The Kervaire conjecture and the minimal complexity of surfaces

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KNOT ONLINE SEMINAR
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Weight of a group

Def: The **weight (or normal rank)** of a group G is the minimal cardinality of $S \subset G$ such that $\langle\langle S \rangle\rangle = G$.

- $\langle\langle S \rangle\rangle$ is the smallest normal subgroup containing S (normal closure)
- When $S = \{w\}$, $\langle\langle w \rangle\rangle$ is generated by conjugates of w (and w^{-1})
- G has weight at most one iff $G = \langle\langle w \rangle\rangle$ for some $w \in G$.

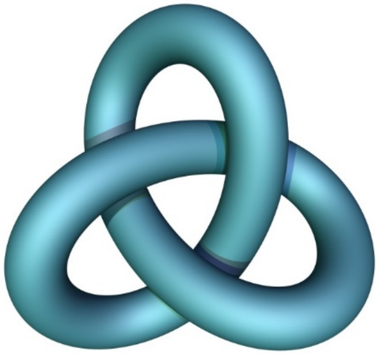
Question: How to find lower bounds of the weight?

Lemma: If $G \twoheadrightarrow H$, then $\mathrm{wt}(G) \geq \mathrm{wt}(H)$.

Examples: $\mathrm{wt}(\mathbb{Z}^n) = n$, $\mathrm{wt}(F_n) = n$

Question (Wiegold): A f.g. perfect group G with $\mathrm{wt}(G) > 1$?

Connection to topology



(higher dimensional) knot group:

- $K \cong S^n$ n-knot in S^{n+2}, $M = S^{n+2} \setminus N(K)$, $n \geq 1$
- Knot group= $\pi_1(M) = \langle\langle w \rangle\rangle$, w =meridian

  $\star$ $1 = \pi_1(S^{n+2}) = \pi_1(M)/\langle\langle w \rangle\rangle$. so $\mathrm{wt}(\pi_1(M)) \leq 1$

Theorem (Kervaire): Fix $n \geq 3$, G is an n-knot group if and only if G is f.p., $\mathrm{wt}(G) \leq 1$, $H_1(G;\mathbb{Z}) \cong \mathbb{Z}$ and $H_2(G;\mathbb{Z}) = 0$.

Question (Kervaire)

Can $G \star \mathbb{Z}$ be an n-knot group?

Conj. 1 (Kervaire '50s): For any group $G \neq 1$, $\mathrm{wt}(G \star \mathbb{Z}) > 1$.

Connection to topology

Cabling Conjecture (Gonzalez-Acuña and Short):
When is Dehn surgery on a knot K in S^3 a connected sum?

- If M is obtained by Dehn surgery on a knot, then $\mathrm{wt}(\pi_1(M)) \le 1$
- M^3 is a connected sum iff $\pi_1(M) = A \star B$

Question: $w \in A \star B$, when is $(A \star B)/\langle\langle w \rangle\rangle$ nontrivial?

One-relator products: $H = (A \star B)/\langle\langle w \rangle\rangle$

Example: $A = \mathbb{Z}/2 = \langle a \mid a^2 = 1 \rangle$, $B = \mathbb{Z}/3 = \langle b \mid b^3 = 1 \rangle$.

$w = aub^{-1}u^{-1}$, $u \in A \star B$. Then $\bar{a}^2 = \bar{a}^3$ in $H \implies \bar{a} = id \in H$

$\implies H = 1$ and $\mathrm{wt}(\mathbb{Z}/2 \star \mathbb{Z}/3) = 1$

The Kervaire conjecture

Question: $w \in A \star B$, when is $(A \star B)/\langle\langle w \rangle\rangle$ nontrivial?

Previous example: Torsion elements may cause problems.

Conjecture: A, B torsion-free, then $(A \star B)/\langle\langle w \rangle\rangle \neq 1$ for any $w \in A \star B$.

Conjecture: $w \in A \star B$, $(A \star B)/\langle\langle w^k \rangle\rangle$ is nontrivial, $k \geq 2$.

Conj. 1 (Kervaire '50s): Group $G \neq 1$, for any $w \in G \star \mathbb{Z}$, the quotient $(G \star \mathbb{Z})/\langle\langle w \rangle\rangle = \langle G, t \mid w \rangle$ is nontrivial.

Special case: $H = \langle F_n \mid w \rangle = \langle x_1, \ldots, x_n \mid w \rangle = (F_{n-1} \star \mathbb{Z})/\langle\langle w \rangle\rangle$

Theorem (Freiheissatz): If w essentially involves x_n, then $\{\bar{x}_1, \ldots, \bar{x}_{n-1}\}$ generates a free subgroup in H.

The Kervaire conjecture

Conj. 1 (Kervaire '50s): Group $G \neq 1$, for any $w \in G \star \mathbb{Z}$, the quotient $(G \star \mathbb{Z})/\langle\langle w \rangle\rangle = \langle G, t \mid w \rangle$ is nontrivial.

Easy for many choices of w.

