Knotted surfaces with infinite cyclic knot group.

Plan

I Closed surfaces in closed 4-manifolds

II Surfaces with boundary in 4-manifolds with boundary

III Proof sketches

(IV An application to rim surgery)

If time permits.

I Closed surfaces in closed 4-manifolds

Motivation: unknotting 2-knots

Convention:
Everything in Top.

Recall that a 2-knot is a locally flat embedding $K : S^2 \hookrightarrow S^4$

Definition: A 2-knot $S^2 \overset{K}{\hookrightarrow} S^4$ is unknotted if it bounds a locally flat embedded 3-ball $B^3 \hookrightarrow S^4$

Note that an unknotted $S^2 \overset{K}{\hookrightarrow} S^4$ has $\pi_1(S^4 \setminus K) = \mathbb{Z}$.

In fact, the converse is true:

The smooth version is open.

Theorem (Freedman Quinn 1990)
Let $K^2 \subset S^4$ be a 2-knot.
K is unknotted if and only if $\pi_1(S^4 \setminus K) = \mathbb{Z}$

Convention: surfaces are assumed to be compact, connected and oriented.

What about surfaces $\Sigma \subset S^4$ of higher genus?

Closed $\mathbb{Z}$-surfaces in S^4

Convention: everything in TOP.

Terminology: • A surface $\Sigma^2 \subset S^4$ is unknotted if it bounds a locally flat embedded handlebody of genus g.

• A $\mathbb{Z}$-surface $\Sigma \subset S^4$ is a surface with $\pi_1(S^4 \setminus \Sigma) = \mathbb{Z}$.

Note that unknotted surfaces are $\mathbb{Z}$-surfaces.

Question: Is the converse true? If $\pi_1(S^4 \setminus \Sigma) = \mathbb{Z}$, is $\Sigma \subset S^4$ unknotted?

Theorem (C. Powell 2020)
Any two closed genus $g \geq 3$ $\mathbb{Z}$-surfaces $\Sigma_0, \Sigma_1 \subset S^4$ are ambient isotopic.

What about surfaces in other 4-manifolds?

In particular a closed $\mathbb{Z}$-surface $\Sigma_g \subset S^4$ is unknotted if $g \neq 1, 2$.

History: Hillman-Kawauchi '1995 claimed the result $\forall g$; relied on 1994 work from Kawauchi.
Hambleton-Teichner '1997 found a gap in $\longrightarrow$
Kawauchi '2013 proposes a fix (?).

$\mathbb{Z}$-surfaces in simply-connected 4-manifolds

closed, connected, oriented

Let X be a closed simply-connected 4-manifold, and let $\Sigma \subset X$ be a $\mathbb{Z}$-surface

$\pi_1(X \setminus \nu\Sigma) = \mathbb{Z}$.

Some notation

- $X_\Sigma := X \setminus \nu\Sigma$, the exterior of Σ.
- $\widetilde{X}_\Sigma :=$ the $\mathbb{Z}$-cover of X_Σ. $=$ the universal cover of X_Σ.

Topological invariants? $\pi_1(X_\Sigma) = \mathbb{Z} \Rightarrow H_1(\widetilde{X}_\Sigma) = 0$, $\pi_2(X_\Sigma) = H_2(\widetilde{X}_\Sigma) \cong \mathbb{Z}[t^{\pm 1}]^{2g}$, $(H_2(X_\Sigma), Q_{X_\Sigma}) = (H_2(X), Q_X) \oplus (\mathbb{Z}^{2g}, (0)^{\oplus 2g})$

Definition:

The <u>equivariant intersection form</u> of X_Σ is the Hermitian, sesquilinear, non-degenerate form

$$\begin{aligned} \lambda_\Sigma : H_2(\widetilde{X}_\Sigma) \times H_2(\widetilde{X}_\Sigma) &\longrightarrow \mathbb{Z}[t^{\pm 1}] \\ (x, y) &\longmapsto \sum_{k \in \mathbb{Z}} (x \cdot t^k y)\, t^{-k}. \end{aligned}$$

Note that if $F : (X, \Sigma_0) \xrightarrow{\cong} (X, \Sigma_1)$ is orientation preserving, then $\lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$ (lift F to $\mathbb{Z}$-covers).

