The simplicity conjecture

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Section 1

Introduction

An old theorem of Fathi

$Homeo_c(D^n, \omega)$: group of volume-preserving homeomorphisms of the n-disc, identity near the boundary.

Theorem (Fathi, 70s)

$Homeo_c(D^n, \omega)$ is simple when $n \geq 3$.

(Definition of simple: no non-trivial proper normal subgroups. Simple $\implies$ no quotient groups.)

Note: $Homeo_c(D^n, \omega) \triangleleft Homeo(D^n, \omega)$.

Today's theorem

Theorem ("Simplicity conjecture"; CG., Humiliere, Seyfadinni)

$Homeo_c(D^2, \omega)$ is not simple.

Why doesn't Fathi's proof work in dim 2?

Le Roux (2009):

- Fathi's proof uses a "fragmentation" property: For any $\varphi \in Homeo_c(D^n, \omega), n \geq 3$, can find φ_1, φ_2 such that
  1. $\varphi = \varphi_1 \varphi_2$.
  2. φ_1, φ_2 are supported in discs of volume $= 3/4$.
- Formulates fragmentation properties FP_ρ for $\rho \in (0,1)$. He proves

Simplicity Conjecture $\Leftrightarrow$ FP_ρ **fails for every** ρ.

Some history; comparisons

(Assumptions: M connected. All maps compactly supported and in the connected component of the identity.)

- Ulam ("Scottish book", 1930s): Is $Homeo_0(S^n)$ simple?
- 30s-60s: $Homeo_0(M)$ simple (Ulam, von Neumann, Anderson, Fisher, Chernavski, Edwards-Kirby)
- Smale (late 60s?): What about $Diff_0^\infty(M)$?
- 70s: $Diff_0^\infty(M)$ simple (Epstein, Herman, Mather, Thurston)

Other cases:

- Volume preserving diffeos: there is a “flux” homomorphism, kernel is simple for $n \geq 3$. (Thurston)
- Symplectic case: there is still the flux homomorphism
  - kernel of flux simple when manifold closed (Banyaga)
  - if not closed, there’s a Calabi homomorphism, kernel of Calabi simple (Banyaga)
- Volume preserving homeomorphisms: there is a “mass flow” homomorphism; kernel is simple for $n \geq 3$ (Fathi).

Our case — comparison

In comparison, our case seems more wild!

- Not simple,
- but (as far as we know) no obvious natural homomorphism out of $Homeo_c(D^2, \omega)$ either
- "Lots of" normal subgroups (Le Roux) ("radically different" from diffeomorphism group)

Section 2

Idea of the proof

The Calabi invariant

Fact: $\mathit{Diffeo}_c(D^2, \omega)$ is not simple. Proof: There is a non-trivial homomorphism **Calabi**

$$\mathit{Cal} : \mathit{Diffeo}_c(D^2, \omega) \longrightarrow \mathbb{R},$$

defined as follows:

- Given $\varphi \in \mathit{Diffeo}_c(D^2, \omega)$, we can write

  $$\varphi = \varphi_H^1,$$

  where $H : S^1 \times D^2 \longrightarrow \mathbb{R}$ is a time-varying Hamiltonian; we demand $H = 0$ near ∂D^2.
- Define $\mathit{Cal}(\varphi) := \int_{D^2} \int_{S^1} H dt \omega$. Can check: doesn't depend on choice of H!

Calabi as average rotation

Calabi measures the "average rotation" of the map φ:

$$Cal(\varphi) = \int\int Var_{t=0}^{t=1} Arg(\varphi_H^t(x) - \varphi_H^t(y))dxdy.$$

Naive idea

There's an inclusion

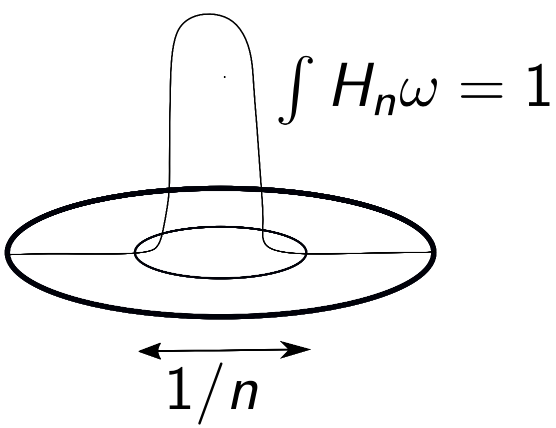
$$Diffeo_c(D^2, \omega) \subset Homeo_c(D^2, \omega),$$

dense in C^0-topology. Can we extend Calabi?