$\bar{p}_{\mathbb{Z}} : (G \star \mathbb{Z})/\langle\langle w \rangle\rangle \twoheadrightarrow \mathbb{Z}/|p_{\mathbb{Z}}(w)|\mathbb{Z}$

- $p_{\mathbb{Z}} : G \star \mathbb{Z} \twoheadrightarrow \mathbb{Z}$

  $G \ni g \mapsto 0$

  $1 \mapsto 1$

- If $|p_{\mathbb{Z}}(w)| \neq 1$, then $\mathbb{Z}/|p_{\mathbb{Z}}(w)|\mathbb{Z} \neq 1$

- **The interesting case: $p_{\mathbb{Z}}(w) = 1$**

The Kervaire–Laudenbach conjecture

When $\boldsymbol{p_{\mathbb{Z}}(w) = 1}$, expect something stronger.

Conj. 2 (Kervaire–Laudenbach): For any $w \in G \star \mathbb{Z}$ with $p_{\mathbb{Z}}(w) = 1$, we have $G \hookrightarrow (G \star \mathbb{Z})/\langle\langle w \rangle\rangle$.

- **Still open** in general
- Similar to Freiheissatz
- Not true in general if $p_{\mathbb{Z}}(w) = 0$

  $\star$ $w = ghtht^{-1}$, $g, h \in G$ have different orders, $\mathbb{Z} = \langle t \rangle$

- Many partial answers by Gonzalez-Acunna, Short, Levin, Gerstenhaber, Rothaus, Stallings, Casson, Duncan, Howie, Klyachko, Fenn, Rourke, Thom, Brodskii, Forester, etc...

Two confirmed cases

Conj. 2 (Kervaire–Laudenbach): For any $w \in G \star \mathbb{Z}$ with $p_{\mathbb{Z}}(w) = 1$, we have $G \hookrightarrow (G \star \mathbb{Z})/\langle\langle w \rangle\rangle$.

Theorem (Gerstenhaber–Rothaus '62):
Conj. 2 holds for G **finite**.

- $\implies$ Conj. 2 holds for G residually finite
- E.g. finitely generated linear groups

Theorem (Klyachko '93): Conj. 2 holds for G **torsion-free**.

- **Clear conceptual reason**?

From equations to surfaces

Suppose $G \not\hookrightarrow (G \star \mathbb{Z}) / \langle\langle w \rangle\rangle$,

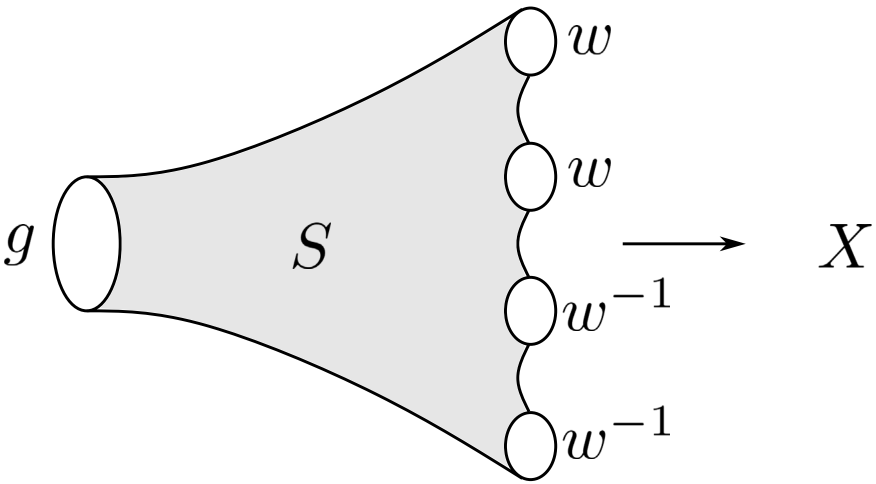
- $g \in \langle\langle w \rangle\rangle$ for some $g \neq 1 \in G$
- $\implies g$ is a product of conjugates of w and w^{-1}
- E.g. $g = awa^{-1} \cdot bwb^{-1} \cdot cw^{-1}c^{-1} \cdot dw^{-1}d^{-1}$ in $G \star \mathbb{Z}$
- An equation in $G \star \mathbb{Z}$, involving conjugacy classes