"isometric" $F : H_2(\widetilde{X}_{\Sigma_0}) \xrightarrow{\cong} H_2(\widetilde{X}_{\Sigma_1})$; $\lambda_{\Sigma_1}(F(x), F(y)) = \lambda_{\Sigma_0}(x, y)$.

$\mathbb{Z}$-surfaces in simply-connected 4-manifolds

closed, connected, oriented

Let X be a closed simply-connected 4-manifold, and let $\Sigma \subset X$ be a $\mathbb{Z}$-surface

We saw that the equivariant intersection form of X_Σ is

$$\begin{aligned} \lambda_\Sigma : H_2(\widetilde{X}_\Sigma) \times H_2(\widetilde{X}_\Sigma) &\longrightarrow \mathbb{Z}[t^{\pm 1}] \\ (x,y) &\longmapsto \sum_{k \in \mathbb{Z}} (x \cdot t^k y) t^{-k}. \end{aligned}$$

We also saw that $(X, \Sigma_0) \xrightarrow{\cong} (X, \Sigma_1)$ o.p. $\Rightarrow \lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$.

Theorem (C-Powell 2020)

Let $\Sigma_0, \Sigma_1 \subset X$ be two closed genus g $\mathbb{Z}$-surfaces. The following assertions are equivalent:

- Σ_0, Σ_1 are equivalent: $\exists$ an orientation-preserving homeo $(X, \Sigma_0) \xrightarrow{\cong} (X, \Sigma_1)$.
- $\lambda_{\Sigma_0}, \lambda_{\Sigma_1}$ are isometric.

An additional technical condition on the isometry provides a criterion for ambient isotopy.

II Surfaces with boundary in 4-manifolds with boundary.

$\mathbb{Z}$-surfaces in S^4 and $\mathbb{D}^4$

A

Theorem (C.-Powell 2020)
Any two closed genus $g \neq 1,2$ $\mathbb{Z}$-surfaces $\Sigma_0, \Sigma_1 \subset S^4$ are ambient isotopic.

← $g=0$: Freedman-Quinn

This will follow from the following result:

$g=0$ C-Powell 2019.

B

Theorem (C-Powell 2020)
Let $K \subset S^3$ be an Alexander polynomial one knot.
Any two genus $g \neq 1,2$ $\mathbb{Z}$-surfaces $\Sigma_0, \Sigma_1 \subset \mathbb{D}^4$ with $\partial \Sigma_i = K$ are ambient isotopic rel boundary.

proof of A assuming B:

. After an isotopy, assume that Σ_0, Σ_1 agree on a $B^2 \subset \Sigma_0 \cap \Sigma_1$

. Remove $(\mathring{B}^4, \mathring{B}^2)$ from $(S^4, \Sigma_0 \cap \Sigma_1)$ to get $\mathring{\Sigma}_0, \mathring{\Sigma}_1 \subset \mathbb{D}^4$ with common boundary the unknot.

. Apply Theorem B to deduce that $\mathring{\Sigma}_0, \mathring{\Sigma}_1$ are isotopic rel boundary.

- Glue the (B^4, B^2) back in to get Theorem A.

$\mathbb{Z}$-surfaces in simply-connected 4-manifolds with boundary

Let X be a closed simply-connected 4-manifold,

$N := X \setminus \mathring{D}^4$ be a punctured X. Note that $\partial N = S^3$.

$\Sigma \subset N$ a $\mathbb{Z}$-surface with boundary $K \subset S^3 = \partial N$.

(always frame $\Sigma \subset X$ so that $H_1(\Sigma) \to H_1(\partial X_\Sigma) \to H_1(X_\Sigma)$ is zero map).

Some notation:

$$\partial N_\Sigma \cong (S^3 \setminus \nu K) \underset{\partial}{\cup} (\Sigma_{g,1} \times S^1)$$

$$\begin{tikzcd} & \partial N_\Sigma \\ \mathbb{Z} \arrow[r, phantom, "\curvearrowright"] & \partial \widetilde{N}_\Sigma \arrow[u] \end{tikzcd}$$

- $N_\Sigma := N \setminus \nu\Sigma$, the exterior of Σ.
- $\widetilde{N}_\Sigma :=$ the $\mathbb{Z}$-cover of N_Σ.

Again, for two such surfaces to be isotopic, their equivariant intersection forms must be isometric.