Problem: Cal not C^0 continuous.

Eg: Take H_n, supported on disc of radius $1/n$, where $\int H_n \, \omega = 1$.

Then, $Cal(\varphi^1_{H_n}) = 1$, but $\varphi^1_{H_n} \xrightarrow{C^0} \mathrm{Id}$.



Battle plan

Idea to get around this:

- For $\varphi \in \mathit{Diffeo}_c$, use "PFH spectral invariants" $c_d(\varphi) \in \mathbb{R}, d \in \mathbb{N}$ defined via "Periodic Floer Homology".
- Show $c_d(\varphi)$ are C^0 continuous, so extend to Homeo_c
- Prove "enough" of Hutchings' conjecture:

$$\mathit{lim}_{d \longrightarrow \infty} \frac{c_d(\varphi)}{d} = \mathit{Cal}(\varphi)$$

on Diffeo_c. (Inspired by "Volume Conjecture" for ECH.)

Section 3

Outline of the argument

Our normal subgroup: Finite energy homeomorphisms

Say $\varphi \in \text{FHomeo}_c(D^2, \omega)$ — "finite Hofer energy homeomorphisms" — if there exists

$$\varphi^1_{H_i} \xrightarrow{C^0} \varphi, \quad ||H_i||_{1,\infty} \leq M,$$

for M independent of i. Here, $||H_i||_{1,\infty}$ is the **Hofer norm**

$$||H_i||_{1,\infty} = \int_0^1 max(H_i) - min(H_i) dt.$$

We show: $\text{FHomeo}_c \trianglelefteq \text{Homeo}_c$. Hard part: showing FHomeo_c is proper.

Remarks:

- Oh-Müller group: $\mathrm{Hameo}_c \subset \text{FHomeo}_c$.
- FHomeo_c contains the commutator subgroup of Homeo_c.

Idea for showing properness

Must find a homeo with "infinite Hofer energy."

Observe: For $\varphi_H^1 \in \mathrm{Diff}_c(D^2, \omega)$, we have

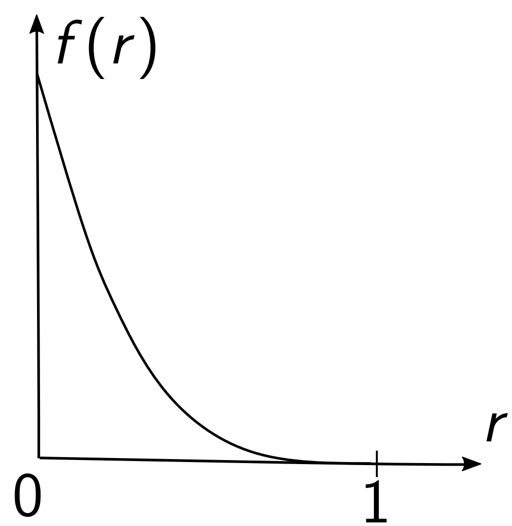
$$\mathrm{Cal}(\varphi_H^1) \leq ||H||_{1,\infty}.$$

