# Variations on Khovanov homology

Vincent Colin

joint with Ko Honda (UCLA) and Yin Tian (Beijing)

Université de Nantes – LMJL

June 17th, 2021

## Introduction

Goal is threefold:

- Set-up the generalized framework for higher-dimensional Heegaard Floer homology.
- Define versions of Khovanov homology for links in $S^3$ and for braids in fibrations over $S^1$.
- Provide combinatorial versions and interesting *surface algebras*.

# Higher-dimensional Heegaard Floer homology

The data:

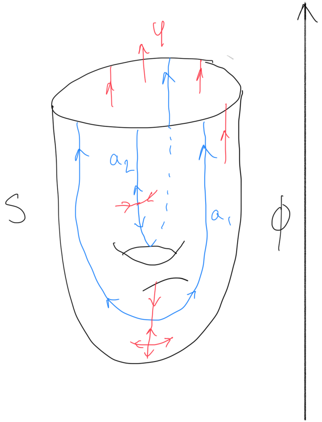
- $(W, \beta, \phi)$ a Weinstein domain of dimension $2n$;
- $h \in Symp(W, \partial W, d\beta)$.

This means that:

- $d\beta$ is symplectic;
- the *Liouville* vector field $Y$, s.t. $i_Y d\beta = \beta$, is transversally exiting $W$ along $\partial W$;
- $\phi$ is a Morse function on $W$, $\partial W$ regular level set and $\phi$ is gradient-like for $Y$.

In particular, $(W, d\beta)$ has a *basis* of Lagrangian $n$-disks $a_1, \dots, a_k$.

# Example of a surface $S = W$ $(n = 1)$.

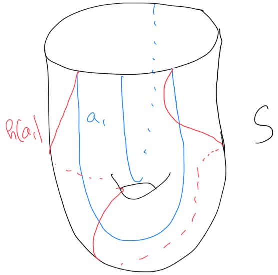


## Higher-dimensional Heegaard Floer homology

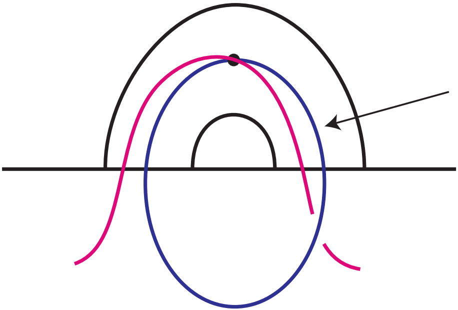
The chains:

$$\widehat{CF}(W, \beta, \phi; h) = \oplus_{\mathbf{y}} \Lambda[\![\hbar]\!] \mathbf{y}$$

$\mathbf{y} = \{y_1 \times [0,1], \ldots, y_k \times [0,1]\}$ with $y_i \in a_i \cap h(a_{\sigma(i)})$, $\sigma \in \mathcal{S}_k$.
$\Lambda$ is a Novikov ring with $\mathbb{Z}$-coefficients over some quotient $\mathcal{A}$ of a relative $H_2$ and $\Lambda[\![\hbar]\!]$ is the ring of formal power series in $\hbar$ over $\Lambda$.
The intersection $a_i \cap h(a_i)$ is perturbed at the boundary so that only two intersection points remain and we keep only the one of lower index $x_i$.



Also possible to glue $n$-handles to $W$ along $\partial a_i$ and extend $a_i$ to a Lagrangian sphere.

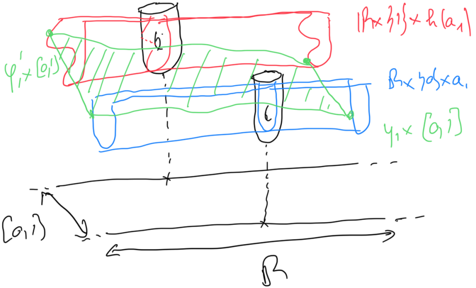


## The differential:

The differential $\widehat{\partial}$ is obtained by a count of pseudoholomorphic sections in $(\mathbb{R}_s \times [0,1]_t \times W, d(e^s(dt+\beta)), J) \to \mathbb{R} \times [0,1]$ with boundary on Lagrangians $\mathbb{R} \times \{1\} \times a_i$ and $\mathbb{R} \times \{0\} \times h(a_i)$ and asymptotic to some $k$-tuples of chords $\mathbf{y}$ and $\mathbf{y}'$ at $\pm\infty$.

The coefficient $\hbar$ registers the euler characteristic of the curve (needed to get compactness). Should also incorporate homological coefficients in $[\mathcal{A}]$ to control energy.