The equivariant intersection form of N_Σ is a sesquilinear, Hermitian, non-degenerate form

$$\lambda_\Sigma : H_2(\widetilde{N}_\Sigma) \times H_2(\widehat{N}_\Sigma) \longrightarrow \mathbb{Z}[t^{\pm 1}] \qquad (x,y) \longmapsto \sum_{k \in \mathbb{Z}} (x \cdot t^k y)\, t^{-k}.$$

The Blanchfield form of ∂N_Σ is a sesquilinear, Hermitian, non-singular form

$$Bl_{\partial N_\Sigma} : \underset{\nwarrow\ \mathbb{Z}[t^{\pm 1}]\text{-torsion}}{H_1(\partial \widetilde{N}_\Sigma)} \times H_1(\partial \widetilde{N}_\Sigma) \longrightarrow \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \qquad ([a],[b]) \longmapsto \frac{1}{p} \sum_{k \in \mathbb{Z}} (F \cdot t^k b)\, t^{-k}.$$

$p \in \mathbb{Z}[t^{\pm 1}].$ $p[a] = 0$

$pa = \partial F^2$

$\mathbb{Z}$-surfaces in simply-connected 4-manifolds with boundary

$\Sigma \subset N$ a $\mathbb{Z}$-surface with boundary $K \subset S^3$.

The *equivariant intesection form* of N_Σ is

$$\lambda_\Sigma : H_2(\widetilde{N}_\Sigma) \times H_2(\widetilde{N}_\Sigma) \longrightarrow \mathbb{Z}[t^{\pm 1}]$$

The *Blanchfield form* of ∂N_Σ is

$$\begin{aligned} Bl_{\partial N_\Sigma} : H_1(\partial \widetilde{N}_\Sigma) &\times H_1(\partial \widetilde{N}_\Sigma) \longrightarrow \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ \| \\ H_1(\widetilde{X}_K) &\oplus H_1(\widetilde{\Sigma_{g,1} \times S^1}) \end{aligned}$$

$$\begin{tikzcd} \partial N_\Sigma \cong \overbrace{\left(S^3 \setminus \nu K\right)}^{:= X_K} \cup_\partial \left(\Sigma_{g,1} \times S^1\right) \\ \partial \widetilde{N}_\Sigma \cong \widetilde{X}_K \cup_\partial \widetilde{\Sigma_{g,1} \times S^1} \arrow[u] \\ \circlearrowleft \quad\quad \circlearrowleft \quad\quad\quad \circlearrowleft \\ \mathbb{Z} \quad\quad\; \mathbb{Z} \quad\quad\quad \mathbb{Z}\,. \end{tikzcd}$$

Facts:

An isometry $F : \left(H_2(\widetilde{N}_{\Sigma_0}), \lambda_{\Sigma_0}\right) \xrightarrow{\cong} \left(H_2(\widetilde{N}_{\Sigma_1}), \lambda_{\Sigma_1}\right)$ induces an isometry

$$\partial F : \left(H_1(\partial \widetilde{N}_{\Sigma_0}), Bl_{\partial N_{\Sigma_0}}\right) \xrightarrow{\cong} \left(H_1(\partial \widetilde{N}_{\Sigma_1}), Bl_{\partial N_{\Sigma_1}}\right).$$

This decomposes as $\partial F = (h_K, h_\Sigma)$

$$h_K : \left(H_1(\widetilde{X}_K), Bl_K\right) \xrightarrow{\cong} \left(H_1(\widetilde{X}_K), Bl_K\right).$$

$$h_\Sigma : \left(H_1(\widetilde{\Sigma_{g,1} \times S^1}), Bl_\Sigma\right) \xrightarrow{\cong} \left(H_1(\widetilde{\Sigma_{g,1} \times S^1}), Bl_\Sigma\right).$$

$\mathbb{Z}$-surfaces in simply-connected 4-manifolds with boundary

Let X be a closed simply-connected 4-manifold,

$N := X \setminus \mathring{\mathbb{D}}^4$ be a punctured X. Note that $\partial N = S^3$.

$\Sigma \subset N$ a $\mathbb{Z}$-surface with boundary $K \subset S^3$.