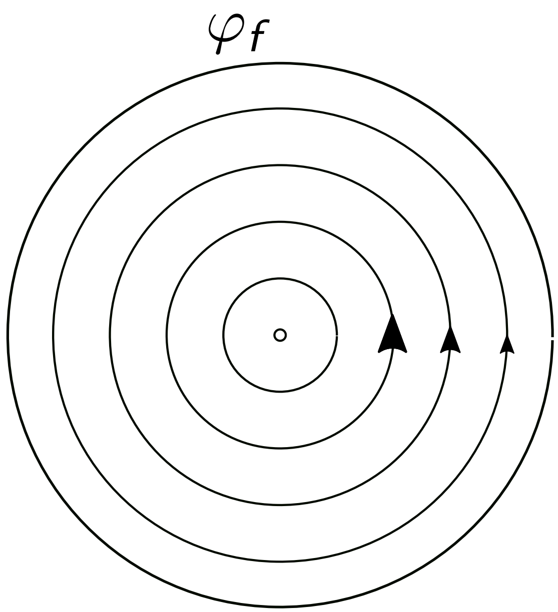
Look for a homeo with "infinite Calabi invariant."

The infinite twist

$f : [0,1] \longrightarrow \mathbb{R}$ smooth, decreasing. Define the **monotone twist** φ_f

$$(r,\theta) \longrightarrow (r, \theta + 2\pi f(r)).$$





The infinite twist

$f : [0,1] \longrightarrow \mathbb{R}$ smooth, decreasing. Define the **monotone twist** φ_f

$$(r,\theta) \longrightarrow (r, \theta + 2\pi f(r)).$$

Simple computation: $\mathrm{Cal}(\varphi_f) = \int_0^1 \int_r^1 sf(s)ds\ r\ dr$.

$f : (0,1] \longrightarrow \mathbb{R}$ smooth, decreasing. Call φ_f an **infinite twist** if

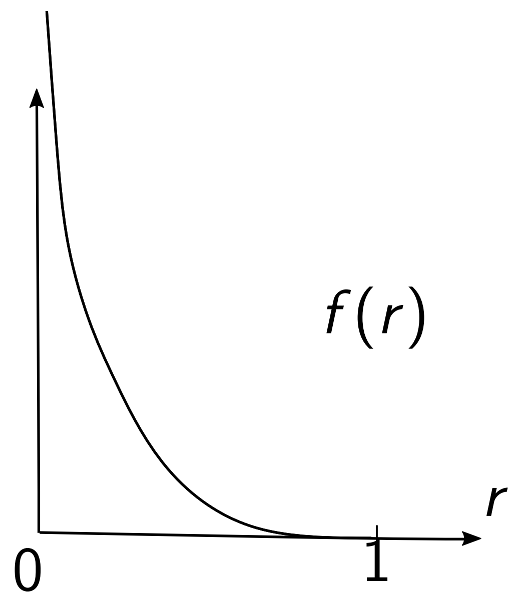
$$\int_0^1 \int_r^1 sf(s)ds\ r\ dr = \infty.$$

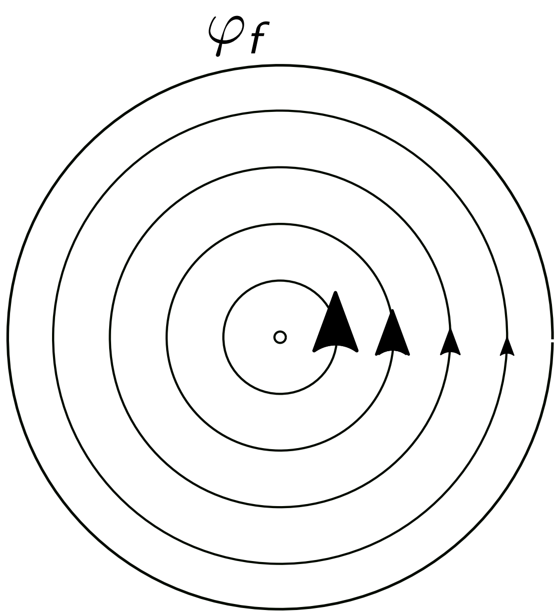
Note: $\lim_{r \longrightarrow 0} f(r) = \infty$.
We will show $\varphi_f \notin \mathrm{FHomeo}_c$.