From equations to surfaces

Equations in $G \star \mathbb{Z}$

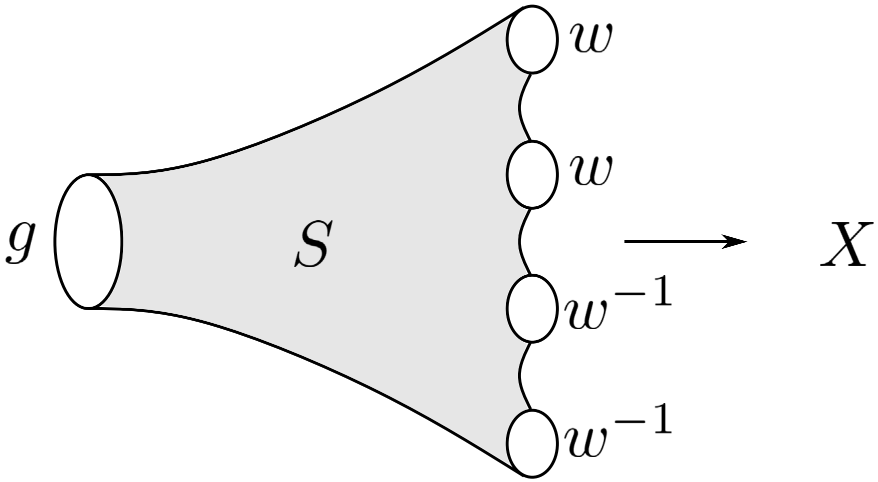
- $g = awa^{-1} \cdot bwb^{-1} \cdot cw^{-1}c^{-1} \cdot dw^{-1}d^{-1}$

Surfaces in X, a space with $\pi_1(X) = G \star \mathbb{Z}$.



What's wrong?

Surfaces in X, a space with $\pi_1(X) = G \star \mathbb{Z}$.



Question: Why should such surfaces not exist?

- $-\chi(S) = n - 1$, $n = \#w + \#w^{-1}$

Our new proof: Show $-\chi(S) \geq n$ if S bounds w, w^{-1} or $g \in G$

- S must be complicated enough compared to its boundary

Minimal complexity

Theorem 1 (C.): For $G \star \mathbb{Z}$ with G torsion-free, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \deg(S).$$

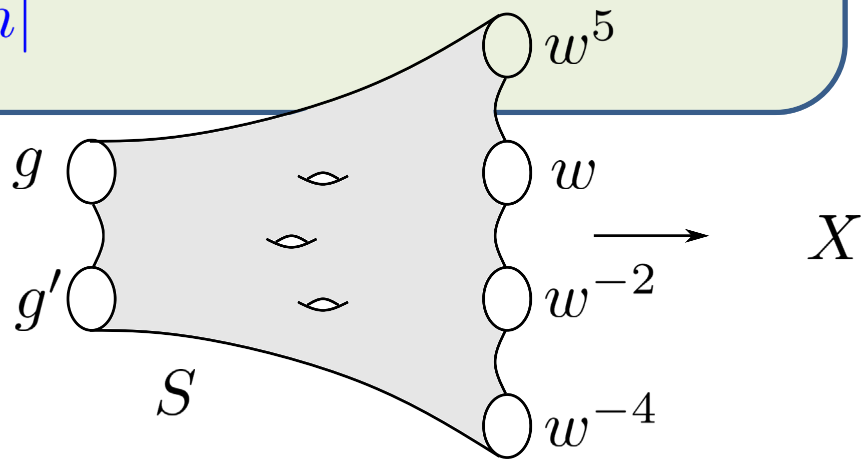
Def: $\pi_1(X) = G \star \mathbb{Z}$, $f : S \to X$ for S compact oriented is w-admissible if each component of ∂S represents

(1) either $g \in G$, (2) or w^n for $n \in \mathbb{Z} \setminus \{0\}$ (conjugation)

Its **degree** $\deg(S) = \sum_{w^n \subset \partial S} |n|$

$$\begin{aligned} \deg(S) &= 5 + 1 + 2 + 4 \\ &= 6 + 6 = 12 \end{aligned}$$

- Not necessarily planar



Irreducibility

Theorem 1 (C.): For $G \star \mathbb{Z}$ with G torsion-free, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