An isometry $F : (H_2(\widetilde{N}_{\Sigma_0}), \lambda_{\Sigma_0}) \xrightarrow{\cong} (H_2(\widetilde{N}_{\Sigma_1}), \lambda_{\Sigma_1})$ induces an isometry

$$\partial F : (H_1(\partial \widetilde{N}_{\Sigma_0}), Bl_{\partial N_{\Sigma_0}}) \xrightarrow{\cong} (H_1(\partial \widetilde{N}_{\Sigma_1}), Bl_{\partial N_{\Sigma_1}})$$

This decomposes as $\partial F = (h_K, h_\Sigma)$, where

$$h_K : (H_1(\widetilde{X}_K), Bl_K) \xrightarrow{\cong} (H_1(\widetilde{X}_K), Bl_K).$$

<u>Theorem</u> : (C.-Powell '2020)

Let $\Sigma_0, \Sigma_1 \subset N$ be $\mathbb{Z}$-surfaces for a knot K and let $F : \lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$ be an isometry.

Set $\partial F = (h_K, h_\Sigma)$. Assume that

$h_K : Bl_K \cong Bl_K$ is realized by an orientation-preserving homeomorphism $h : X_K \xrightarrow{\cong} X_K$ with $h|_\partial = id$

Then there is an orientation-preserving homeomorphism $(N, \Sigma_0) \xrightarrow{\cong} (N, \Sigma_1)$ that induces F.

If additionally $N = \mathbb{D}^4$, then we get an ambient isotopy.

$\mathbb{Z}$-surfaces in simply-connected 4-manifolds with boundary

Theorem : (C.-Powell '2020)
Let $\Sigma_0, \Sigma_1 \subset N$ be $\mathbb{Z}$-surfaces for a knot K and let $F : \lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$ be an isometry.
Set $\partial F = (h_K, h_\Sigma)$. Assume that
(*) $h_K : Bl_K \cong Bl_K$ is realized by an orientation-preserving homeomorphism $h : X_K \xrightarrow{\cong} X_K$ with $h|_\partial = id$
Then there is an orientation-preserving homeomorphism $\Phi (N, \Sigma_0) \xrightarrow{\cong} (N, \Sigma_1)$ that induces F.
If additionally $N = D^4$, then we get an ambient isotopy.

Remarks :

- Take away : Here is a recipe to show that $\mathbb{Z}$-surfaces Σ_0, Σ_1 for K are equivalent/isotopic
  - a) Show that $\lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$.
  - b) Study the isometries of Bl_K
- If $\Delta_K = 1$, then $H_1(\tilde{X}_K) = 0$ and so one can take $h = id$ in (*). ($\Rightarrow$ closed case, next slide).

NB: In this case $\Phi|_{\partial N} = id$.

$\mathbb{Z}$-surfaces in simply-connected 4-manifolds with boundary

<u>Theorem</u> : (C.-Powell '2020)
Let $\Sigma_0, \Sigma_1 \subset N$ be $\mathbb{Z}$-surfaces for a knot K and let $F : \lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$ be an isometry.
Set $\partial F = (h_K, h_\Sigma)$. Assume that
(*) $h_K : Bl_K \cong Bl_K$ is realized by an orientation-preserving homeomorphism $h : X_K \xrightarrow{\cong} X_K$ with $h|_\partial = \mathrm{id}$
Then there is an orientation-preserving homeomorphism $(N, \Sigma_0) \xrightarrow{\cong} (N, \Sigma_1)$ that induces F.
If additionally $N = D^4$, then we get an ambient isotopy.

- If $\Delta_K = 1$, then $H_1(\widetilde{X}_K) = 0$ and so one can take $h = \mathrm{id}$ in (*). ($\Rightarrow$ closed case; next slide).

B

<u>Theorem</u> (C.-Powell 2020)
Let $K \subset S^3$ be an Alexander polynomial one knot.
Any two genus $g \neq 1,2$ $\mathbb{Z}$-surfaces $\Sigma_0, \Sigma_1 \subset D^4$ with $\partial \Sigma_i = K$ are ambient isotopic rel boundary.