The infinite twist

$\varphi_f(r,\theta) = (r, \theta + 2\pi f(r))$, where $f : (0,1] \longrightarrow \mathbb{R}$ is smooth, decreasing and

$$\int_0^1 \int_r^1 sf(s)ds\, r\, dr = \infty.$$





Asymptotic arguments

We need to show: $\varphi_f \notin FHomeo_c$.

The argument will go like this:

- (A) For any $\varphi \in FHomeo_c$, there exists a constant M with

$$c_d(\varphi) \leq Md.$$

- (B) For any infinite twist φ_f,

$$lim_{d \longrightarrow \infty} \frac{c_d(\varphi)}{d} = +\infty.$$

(A) — Hofer continuity

To prove (A) $[c_d(\varphi) \leq Md$ when $\varphi \in FHomeo_c]$,

we prove the following "Hofer continuity" property:

$$|c_d(\varphi^1_H) - c_d(\varphi^1_K)| \leq d||H - K||_{1,\infty}.$$

Then, (A) follows easily from C^0 continuity and the fact that $c_d(id) = 0$, since $id = \varphi^1_K$ for $K = 0$.

(B) — part i: Monotonicity

To prove (B) [$c_d(\varphi_f)/d \longrightarrow \infty$],

we first prove a general "Monotonicity property"

$$H \leq K \implies c_d(\varphi^1_H) \leq c_d(\varphi^1_K),$$

We then approximate φ_f with smooth φ_{f_i} such that:

$$f_i \leq f_{i+1},$$

hence

$$\frac{c_d(\varphi_f)}{d} \geq \frac{c_d(\varphi_{f_i})}{d}.$$

(B) — part ii: Hutchings' conjecture

We have

$$\lim_{d \longrightarrow \infty} \frac{c_d(\varphi_{f_i})}{d} \leq \lim_{d \longrightarrow \infty} \frac{c_d(\varphi_f)}{d}.$$

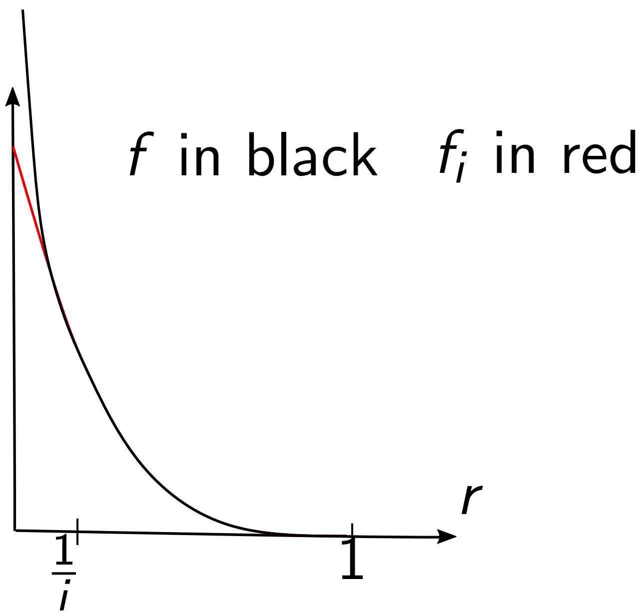
Hutchings' conjecture, which we prove for monotone twists, gives

$$\mathrm{Cal}(\varphi_{f_i}) \leq \lim_{d \longrightarrow \infty} \frac{c_d(\varphi_f)}{d}.$$

We pick f_i agreeing with f except on $[0, \frac{1}{i}]$. Thus,

$$Cal(\varphi_{f_i}) \longrightarrow \infty.$$

Hence, $\lim_{d \longrightarrow \infty} \frac{c_d(\varphi_f)}{d} = \infty$.



$$f - f_i \text{ supported in } [0, \tfrac{1}{i}] \Longrightarrow \mathrm{Cal}(\varphi_{f_i}) \longrightarrow \infty, \ \varphi_{f_i} \xrightarrow{C^0} \varphi_f.$$

Recap: to-do list

To recap, to prove $Homeo_c(D^2, \omega)$ is not simple, we have to:

- Define PFH spectral invariants
- Establish C^0 continuity, Hofer continuity, monotonicity for these invariants
- Prove Hutchings' conjecture for monotone twists
- Put it all together, as explained above.