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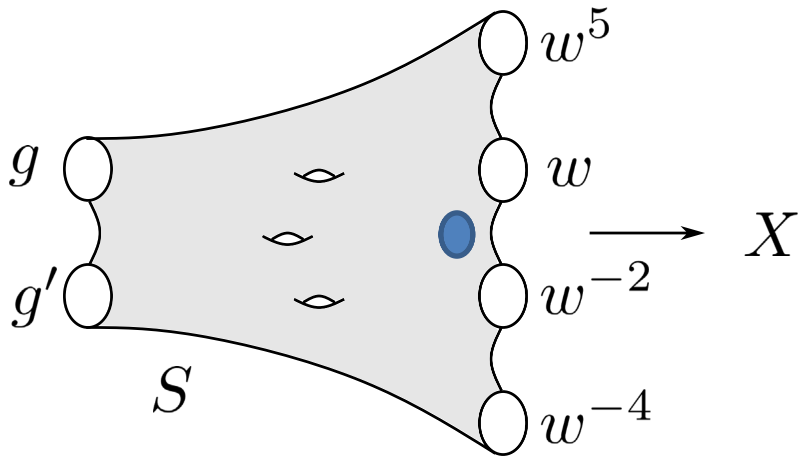
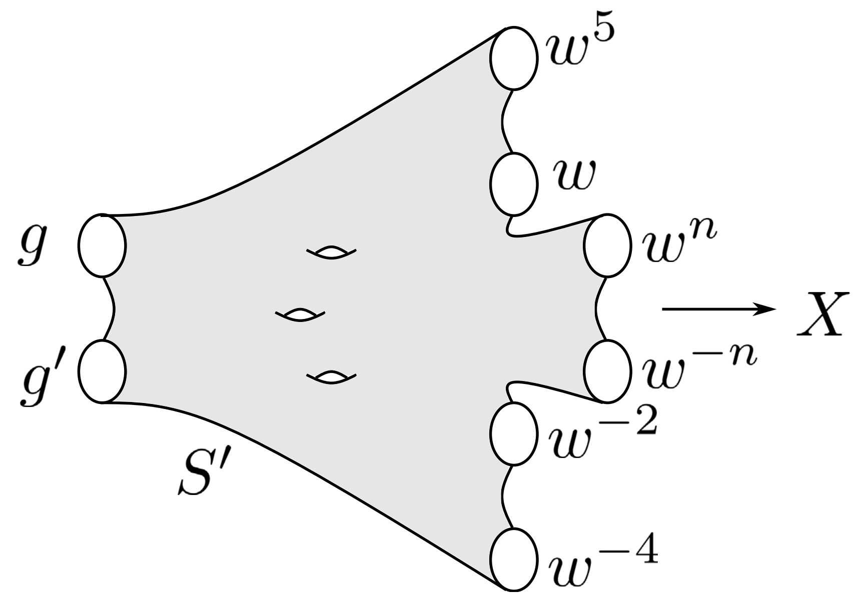
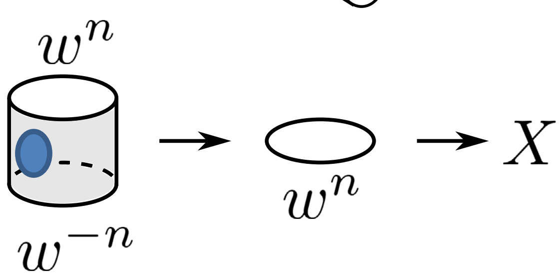






$$-\chi(S') = -\chi(S) + 2$$
$$\deg(S') = \deg(S) + 2n$$

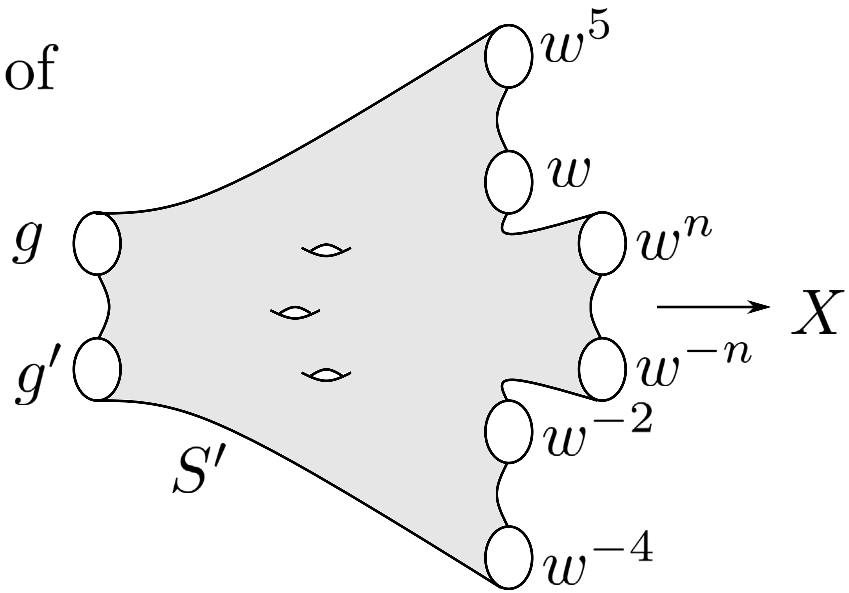
Irreducibility

Theorem 1 (C.): For $G \star \mathbb{Z}$ with G torsion-free, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \deg(S).$$

Def: S is irreducible if no $w^n, w^{-m} \subset \partial S$ with $m, n > 0$ can be merged to represent w^{n-m}.

Lie in different conjugates of the cyclic group $\langle w \rangle$



Theorem 1 implies Klyachko

Theorem 1 (C.): For $G \star \mathbb{Z}$ with G torsion-free, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \deg(S).$$

Allows genus

Theorem (Klyachko):

weaken $p_{\mathbb{Z}}(w) = 1$?

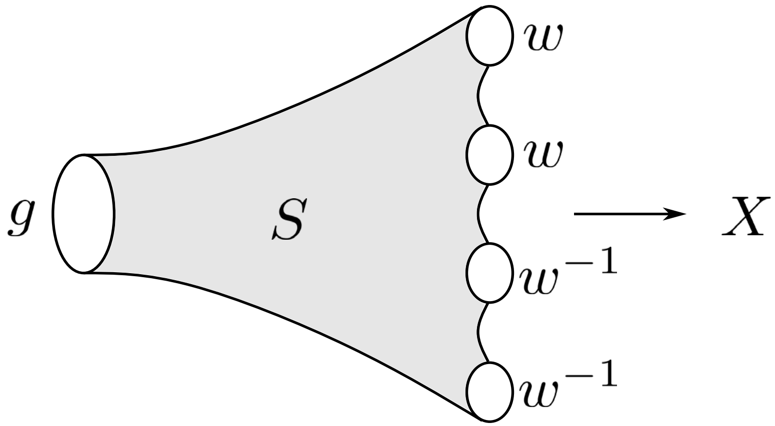
$G \hookrightarrow (G \star \mathbb{Z})/\langle\langle w \rangle\rangle$ if G torsion-free and $p_{\mathbb{Z}}(w) = 1$.