<u>proof sketch</u> :
- Show that $\lambda_{\Sigma_0} \cong \lambda_{\Sigma_1} \cong \begin{pmatrix} 0 & t-1 \\ t^{-1}-1 & 0 \end{pmatrix}^{\oplus g}$ for $g \geq 3$.
- Apply the theorem (because $\Delta_K = 1$)

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$\mathbb{Z}$-surfaces in simply-connected 4-manifolds

Let X be a closed simply-connected 4-manifold,

Theorem (C-Powell 2020)

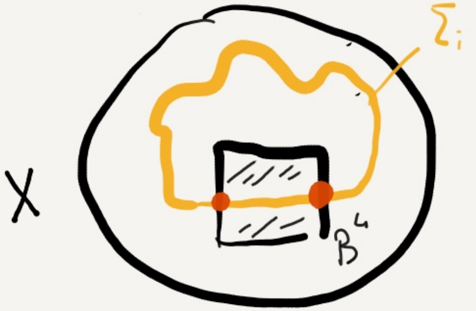
Let $\Sigma_0, \Sigma_1 \subset X$ be two closed genus g $\mathbb{Z}$-surfaces. The following assertions are equivalent:

1) Σ_0, Σ_1 are equivalent: $\exists$ an orientation-preserving homeo $(X, \Sigma_0) \xrightarrow{\cong} (X, \Sigma_1)$.

2) $\lambda_{\Sigma_0}, \lambda_{\Sigma_1}$ are isometric.

proof (assuming $\partial \neq \emptyset$ case, proof of 2) $\Rightarrow$ 1))

- After an isotopy, assume that Σ_0, Σ_1 agree on a $B^2 \subset \Sigma_0 \cap \Sigma_1$
- Remove $(\mathring{B}^4, \mathring{B}^2)$ from $(X, \Sigma_0 \cap \Sigma_1)$ to get $\mathring{\Sigma}_0, \mathring{\Sigma}_1 \subset N$ with common boundary the unknot.
- Observe that $N_{\Sigma_i} \cong X_{\Sigma_i}$ and F gives an isometry $\lambda_{\mathring{\Sigma}_0} \cong \lambda_{\mathring{\Sigma}_1}$.
- Apply Theorem $\partial \neq \emptyset$ to deduce that $\Phi : (N, \mathring{\Sigma}_0) \xrightarrow{\cong} (N, \mathring{\Sigma}_1)$. $\Phi|_{\partial N} = id$.
- Glue the (B^4, B^2) back in to get the result.



III A proof sketch and a classification result.

A question about 4-manifolds

Lemma :

If an orientation preserving homeomorphism

$$\overbrace{X_K \cup_\partial (\Sigma_{g,1} \times S^1)}^{=\partial N_{\Sigma_0}} \xrightarrow{h \cup (j \times id_{S^1})} \overbrace{X_K \cup_\partial (\Sigma_{g,1} \times S^1)}^{=\partial N_{\Sigma_1}}$$

extends to an orientation-preserving homeomorphism

$$\Phi : N_{\Sigma_0} \xrightarrow{\cong} N_{\Sigma_1}$$

Then there is an orientation preserving homeomorphism $(N, \Sigma_0) \xrightarrow{\cong} (N, \Sigma_1)$

Take away: (simplified)

Want to show that

$$N_{\Sigma_0} \cong N_{\Sigma_1}$$

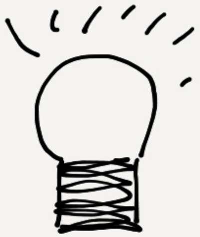
4-manifolds M_0, M_1 with

- $\pi_1(M_i) = \mathbb{Z}$,
- $\pi_1(\partial M_i) \twoheadrightarrow \pi_1(M_i)$,
- $H_1(\partial \widetilde{M}_i)$ is $\mathbb{Z}[t^{\pm 1}]$-torsion.

proof :

Take $\overbrace{N_{\Sigma_0} \cup_\partial (\Sigma_{g,1} \times D^2)}^{=N} \xrightarrow{\Phi \cup (j \times id_{D^2})} \overbrace{N_{\Sigma_1} \cup_\partial (\Sigma_{g,1} \times D^2)}^{=N}$.

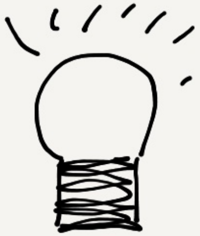
□



We must therefore understand when a homeo $\partial N_{\Sigma_0} \xrightarrow{\cong} \partial N_{\Sigma_1}$ extends to a homeo $N_{\Sigma_0} \xrightarrow{\cong} N_{\Sigma_1}$.

$\partial M_0 \xrightarrow{\cong} \partial M_1$

$M_0 \xrightarrow{\cong} M_1$



spin for simplicity.

A classification result for 4-manifolds with $\pi_1 = \mathbb{Z}$.

Theorem : (C-Powell 2020).

