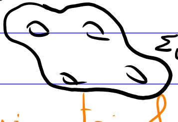
The simplicial volume of surface bundles over surfaces, and other invariants

joint work with Michelle Bucher

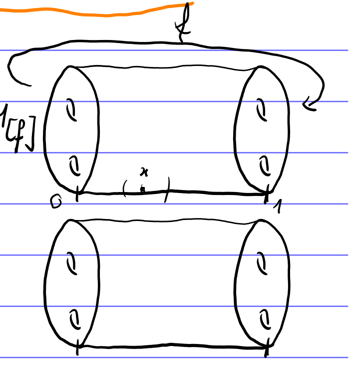
Σ_g, Σ_h closed or. connected surfaces of genus $g, h \geq 2$.



Mapping tori of surfaces = Surface bundles over S^1

$[f] \in \mathrm{Homeo}^+(\Sigma_g)/\mathrm{Homeo}_0(\Sigma_g) = \mathrm{Mod}(\Sigma_g)$ mapping class group

$$\begin{tikzcd}
\Sigma_g \arrow[r, hook] & M_{[f]} = \Sigma_g \times [0,1]/_{(x,0)\sim(f(x),1)} \arrow[d, "\pi"] \\
& S^1
\end{tikzcd}$$



Examples:

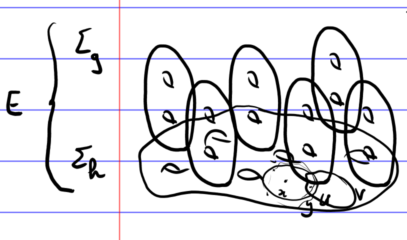
- If $[f] = id$, $M_{[f]} \cong \Sigma_g \times S^1$.
- (Thurston) $M_{[f]}$ is hyperbolic $\Leftrightarrow$ $[f]$ is pseudo-Anosov.
- (Agol) Every clo. or. hyp. 3-mfld is finitely covered by such an $M_{[f]}$.

Surface bundles over surfaces

$\Sigma_g \to E \xrightarrow{\pi} \Sigma_h$ s.t.

$\forall x \in \Sigma_h \ \exists U \ni x$ open nbhood s.t.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi_U", "\cong"'] \arrow[dr, "\pi"'] & & U \times \Sigma_g \arrow[dl, "p_U"] \\
& U &
\end{tikzcd}$$



Examples:

- $E = \Sigma_g \times \Sigma_h$
- See later non-trivial examples.

Fact: $\{\Sigma_g\text{-bdles over } \Sigma_h\}/_{\text{bdle isom}} \overset{1:1}{\longleftrightarrow} \{\text{conj. classes of homo. } \pi_1(\Sigma_h) \xrightarrow{\rho} \mathrm{Mod}(\Sigma_g)\}$

monodromy of E

Question: (open) Is there a $\Sigma_g \hookrightarrow E \to \Sigma_h$ with a hyperbolic metric?

Invariants

A) Euler characteristic : $\chi(E) = \chi(\Sigma_g)\chi(\Sigma_h) = \chi(\Sigma_g \times \Sigma_h)$

B) Signature : $\cup : H^2(E) \times H^2(E) \longrightarrow H^4(E) \cong \mathbb{R}$

$(\alpha, \beta) \longmapsto \alpha \cup \beta \longmapsto (\alpha\cup\beta) \cap [E]$

bilinear form

symmetric : $\beta \cup \alpha = (-1)^{2\cdot 2} \alpha \cup \beta$

diagonalizable over $\mathbb{R}$ with eigenvalues in $\mathbb{R}$

$\sigma(E) := \sigma(\cup) = n_+ - n_- \in \mathbb{Z}$ signature

Facts:
- $\sigma(\Sigma_g \times \Sigma_h) = 0$
- If E is hyperbolic, $\sigma(E) = 0$.
- Chern-Hirzebruch-Serre '59: If $\pi_1(\Sigma_h) \xrightarrow{\rho} \mathrm{Mod}(\Sigma_h) \twoheadrightarrow Sp(2g, \mathbb{Z})$ is trivial, then $\sigma(E) = 0$.

homot. classes of curves

Σ_g

$(H_1(\Sigma_g, \mathbb{Z}), \cdot)$

Question: Is there $\Sigma_g \hookrightarrow E \to \Sigma_h$ with $\sigma(E) \neq 0$?