Section 4

Bonus 1: PFH spectral invariants — impressionistic sketch

We define PFH spectral invariants by embedding D^2 as the northern hemisphere of S^2, and then using the periodic Floer homology of S^2.

The PFH of S^2: the setup

Let $\varphi \in \mathit{Diffeo}_0(S^2, \omega)$. Recall the **mapping torus**

$$Y_\varphi = S^2_x \times [0,1]_t / \sim, \quad (x,1) \sim (\varphi(x), 0).$$

Has a canonical vector field

$$R := \partial_t,$$

and a canonical two-form ω_φ induced by ω.

The PFH of S^2

The $\mathbb{Z}_2$ vector space $PFH(\varphi)$ is homology of a chain complex $PFC(\varphi)$, for nondegenerate φ.

Details of $PFC(\varphi)$:

- Generated by sets $\{(\alpha_i, m_i)\}$, where
  - α_i distinct, embedded closed periodic orbits of R
  - m_i positive integer; $m_i = 1$ if α_i is hyperbolic
- Differential ∂ counts $I = 1$ J-holomorphic curves in $\mathbb{R} \times Y_\varphi$, for generic J, where I is the "ECH index"
- ECH index beyond scope of talk; basic idea: $I = 1$ forces curves to be mostly embedded,

The PFH differential

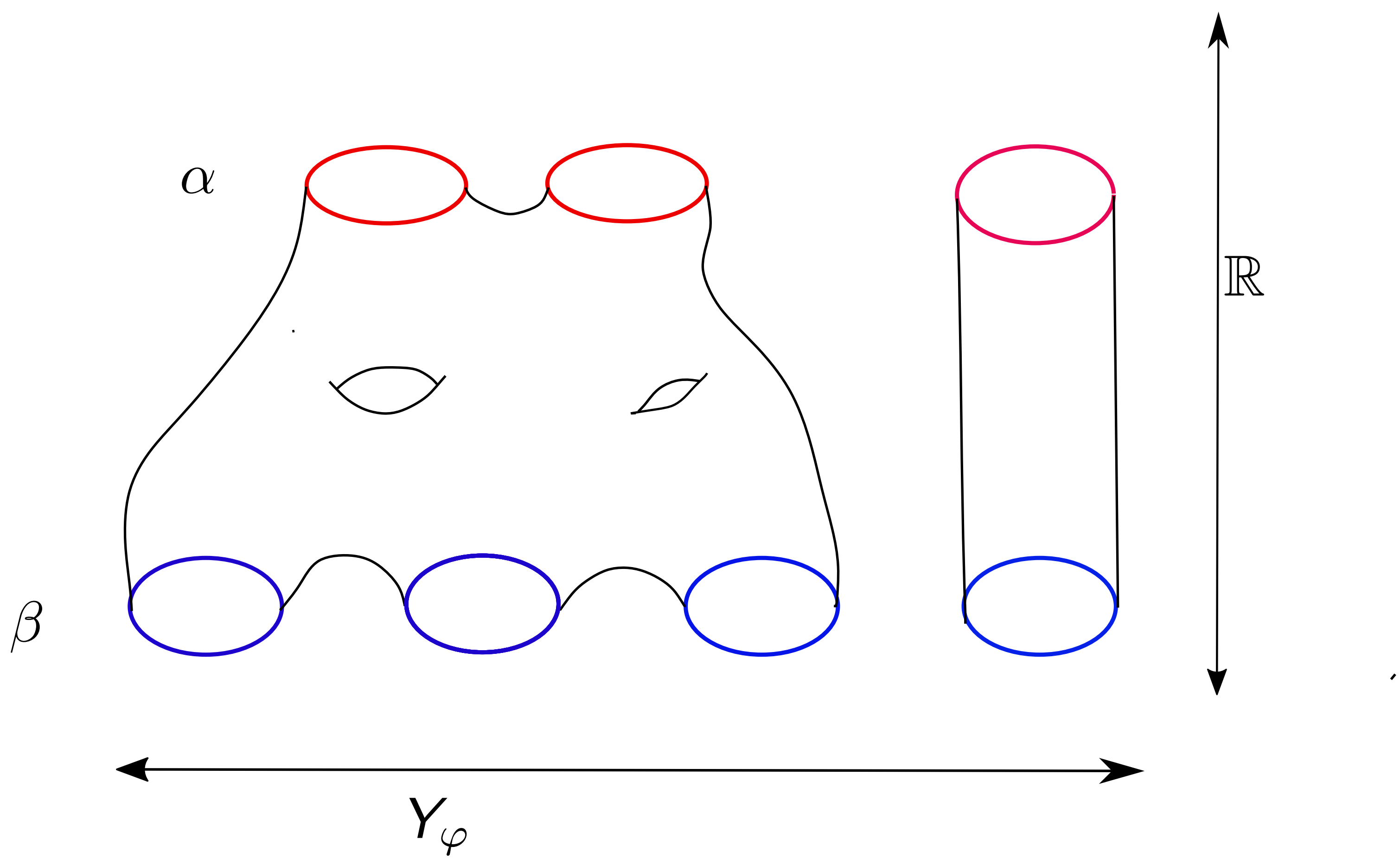


Figure: A J-hol curve contributing to $\langle \partial\alpha, \beta \rangle$.

More about PFH

$PFH(\varphi)$ homology of $PFC(\varphi, \partial)$.

There's a splitting

$$PFH(\varphi) = \oplus_d PFH(\varphi, d),$$

where $PFH(\varphi, d)$ homology of subcomplex generated by degree d orbit sets.

The spectral invariants:

Two auxiliary structures on $\widetilde{PFH}$:

- "The action": $\mathcal{A}(\alpha, Z) = \int_Z \omega_\varphi$
- "The grading": $gr(\alpha, Z) = I(Z)$

We now define $c_d(\varphi)$ to be the minimum action of a homology class with grading 0 and degree d.

Section 5