Let M_0, M_1 be spin 4-manifolds with $\begin{cases} \bullet\ \pi_1(M_i) = \mathbb{Z} \\ \bullet\ \pi_1(\partial M_i) \twoheadrightarrow \pi_1(M_i) \\ \bullet\ H_1(\partial \widetilde{M}_i)\ \ \mathbb{Z}[t^{\pm 1}]\text{-torsion} \end{cases}$

$f : \partial M_0 \xrightarrow{\cong} \partial M_1$ be an orientation preserving homeomorphism with

$$\begin{tikzcd} \pi_1(\partial M_0) \arrow[rr, "f_*"] \arrow[dr] & \cdot & \pi_1(\partial M_1) \arrow[dl] \\ & \mathbb{Z} & \end{tikzcd}$$

$F : \lambda_{M_0} \cong \lambda_{M_1}$ be an isometry.

The following assertions are equivalent :

1) $\partial F = \tilde{f}_* : H_1(\partial \widetilde{M}_0) \to H_1(\partial \widetilde{M}_1)$.

2) The homeomorphism f extends to an orientation-preserving homeo

$$\Phi : M_0 \xrightarrow{\cong} M_1$$

that induces F.

- <u>proof sketch</u> (1) ⇒ 2)).
- Use 1) to show that $\lambda_{M_0 \cup -M_1}$ is hyperbolic $\cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus b}$ and that $M_0 \cup -M_1$ is spin (with $\pi_1 = \mathbb{Z}$). $=$ W
- Closed classification ⇒ $M_0 \cup -M_1 \cong S^1 \times S^3 \# \#_b S^2 \times S^2$. $= \partial(\underbrace{S^1 \times D^4 \natural \natural_b S^2 \times D^3}_{W})$.
- Show that W is a relative s-cob. Apply the 5d s-cob theorem ($\mathbb{Z}$ is good).

$\mathbb{Z}$-surfaces in simply-connected 4-manifolds with boundary

Theorem : (C.-Powell '2020)
Let $\Sigma_0, \Sigma_1 \subset N$ be $\mathbb{Z}$-surfaces for a knot K and let $F : \lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$ be an isometry.
Set $\partial F = (h_K, h_\Sigma)$. Assume that
(*) $h_K : Bl_K \cong Bl_K$ is realized by an orientation-preserving homeomorphism $h : X_K \xrightarrow{\cong} X_K$ with $h|_\partial = id$
Then there is an orientation-preserving homeomorphism $(N,\Sigma_0) \xrightarrow{\cong} (N,\Sigma_1)$ that induces F.

· proof sketch

- Find f with $\partial F = \tilde{f}_*$ where $f = h \cup (j \times id_{S^1}) : \overbrace{X_K \cup (\Sigma_{g,1} \times S^1)}^{\partial N_{\Sigma_0}} \xrightarrow{\cong} \overbrace{X_K \cup (\Sigma_{g,1} \times S^1)}^{\partial N_{\Sigma_1}}$.
- Classification theorem $\Rightarrow$ f extends to a homeo $\Phi : N_{\Sigma_0} \xrightarrow{\cong} N_{\Sigma_1}$
- Lemma $\Rightarrow$ $(N,\Sigma_0) \xrightarrow{\cong} (N,\Sigma_1)$.

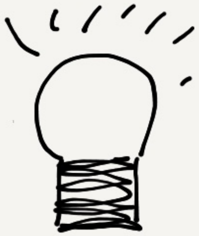
IV An application to rim surgery

Assume X is smooth
Σ smoothly embedded

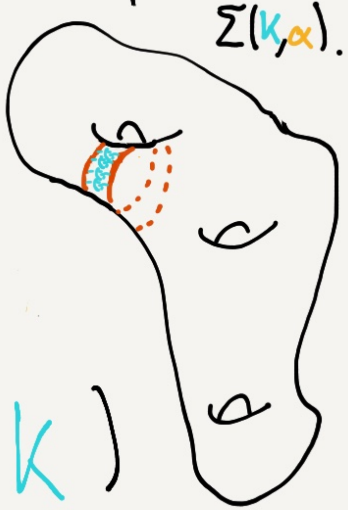
Rim Surgery.

Given a (closed) surface $\Sigma \subset X$ and a knotted arc $K \subset B^3$, rim surgery outputs a new surface $\Sigma_K \subset X$.

