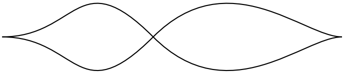

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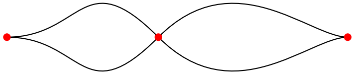

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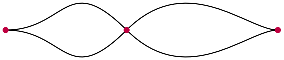

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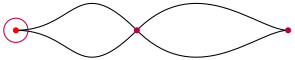

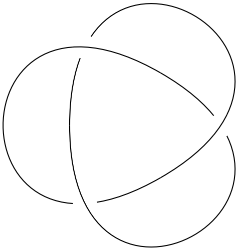

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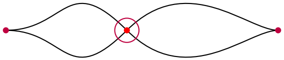

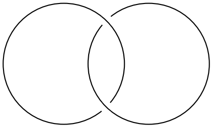

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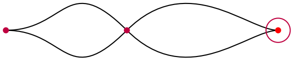

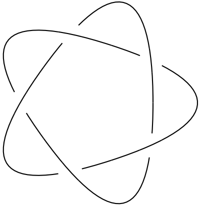

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Given a collection of links of singular points, does there exist a genus g curve having singularities with precisely these links?

Questions we ask

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Question (Classification problem)

Given topological data (genus, degree), find all possible configurations of singular points.

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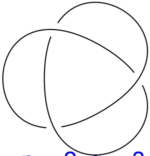

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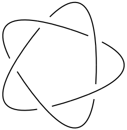
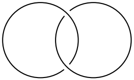


$p = 2, q = 5$



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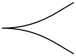

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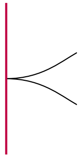

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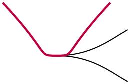

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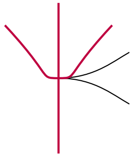

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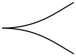

Example

If $C = \{x^p - y^q = 0\}$, the semigroup is generated by p, q.

Semigroup and the Alexander polynomial

Theorem

If S_z is the semigroup of a cuspidal singular point, then expression

$$\Delta = 1 + (t-1)(t^{g_1} + \cdots + t^{g_s}),$$

where $\{g_1, \ldots, g_s\} = \mathbb{Z}_{\geq 0} \setminus S_z$ is the Alexander polynomial of the link of singularity.

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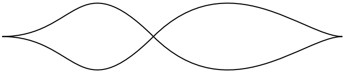
Example

For $\{x^4 - y^7 = 0\}$, the set $\{g_1, \ldots, g_s\}$ is given by $\{1, 2, 3, 5, 6, 9, 10, 13, 17\}$. We have

$$1 + (t-1)(t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17}) =$$
$$1 - t + t^4 + t^5 - t^7 - t^9 + t^{11} - t^{13} + t^{14} - t^{17} + t^{18} =$$
$$\frac{(t^{28}-1)(t-1)}{(t^4-1)(t^7-1)}.$$

A complement of a curve



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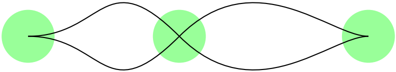

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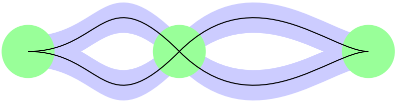

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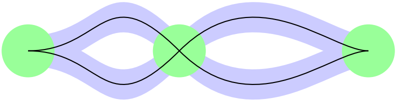

We let N denote the neighborhood of C, $Y = \partial N$, $X = \mathbb{C}P^2 \setminus N$.

Attack plans

- Describe Y in terms of links of C;

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- Compute topology of X (like homology groups);
- Determine whether Y can bound a manifold with topology like X;
- First two steps are standard, the third one involves Heegaard Floer theory.

Theorem (—,Livingston, 2014)

Suppose C is a rational (genus is zero) and cuspidal (all singularities have one branch) curve. Then Y is the d^2 surgery on the connected sum of links of singularities.

Topology of the complement. Rational cuspidal case.

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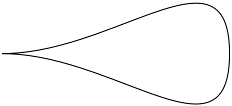

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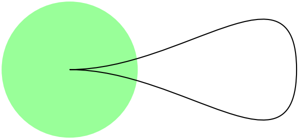

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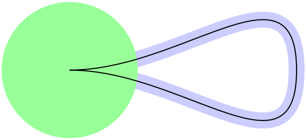

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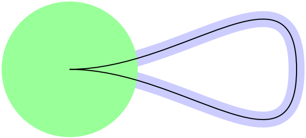

The blue part is a 2-handle attached along K to the four-ball. The framing is C^2.