Non-trivial bundles || Yes! Kodaira '67 Atiyah 69 Morita '99

Many others ...

C) Simplicial Volume (Gromov '82)

M^n or. closed mfld

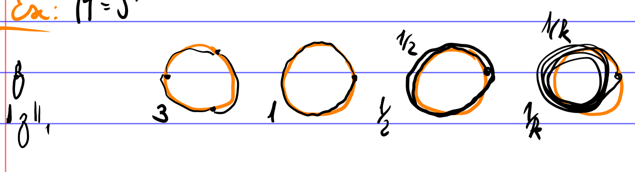
$H_n(M, \mathbb{Z}) \ni [M] \longmapsto [M] \in H_n(M, \mathbb{R})$

$\|M\| := \inf\left\{\sum_{i=1}^k |a_i| \;\middle|\; \left[\sum_{i=1}^k a_i \sigma_i\right] = [M] \in H_n(M,\mathbb{R})\right\} \in \mathbb{R}_{\geq 0}$

simplicial volume of M

homotopy invariant

Ex: $M = S^1$



inf $\Rightarrow \|S^1\| = 0$

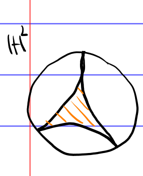
Fact: $M \xrightarrow{f} N \Rightarrow \|M\| \geq |\deg(f)| \, \|N\|$, = if f is a covering

Cor: $\|S^n\|, \|T^n\|, \|S^n \times M\|, \|T^n \times M\|, \ldots = 0 \quad \forall n \geq 1$

Thm: (Gromov, Thurston) If M^n is closed or. hyp.,

$\|M\| = \frac{\mathrm{Vol}(M)}{v_n}$, v_n = vol. of id. reg. simplex in $\mathbb{H}^n$

Rem:
- $n \geq 3$ Mostow rigidity: Vol is a top. invariant
- $n = 2$ Gauß-Bonnet: $\mathrm{Vol}(\Sigma_g) = \int_{\Sigma_g} |-1| \, d\mathrm{Vol} = 2\pi |\chi(\Sigma_g)|$



$v_2 = \pi \quad \Rightarrow \|\Sigma_g\| = 2|\chi(\Sigma_g)|$.

Relations A)-B), A)-C), B)-C)

$\Sigma_g \hookrightarrow E \to \Sigma_h$

- Kotschick '98: $2|\sigma(E)| \leq \chi(E)$; in some cases $3|\sigma(E)| \leq \chi(E)$.

a) - Hamenstädt '20: $3|\sigma(E)| \leq \chi(E) \quad \forall E$.

All known examples admit bigger constants; smallest known is $\frac{9}{2}$ (Catanese Rollenske '06)

- Hoster-Kotschick '00: $\|E\| \geq 4\chi(E)$.

b) - Bucher '09: $\|E\| \geq 6\chi(E)$, = if $E = \Sigma_g \times \Sigma_h$.

If $E = \Sigma_g \times \Sigma_g \xrightarrow[\text{covering}]{\text{finite}} E'$, $\|E'\| = 6\chi(E')$

$\|E\| = d\|E'\|$

$6\chi(E) = 6d\chi(E')$

Question: Is there $\Sigma_g \hookrightarrow E \to \Sigma_h$ with $\|E\| > 6\chi(E)$?

① - Bucher-C '16: There is E with $\|E\| > 6\chi(E)$.

② - Bucher-C '16: $\|E\| \geq 36|\sigma(E)|$.

Rem: a) + b) $\|E\| \underset{b)}{\geq} 6\chi(E) \underset{a)}{\geq} 6 \cdot 3|\sigma(E)|$.

How to get such results?