The homology is denoted $\widehat{HF}(W, \beta, \phi; h)$.



## Some results

**Theorem 1**

*$\widehat{HF}(W, \beta, \phi; h)$ is independant on the choice of $J$ and of $h$ up to isotopy in Symp$(W, \partial W, d\beta)$. The element $\mathbf{x} = \{x_1, \dots, x_k\}$ is a cycle whose homology class is called the contact class.*

In general $\widehat{HF}(W, \beta, \phi; h)$ depends on the choice of $\beta$, i.e. on the basis of Lagrangians.

**Theorem 2**

*If $(W, \beta, \phi; h)$ is an open book decomposition supporting a contact manifold $(M, \xi)$ and whose contact class vanishes, then $(M, \xi)$ is not Liouville fillable and satisfies the Weinstein conjecture.*
*Every overtwisted contact manifold is carried by an open book decomposition whose contact class is $0$.*

## Open book decomposition [Giroux]

To $(W^{2n}, \beta, \phi; h)$ can associate a contact $(2n+1)$-manifold $(M, \xi)$.

$$M = W \times [0,1]_t / (h(x), 0) \sim (x, 1) \textit{ and } (y, t) \sim (y, t')$$

for every $y \in \partial W$.
Roughly,

$$\xi = \ker(dt + \epsilon\beta)$$

on $W_\epsilon \times [0,1]/\sim$, where $W_\epsilon$ small retraction of $W$ and

$$\xi = \ker(\beta|_{\partial W} + r^2 d\theta)$$

in $N(K) = K \times D^2_{(r,\theta)}$, where $K = \partial W \times [0,1]/\sim$.

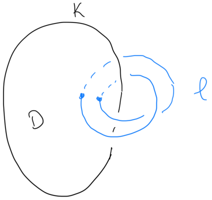
**Theorem (Giroux-Mohsen)**

*Every contact manifold is obtained in this way.*

## Variation on symplectic Khovanov homology

Based on the symplectic reformulation of Khovanov homology by Seidel-Smith with an input by Manolescu.
We formulate a new variation as a special 5-dimensional Heegaard Floer homology.

Given a link $\ell \subset S^3$, put it in braid position $\sigma$ with respect to the standard open book decomposition with unknotted binding $K$. It intersects a disk page in $k$ points.

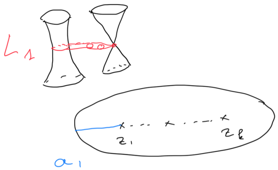


## The associated open book

Consider the 4-dimensional manifold $W_k = \{u^2 + v^2 = P(z)\} \subset \mathbb{C}^3$ where $P$ degree $k$ polynomial with distinct roots $z_1, \dots, z_k$ ($A_k$-singularity). It has a projection $(u, v, z) \mapsto z$ to $\mathbb{C}$ which is a Lefschetz fibration to the disk $D$ with $k$ singular values $z_1, \dots, z_k$. This gives a Weinstein domain $(W_k, \beta_k, \phi_k)$ with a Lagrangian basis obtained by lifts of $k$ arcs from $z_i$ to $\partial D$ (*Lefschetz thimbles*).

The braid $\sigma$ gives a monodromy map of $\mathit{Diff}(D, \{z_1, \dots, z_k\})$ that lifts to $h \in \mathit{Symp}(W_k, \beta_k)$.

Define $Kh^\sharp(W_k, \beta_k, \phi_k; h) := \widehat{HF}(W_k, \beta_k, \phi_k; h)$ with coefficients $\mathbb{Z}/2[\mathcal{A}][\![\hbar, \hbar^{-1}]\!]$.



# Invariance

**Theorem 3**

*$Kh^\sharp(W_k, \beta_k, \phi_k; h)$ is a link invariant, i.e. independent on the choice of arcs in D and of the choice of braid representative $\sigma$ up to positive and negative Markov stabilizations.*

Denote by **x** the contact class.

**Theorem 4**

*The contact class $\mathbf{x} =: \psi^\sharp(\sigma)$ is invariant of the braid up to positive Markov stabilization.*

Similar to the Plamenevskaia invariant of a link transverse to the standard contact plane.

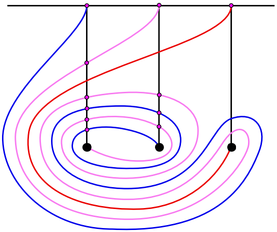
## Some computations

Consider the 3-braid $\sigma_{BP,p}$ (Baldwin-Plamenevskaia) with monodromy $\sigma_1^{-p} \circ \sigma_2 \circ \sigma_1^2 \circ \sigma_2$.