Proof: Suppose $G \not\hookrightarrow (G \star \mathbb{Z})/\langle\langle w \rangle\rangle$

Find $1 \neq g \in \langle\langle w \rangle\rangle \cap G$

Simplest equation $\implies$ S irreducible

$$n - 1 = -\chi(S) \overset{\text{Thm1}}{\geq} n = \deg(S).$$



Torsion

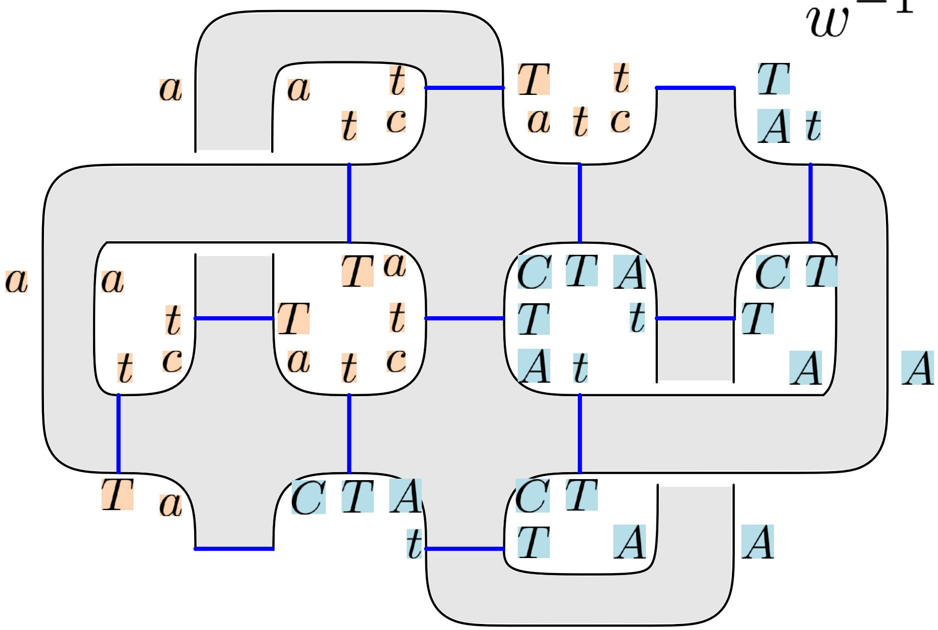
Theorem 1 (C.): For $G \star \mathbb{Z}$ with G torsion-free, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

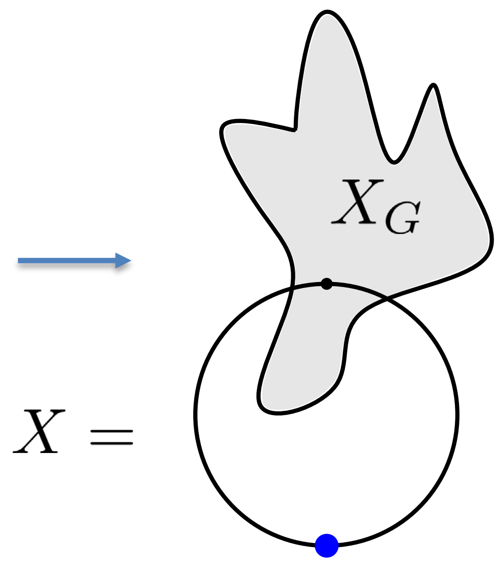
$$-\chi(S) \geq \deg(S).$$

This fails if G has torsion.