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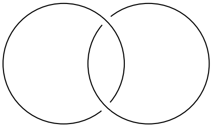

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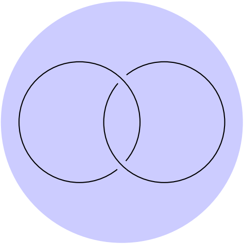

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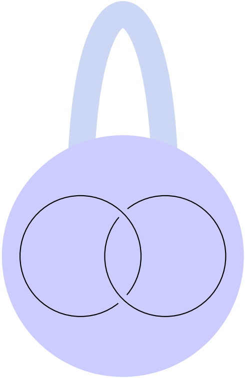

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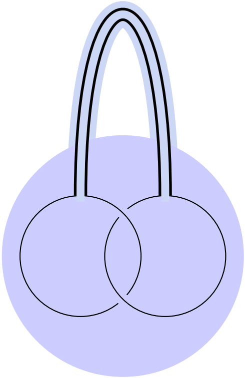

Neighborhood of a non-rational, non-cuspidal curve

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- *Y is a result of a surgery on $K = \widehat{L}_1 \# \ldots \# \widehat{L}_n \#^g \widehat{B}$ in $\#^r S^2 \times S^1$, where $r = 2g + \sum(r_i - 1)$ and B is the Borromean rings.*

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- *The intersection form on $X = \mathbb{C}P^2 \setminus N$ is trivial. We have $b_2(X) = r$.*

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The d-*invariant* is a rational number associated with a rational homology 3-sphere Y with a choice of a spin-c structure $\mathfrak{s}$.

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Remark

If $b_1(Y) > 0$ and $c_1(\mathfrak{s})$ is torsion, we can associate with Y several d-invariants, among which $d^{top}(Y,\mathfrak{s})$ and $d^{bot}(Y,\mathfrak{s})$ play a special role.

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We need to understand the action of $H_1(Y)$ on HF^- to compute these invariants.

d-invariant inequalities

Theorem (Ozsváth, Szabó, 2003)

If X is a rational homology ball and $\partial X = Y$, then $d(Y, \mathfrak{s}) = 0$ for all spin-c structure $\mathfrak{s}$ over Y that extend over X.

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Proposition

With X, Y as in our situation, $d^{bot}(Y, \mathfrak{s}) \geq -\frac{r}{2}$, $d^{top}(Y, \mathfrak{s}) \leq \frac{r}{2}$.

Computing d-invariants for surgeries on algebraic knots

Proposition (Ozsváth, Szabó, 2003)

If K is an L-space knot, then d-invariants of $S^3_q(K)$, with $q > 2g(K) - 1$ can be effectively computed from the Alexander polynomial of K.

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Remark

In general, d-invariants of large surgeries can be computed from $CF^-(K)$.

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Remark

The main problem is to understand the action of $H_1(Y)$. The key tool is Zemke's approach to Heegaard Floer theory and naturality.

A technical step

Proposition

The link Floer chain complex for the knotification of $T(2,2n)$ *splits as a direct sum of two complexes, the action of* $H_1(S^2 \times S^1)$ *intertwines the two complexes.*

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Usually the action on $H_1(S^2 \times S^1)$ is defined only at the level of homology.

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Remark

Usually the action on $H_1(S^2 \times S^1)$ is defined only at the level of homology.

Question

Do the complexes for knotifications of $T(p,q)$ torus links share this property, or it is typical for $T(2,2n)$?

Main result

Theorem (—, Liu, Zemke 2021)

Let C be a degree d curve with genus g, h double points and one cuspidal singular point z. Then, for all $k = 1, \dots, d-2$:

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Generalizations apply for more than one cuspidal singular point.

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Let C be a degree d curve with genus g, ℓ negative double points and one cuspidal singular point z. Then, for all $k = 1, \dots, d-2$:

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Generalizations do not apply (yet?) for more than one cuspidal singular point.

Comparing double points and genus

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This gives obstructions for ‘trading genus for double points’.

Orevkov's curve

Proposition (Orevkov 2004, —, Hedden, Livingston 2016)

For any $n > 0$ *there exists a complex curve in* $\mathbb{C}P^2$ *of degree* ϕ_{4n} *with genus* 1 *and a single singular point whose link is* $T(\phi_{4n-2}, \phi_{4n+2})$ *torus knot.*

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Open problem. Can we trade for a positive double point?

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There are no known obstructions for the existence of a curve of degree 27 with genus 1 and singularity $T(10,73)$ as well as a curve with degree 33 and a singular point $T(12,91)$.

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Remark

There is no known construction of such curve.

Proposition (—, Liu, Zemke 2021)

If such curve exists, then the genus cannot be traded for a positive double point.