**Theorem 5**

$\psi^\sharp(\sigma_{BP,p}) = 0$, *hence for* $p \geq 3$, *the contact* 5*-manifolds given by the corresponding open book decompositions* $(A_p, \beta_p, \phi_p; \widehat{\sigma_{BP,p}})$ *are not Liouville fillable and satisfy the Weinstein conjecture.*

The braids $\sigma_{BP,p}$ are *right-veering* and the associated 5-contact manifolds might be tight.



## Relation with Khovanov homology

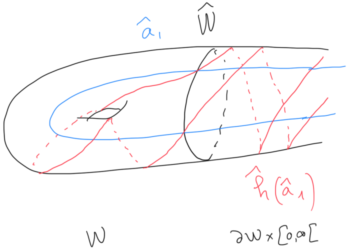
Consider projections $v_1, v_2$ of $\mathbb{R} \times [0,1] \times A_k$ to respectively $\mathbb{R} \times [0,1]$ and $D$.
By Manolescu, differential in symplectic Khovanov homology $Kh_{symp}$ counts, for good choice of $J$, holomorphic curves
$u : F \to \mathbb{R} \times [0,1] \times A_k$ such that the map $(v_1 \circ u, v_2 \circ u)$ is an embedding.
By positive intersection for holomorphic curves, the count of the degree of singularities of $(v_1 \circ u, v_2 \circ u)$ can be turned into a filtration.
Have a spectral sequence whose $E_1$-page is $Kh_{symp}$ and expected to converge to $Kh^\sharp$.

## The fully wrapped version of higher-dimensional HF

Back to $(W, \beta, \phi; h)$ as in higher-dimensional HF.
Extend $W$ to its completion

$$(\widehat{W}, \widehat{\beta}) = (W, \beta) \cup_{\partial W} ([0, \infty[_s \times \partial W, e^s \beta|_{\partial W}).$$

The Lagrangian basis $a_1, \ldots, a_n$ extended cylindrically by $\widehat{a}_1, \ldots, \widehat{a}_n$.
Extend $h$ to $\widehat{h}$ obtained by composing with the time-1 of a quadratic Hamiltonian in $e^s$.
This time we allow ourselves to take parallel copies of the $\widehat{a}_1, \ldots, \widehat{a}_n$ and maybe not all of them.



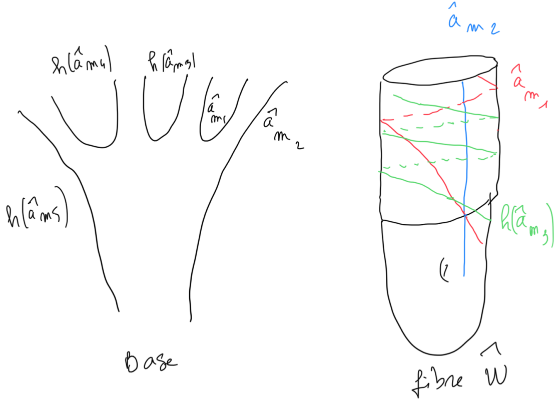
Let $k$ be an integer, $\mathbf{m} = (m_1, \ldots, m_\kappa)$ partition of $k$. We write $|\mathbf{m}| = \sum_{i=1}^{\kappa} m_i = k$
Algebras:

$$R^f(W, \beta, \phi; k) := \oplus_{|\mathbf{m}|=|\mathbf{m}'|=k} \operatorname{Hom}(\widehat{a}_{\mathbf{m}}, \widehat{a}_{\mathbf{m}'}),$$
$$R^f(W, \beta, \phi) := \oplus_{k \geq 0} R^f(W, \beta, \phi; k),$$

where $\operatorname{Hom}(\widehat{a}_{\mathbf{m}}, \widehat{a}_{\mathbf{m}'})$ is the Heegaard Floer homology group $CF(\widehat{Id}(\widehat{a}_{\mathbf{m}}), \widehat{a}_{\mathbf{m}'})$, i.e., we are fully wrapping the first term. Here $\widehat{a}_{\mathbf{m}}$ contains $m_i$ copies of $\widehat{a}_i$.
Similarly we define the $R^f(W, \beta, \phi; k)$- and $R^f(W, \beta, \phi)$-bimodules

$$B^f(W, \beta, \phi; h; k) := \oplus_{|\mathbf{m}|=|\mathbf{m}'|=k} \operatorname{Hom}(\widehat{h}(\widehat{a}_{\mathbf{m}}), \widehat{a}_{\mathbf{m}'}),$$
$$B^f(W, \beta, \phi; h) := \oplus_{k \geq 0} B^f(W, \beta, \phi; h; k).$$

Product structures are obtained by counting multisection of $D_m \times \widehat{W} \to D_m$ where $D_m$ is closed disk with $m+1$ punctures on boundary and with suitable Lagrangian boundary conditions and wrapping scheme.