This construction, due to Fintushel-Stern '1997, may change the diffeomorphism type of $\Sigma \hookrightarrow X$



Rim surgery is used to produce exotic surfaces
$$(X,\Sigma) \cong_{Top} (X,\Sigma') \text{ but } (X,\Sigma) \not\cong_{DIFF} (X,\Sigma').$$

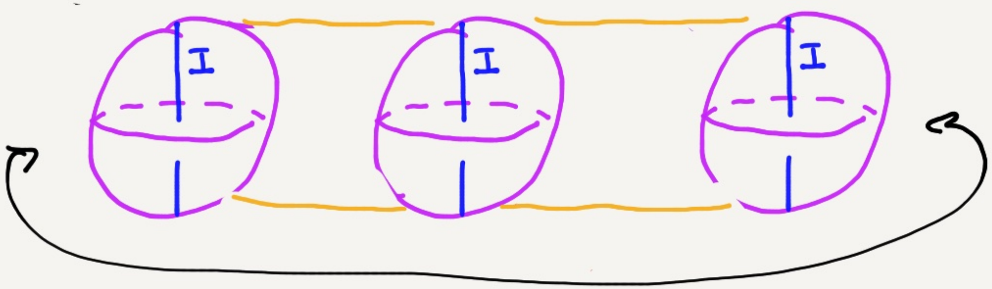


The construction:

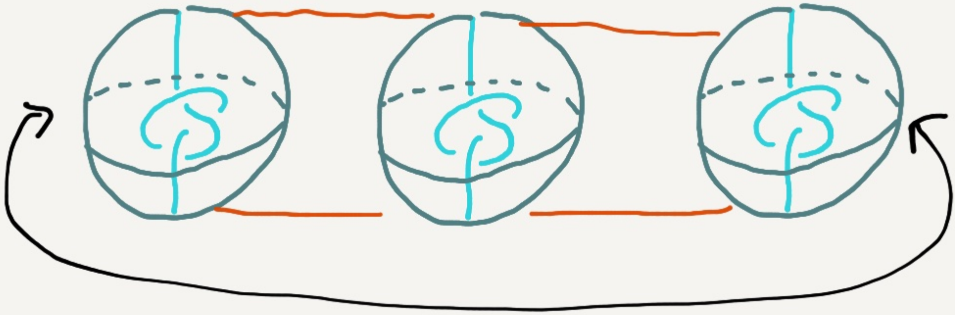
$$(X, \Sigma(K,\alpha)) = (X,\Sigma) \setminus \alpha \times (B^3, I) \cup_{\partial} S^1 \times (B^3, K)$$

$$= (X, \Sigma \setminus (\alpha \times I) \cup (S^1 \times K)).$$

$\alpha \times (B^3, I)$



$S^1 \times (B^3, K)$

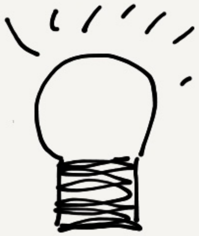


Assume X is smooth
Σ smoothly embedded

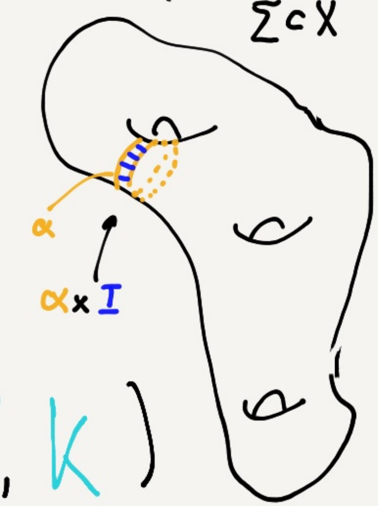
m-twist Rim Surgery.

Given a (closed) surface $\Sigma \subset X$ and a knotted arc $K \subset B^3$, m-twist rim surgery outputs a new surface $\Sigma_K \subset X$.

This construction, due to H.J. Kim' 2006, may change the diffeomorphism type of $\Sigma \hookrightarrow X$



m-twist Rim surgery is used to produce exotic surfaces
$(X,\Sigma) \cong_{Top} (X,\Sigma')$ but $(X,\Sigma) \not\cong_{DIFF} (X,\Sigma')$.