Example: $a \in G$ has order 2, $w = aTatct$, $T = t^{-1}, c = C = id$

$$w^{-1} = TCTAtA$$





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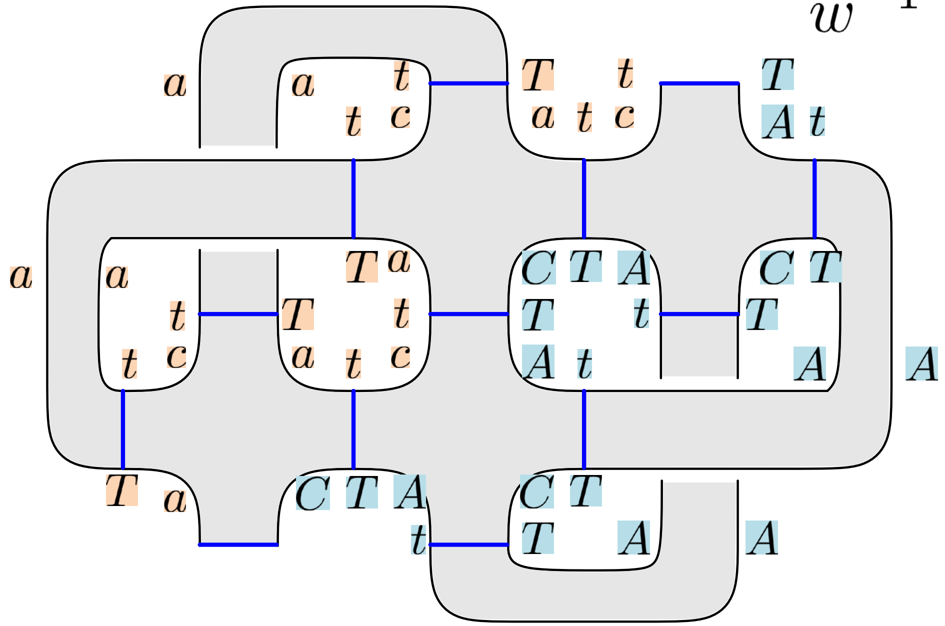


∂S two components:

w^4 and w^{-4}

$\deg(S) = 8$

$-\chi(S) = 4 = \frac{1}{2}\deg(S)$

S non-planar (genus 2)

Torsion

Theorem 1 (C.): For $G \star \mathbb{Z}$ with G torsion-free, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \deg(S).$$

Theorem 2 (C.): For $G \star \mathbb{Z}$, if G has no k-torsion $\forall k < n$, then any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \left(1 - \frac{1}{n}\right)\deg(S).$$

Theorem 2 (special case): For $G \star \mathbb{Z}$ with G arbitrary, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \frac{1}{2}\deg(S).$$

Proper powers

Theorem 2 (special case): For $G \star \mathbb{Z}$ with G arbitrary, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \frac{1}{2}\deg(S).$$

Theorem 3 (C.): $G \hookrightarrow (G \star \mathbb{Z})/\langle\langle w^k \rangle\rangle$ for any G and $k > 1$ if $p_{\mathbb{Z}}(w) = 1$. Klyachko–Lurye proved this in 2010

Conjecture: $A, B \hookrightarrow (A \star B)/\langle\langle w^k \rangle\rangle$ if $k > 1$ and $|w| \geq 2$.

- Known for $k \geq 4$ due to Howie.

Proper powers

Theorem 2 (special case): For $G \star \mathbb{Z}$ with G arbitrary, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \frac{1}{2}\deg(S).$$

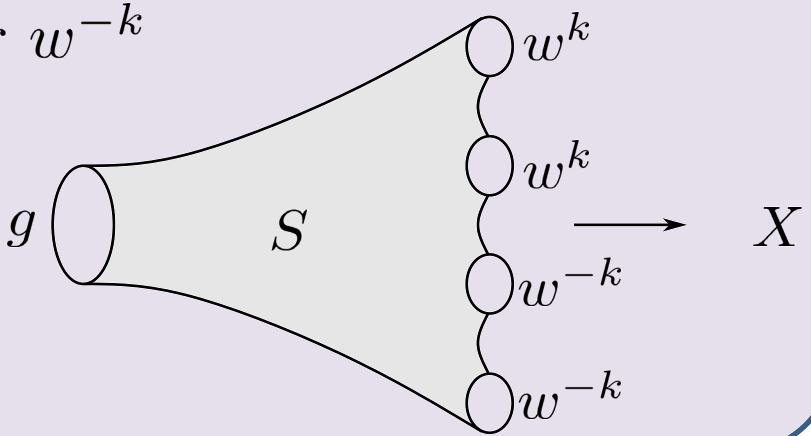
Theorem 3 (C.): $G \hookrightarrow (G \star \mathbb{Z})/\langle\!\langle w^k \rangle\!\rangle$ for any G and $k > 1$ if $p_{\mathbb{Z}}(w) = 1$.

Proof: Minimal counterexample as a w-admissible surface S

$n = \#$ components around w^k or w^{-k}

$$n - 1 = -\chi(S) \overset{\text{Thm2}}{\geq} \tfrac{1}{2}\deg(S) = \tfrac{1}{2}kn \geq n.$$

Since $k \geq 2$.