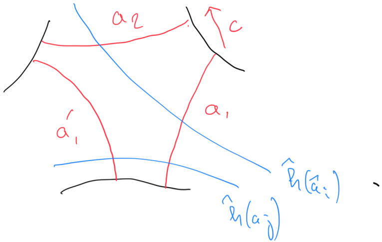


## Invariance

### Theorem 6

*The Hochschild homology of the $A_\infty$-bimodule $\mathcal{B}^f(W, \beta, \phi; h)$ is invariant under handleslides, i.e., Weinstein homotopies $(\beta_t, \phi_t)$, $t \in [0, 1]$.*

Handlesliding along a chord corresponds to taking the cone of the multiplication by the corresponding intersection point/chord (Seidel $+\epsilon$).



## Braid in a surface bundle over $S^1$

The datum:

- $\overline{S}$ closed surface, $\overline{h} \in Diff(\overline{S})$.
- $N_{(\overline{S},\overline{h})} = \overline{S} \times [0,1]/(\overline{h}(x),0) \sim (x,1)$ mapping torus.
- $\ell$ link in $N_{(\overline{S},\overline{h})}$ transverse to fibers $\overline{S} \times \{t\}$: a *braid*.

Let $\ell \cap \overline{S} \times \{0\} = \mathbf{z} = \{z_1, \dots, z_n\}$. Assume $\overline{h} \in Diff(\overline{S}, \mathbf{z} \cup D)$ where $D$ is an open disk in $\overline{S} \setminus \mathbf{z}$.

**Remark**

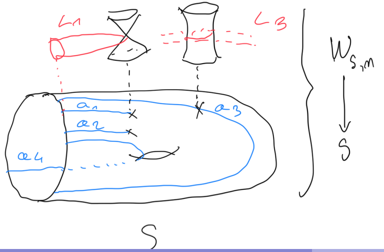
*Choice of D equivalent to a 1-strand braid K in the complement of $\ell$. Our invariant will depend on the isotopy class of K in $N_{(\overline{S},\overline{h})}$ but not on its relative position with $\ell$.*

Let $S := \overline{S} \setminus D$; $h \in Diff(S, \mathbf{z})$ the restriction of $\overline{h}$ to $S$.

## Consider:

- symplectic Lefschetz fibration $\pi : W_{S,n} \to S$ with base $S$ instead of $D^2$, regular fiber $A = S^1 \times [-1,1] \subset T^*S^1$, and $n$ critical values $z_1, \dots, z_n$.
- $\{a_1, \dots, a_n\}$ of pairwise disjoint half-arcs in $S$ joining the critical values of $\pi$ to $\partial S$ augmented by a basis $\{a_{n+1}, \dots, a_{n+s}\}$ of $s$ properly embedded pairwise disjoint arcs in $S$ with endpoints on $\partial S$.

Lift $\{a_1, \dots, a_n\}$ to Lagrangian disks and $\{a_{n+1}, \dots, a_{n+s}\}$ to Lagrangian cylinders $a_i \times 0_{T^*S^1}$ (also could choose disks $a_i \times \mathbb{R} \times \{\theta_0\}$). Get $L_1, \dots, L_{n+s}$.



Consider possibly parallel copies of arcs, but at most 1 copy of half-arcs, to form an $A_\infty$-category $\mathcal{R}^f(S, n, \mathbf{a})$, whose objects are $L_{\mathbf{m}}$ and whose morphisms are $CF^f(L_{\mathbf{m}}, L_{\mathbf{m}'})$.

$$R^f(S, n, \mathbf{a}) := \oplus_{k \geq 0, |\mathbf{m}|=|\mathbf{m}'|=k} CF^f(L_{\mathbf{m}}, L_{\mathbf{m}'}). \tag{1}$$

**Theorem 7**

*The Hochschild homology $HH^f(S, n, \mathbf{a}; h)$ is an invariant of $\ell$ as a transverse link in $N_{(\overline{S}, \overline{h})}$. It depends on the auxiliary data $K$ but not on the relative positions of $\ell$ and $K$ in $M$.*

**Remark**

*To upgrade from a transverse link invariant to a genuine link invariant (of a fibration), need to introduce cup- and cap-bimodules and show bimodule equivalences corresponding to standard moves.*

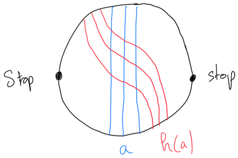
The independance from the choice of arcs and half-arcs comes from Theorem 6 (independance from the basis of Lagrangian in the wrapped higher-HF).
Handlesliding over half-arcs is invariant on the nose, as in the $Kh^\sharp$ case.
Handlesliding over arcs introduces cones so requires to take Hochschild homology and several copies.
Isotopying $\ell$ through $K$ corresponds to composing $h$ with a Dehn twist parallel to the boundary. Since we are doing the fully wrapped version, it does not change the result.