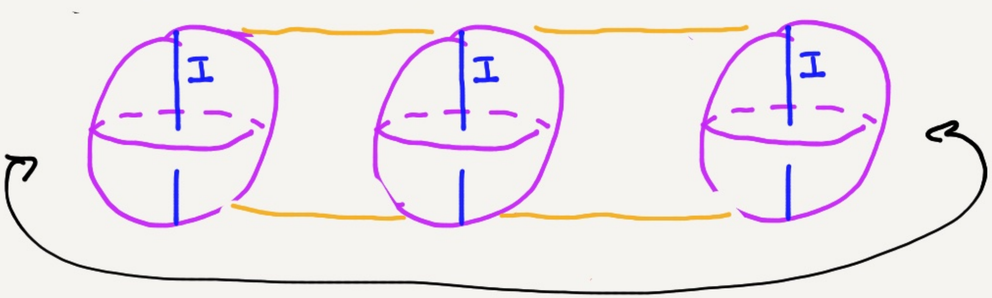


The construction:

$$(X, \Sigma_m(K,\alpha)) = (X,\Sigma) \setminus \alpha \times (B^3, I) \cup_{\partial} S^1 \times_{\tau^m} (B^3, K)$$

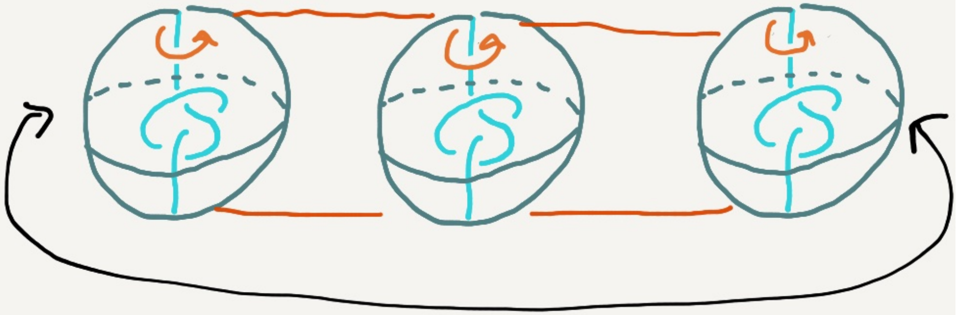
$$= (X, \Sigma \setminus (\alpha \times I) \cup_{\tau^m} (S^1 \times K)).$$

$\alpha \times (B^3, I)$



rotate m times

$S^1 \times (B^3, k)$



When is $(X,\Sigma) \cong (X, \Sigma_m(K,\alpha))$?

"1-twist doesn't change π_1"

Proposition (Kim-Ruberman)

If $H_1(X\setminus\Sigma)=\mathbb{Z}_d$ and $(d,m)=1$

Then $\pi_1(X\setminus\Sigma_m(K,\alpha))=\pi_1(X\setminus\Sigma)$.

Assume that X is simply connected.

$\partial=\emptyset$.

1997 : Fintushel-Stern : when $\pi_1(X\setminus\Sigma)=1$ $m=0$.

2008 Kim-Ruberman : when $\pi_1(X\setminus\Sigma)=\mathbb{Z}_d$, $(d,\overset{\neq 0}{m})=1$

2008 Kim-Ruberman : when $\pi_1(X\setminus\Sigma)$ good, $H_1(X\setminus\Sigma)=\mathbb{Z}_d$
$(d,m)=1$, $H_1(\Sigma_d(K))=0$ and K ribbon.

In particular for $\pi_1(X\setminus\Sigma)=\mathbb{Z}$ and K slice. $\Rightarrow$

- $m=1$, $\pi_1(X\setminus\Sigma)$ good, K slice, $\alpha\subset\Sigma$ has a null homotopic push-off in $X\setminus\Sigma$.

$\partial\neq\emptyset$. 2020 Juhasz-Miller-Zemke : when α bounds a locally flat disc in $X\setminus\Sigma$.

Theorem (C.-Powell 2020)

Let X be a simply-connected 4-manifold and set $N:=X\setminus\mathring{B}^4$.

- If $\Sigma\subset X$ is a closed $\mathbb{Z}$-surface, then Σ and $\Sigma_1(K,\alpha)$ are ambient isotopic.
- If $\Sigma\subset N$ is a properly embedded $\mathbb{Z}$-surface, then $(N,\Sigma)\cong(N,\Sigma_1(K,\alpha))$

For $N=D^4$, we get an ambient isotopy rel ∂