Proof idea

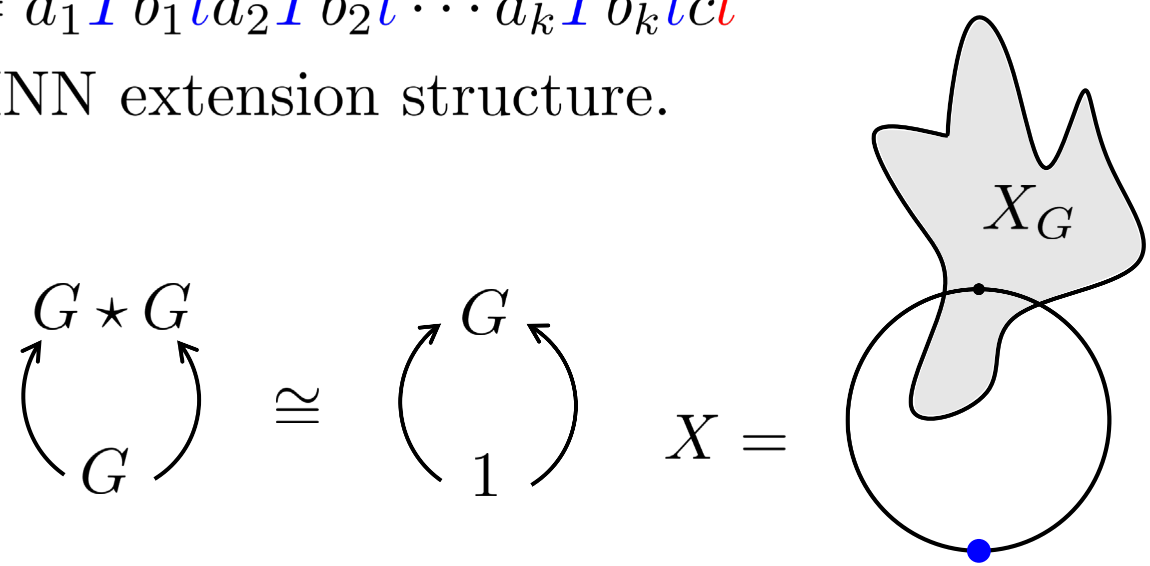
> **Theorem 1 (C.)**: For $G \star \mathbb{Z}$ with G torsion-free, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has
> $$-\chi(S) \geq \deg(S).$$

Outline of proof:

Step 1: Reduce to the case where w has a specific form

$$w = a_1 T b_1 t a_2 T b_2 t \cdots a_k T b_k t c t$$

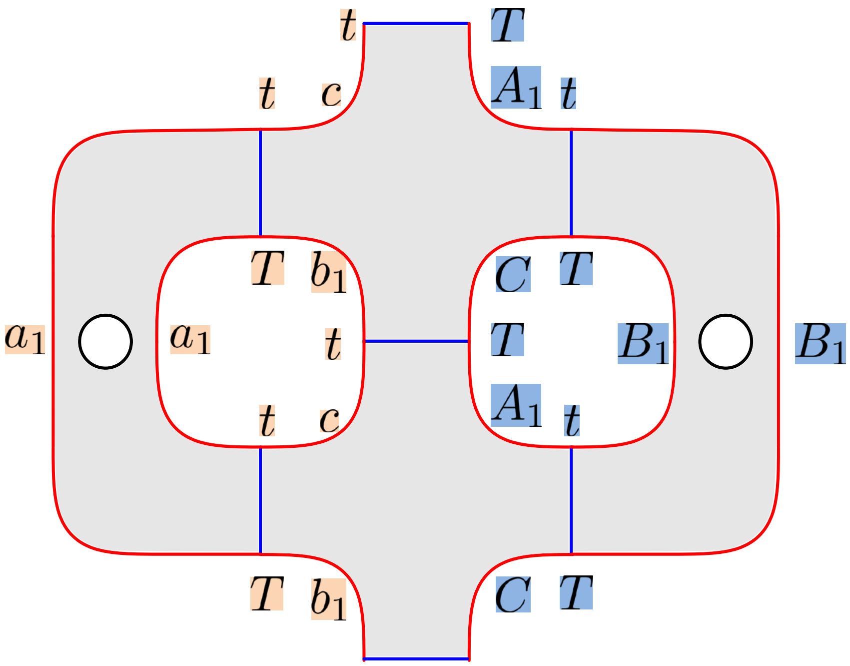
by changing the HNN extension structure.