Note: We could also add stops along $\partial S$ and get invariants of braids in sutured manifolds.

Currently developping some combinatorial models for the (stopped or not) surface algebras that look interesting.

## Local model (basic version)

$S = D^2$, no singularity and 2 stops.




As a graded algebra, $R_k$ is isomorphic to $\mathbb{F}[S_k] \rtimes \Lambda_k$, where $\mathbb{F}[S_k]$ is the group algebra of the symmetric group $S_k$, $\Lambda_k$ is the exterior algebra of $k$ generators of degree 1 and the symmetric group $S_k$, whose elements are of degree 0, acts on $\Lambda_k$ by permuting the generators $\xi_i$.

**Proposition**

*The cohomology algebra $H(R_k)$ is isomorphic to $\Lambda_k$.*

We conjecture that $R_k$ is a formal dg algebra, i.e., it is quasi-isomorphic to $H(R_k)$.

The graded algebra $\mathbb{F}[S_k] \rtimes \Lambda_k$ is generated by $s_i, \xi_j$ for $1 \le i \le k-1, 1 \le j \le k$, and satisfies the following defining relations:

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_{i'} = s_{i'} s_i \text{ for } |i - i'| > 1; \qquad (2)$$
$$\xi_j^2 = 0, \quad \xi_j \xi_{j'} = -\xi_{j'} \xi_j, \text{ for } j \neq j';$$
$$\xi_i s_i = s_i \xi_{i+1}, \quad \xi_{i+1} s_i = s_i \xi_i, \quad \xi_j s_i = s_i \xi_j \text{ for } j \neq i, i+1.$$

It is a finite-dimensional algebra of rank $2^k k!$.
Define a dg algebra $R_k = (\mathbb{F}[S_k] \rtimes \Lambda_k, d)$ by introducing a differential $d$ on the generators as

$$d(\mathbf{1}_k) = 0, \qquad d(s_i) = \xi_i - \xi_{i+1}, \qquad d(\xi_j) = 0,$$

and extended by the Leibniz rule: $d(ab) = d(a)b + (-1)^{\deg(a)} a\, d(b)$.
The degree on the generators is given by $\deg(\xi_j) = 1$ and $\deg(s_i) = 0$.

# Diagrammatic presentation

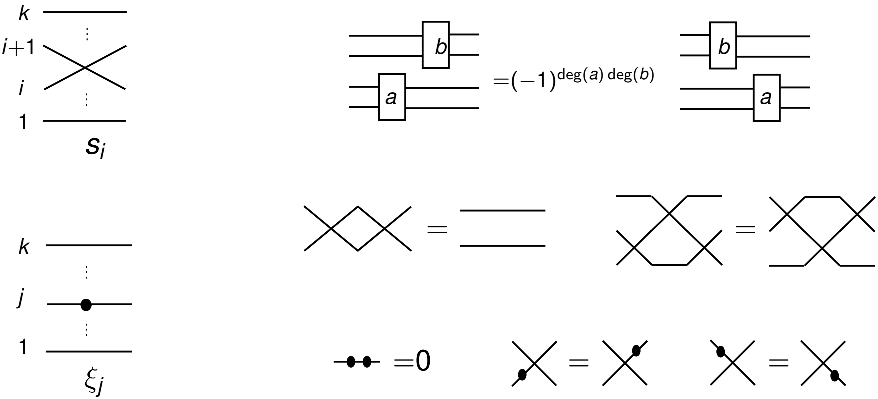



**Figure:** The generators $s_i, \xi_j$ of $R_k$ are on the left, and the relations are on the right.

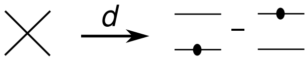



**Figure:** The differential of a crossing.

## Remark

*The dg algebra $R_k$ is the higher-dimensional analogue of the LOT dg algebra. They diagrammatically differ in the following ways:*

1. *the degree of a single crossing is zero in $R_k$, while it is minus one in the LOT algebra;*
2. *the double crossing is the identity in $R_k$, while it is zero in the LOT algebra;*
3. *the dot generators of $R_k$ do not exist in the LOT algebra;*
4. *the differential of a single crossing has two terms with dots, while it has one term in the LOT algebra.*