Two ingredients

1) Bounded cohomology (Gromov, Ivanov)

X top. space

$0 \to C_b^0(X, \mathbb{R}) \xrightarrow{\delta_b} C_b^1(X, \mathbb{R}) \xrightarrow{\delta_b} C_b^2(X, \mathbb{R}) \xrightarrow{\delta_b} \ldots$ cochain complex

$f \in C^q(X, \mathbb{R})$ is bdd if $\|f\|_\infty = \sup_{\sigma \in S_q(X)} |f(\sigma)| < \infty$

$H_b^*(X, \mathbb{R}) = H^*(C_b^*(X, \mathbb{R}), \delta_b)$ Bdd coh. of X

Induced semi-norm on $H_b^*(X, \mathbb{R})$: $\|[f]\|_\infty = \inf\{\|g\|_\infty \mid g \in [f]\} \in \mathbb{R}_{\geq 0}$

Thm: (Gromov, Ivanov) $(H_b^*(X, \mathbb{R}), \|\cdot\|_\infty)$ is a homotopy invariant.

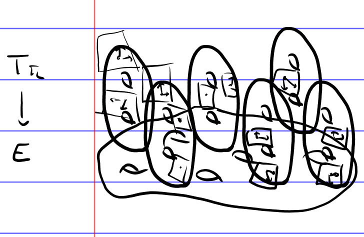
Duality principle: If M^n closed or. and $\beta \in H_b^n(M, \mathbb{R})$

$|\langle \beta, [M] \rangle| = \|\beta\|_\infty \|M\|$.

2) Euler class

Def: The Euler class of the bdle $\Sigma_g \hookrightarrow E \xrightarrow{\pi} \Sigma_h$ is the Euler class of the rk 2 vector bdle $T\pi \to E$, denoted by $e \in H^2(E, \mathbb{Z})$.

$$\begin{tikzcd}
T\pi \arrow[r] \arrow[dr] & TE \arrow[r] \arrow[d] & \pi^* T\Sigma_h \arrow[dl] \\
& E &
\end{tikzcd}$$

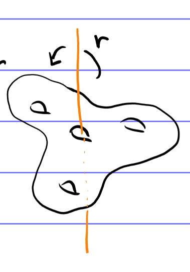


Facts:
- Morita: $\|e\|_\infty \leq \frac{1}{2}$
- Bucher: $\forall \alpha \in H_b^2(E, \mathbb{R})$, $\|\alpha \cup e\|_\infty \leq \frac{1}{3}\|\alpha\|_\infty$.

Bucher: $\|e \cup e\|_\infty \leq \frac{1}{12}$.

Ad ①: Construction of Morita

Σ closed or. surface with an autom. r of finite order d without fixed pts.



$\mathrm{Ker}\left(\pi_1(\Sigma) \twoheadrightarrow H_1(\Sigma, \mathbb{Z}) \twoheadrightarrow H_1(\Sigma, \mathbb{Z}/d\mathbb{Z})\right) \underset{f.i.}{\trianglelefteq} \pi_1(\Sigma)$

$\rightsquigarrow \Sigma' \xrightarrow[\text{finite covering}]{p} \Sigma$ of degree d'

branched covering branched along D

$$\begin{tikzcd}
f^{-1}(\Sigma') \arrow[r, hook] & E \arrow[r, "f", "d"'] \arrow[dr, "\pi"'] \arrow[drr, "\pi'"] & \Sigma' \times \Sigma \arrow[d, "p_\Sigma"] \arrow[dr, "p_{\Sigma'}"] & \supset D = \{(x,y) \in \Sigma' \times \Sigma \mid y = r^i \circ p(x), \ 0 \leq i \leq d-1\} \\
f^{-1}(\Sigma) \arrow[ur, hook] & & \Sigma & \Sigma'
\end{tikzcd}$$

(E, π) is a bdle over Σ $\quad e \in H^2(E, \mathbb{Z})$

(E, π') " Σ' $\quad e' \in H^2(E, \mathbb{Z})$

$$|\langle e \cup e', [E] \rangle| = \|e \cup e'\|_\infty \|E\| \underset{\substack{\text{Bucher} \\ \text{Morita}}}{\leq} \frac{1}{3} \cdot \frac{1}{2} \|E\|$$

$\langle e \cup e', [E] \rangle = \chi(E) + (d-1)|\chi(\Sigma')|$

$$\Rightarrow \|E\| \geq 6\chi(E) + 6\underbrace{(d-1)}_{>0}\underbrace{|\chi(\Sigma')|}_{>0}. \quad \square$$