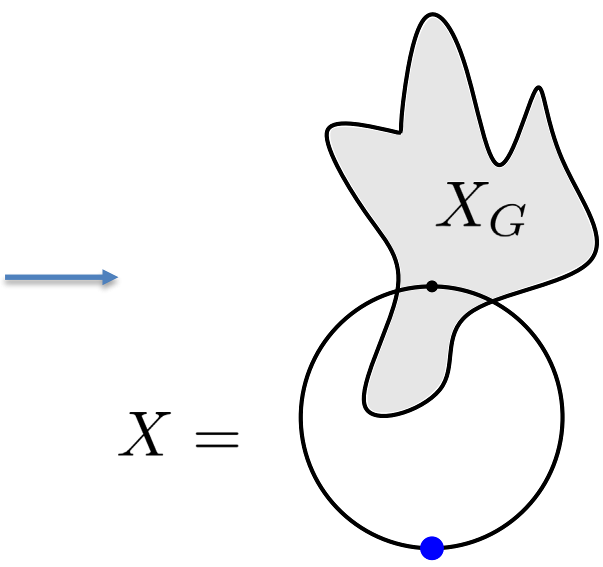


Proof idea: pieces of S

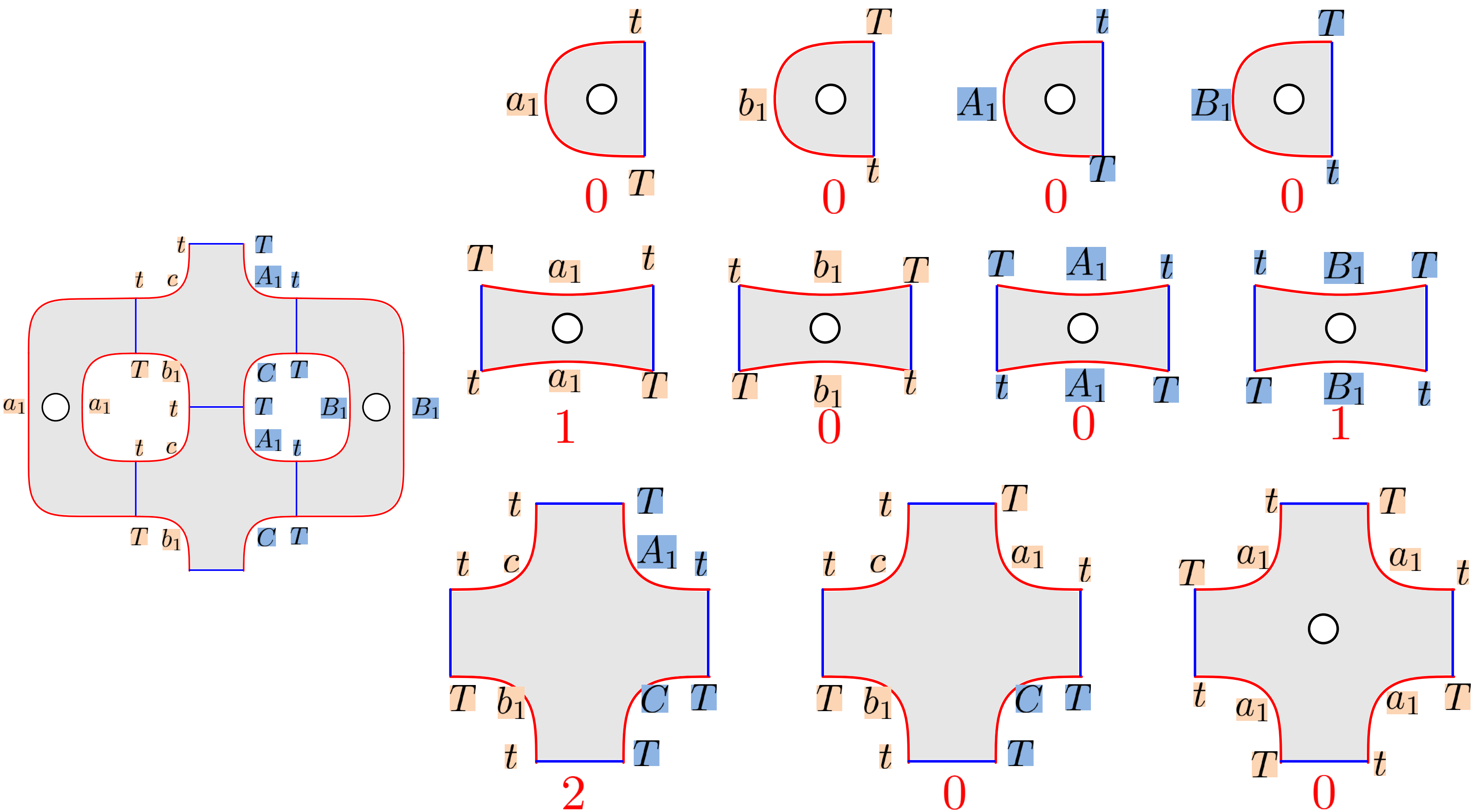
Step 2: Use the edge space to decompose S into pieces,

- Simplify so that each piece is a disk or annulus
- E.g. $w = a_1 T b_1 t c t$, $w^{-1} = T C T B_1 t A_1$





Proof idea: linear programming



Euler characteristic is linear

Proof idea: LP duality

Theorem 1 (C.): For $G \star \mathbb{Z}$ with G torsion-free, any irreducible w-admissible surface S with $p_{\mathbb{Z}}(w) = 1$ has

$$-\chi(S) \geq \deg(S).$$

Step 3: Estimate $-\chi(S)$ using linear programming duality

- Minimizing $-\chi(S)$ is a linear programming problem

$$\min_x \langle c, x \rangle$$

$$Ax \geq b, x \geq 0$$

- Use the dual problem to estimate

$$\max_y \langle b, y \rangle$$

$$A^T y \leq c, y \geq 0$$

$$\begin{aligned} \langle c, x \rangle &\geq \langle A^T y, x \rangle \\ &= \langle y, Ax \rangle \\ &\geq \langle y, b \rangle \end{aligned}$$

  ⋆ Any feasible dual solution gives a lower bound

- Miracle: Uniform dual solution only depending on the specific form

Thank you!