Bonus 2: Remarks on the rest of the proof

Still remains to explain Hofer continuity, monotonicity, C^0-continuity, Hutchings' conjecture in twist case...key ideas:

- Hofer continuity, monotonicity: cobordism map argument inspired by work of Hutchings-Taubes
- C^0 continuity inspired by proof of C^0 continuity of barcodes for Ham. Floer homology
- Hutchings' conjecture in twist case works by direct computation: can write down all closed orbits, curves
    - — get a combinatorial model, involving lattice paths, lattice regions, inspired by work of Hutchings-Sullivan

Section 6

Bonus 3: Beyond the simplicity conjecture

Higher genus surfaces

In more recent joint work, we proved a generalization of the Simplicity Conjecture.

Let (Σ_g, ω) be a closed surface with an area-form. Fathi defined a *mass-flow* homomorphism

$$Homeo_0(\Sigma_g, \omega) \longrightarrow H_1(M)/\Gamma,$$

where Γ is a certain discrete subgroup.

Theorem (CG., Humiliere, Mak, Seyfaddini, Smith.)

The kernel of mass-flow on (Σ_g, ω) is never simple.

Proof uses "link spectral invariants" via a kind of quantitative variant of Heegaard Floer homology. Polterovich-Shelukhin have another (related) proof.

Quasimorphisms

The group $Homeo_0(S^2, \omega)$ does not admit any continuous homomorphisms to $\mathbb{R}$. However, we showed:

Theorem (CG., Humiliere, Mak, Seyfaddini, Smith)

The space of homogeneous quasimorphisms on $Homeo_0(S^2, \omega)$ is infinite dimensional.

Recall that a homogeneous quasimorphism on a group G is a map

$$\mu : G \longrightarrow \mathbb{R}$$

such that

$$\mu(g^d) = d\mu(g), \quad |\mu(gh) - \mu(g) - \mu(h)| \leq D.$$

Some open questions

We also showed (Polterovich-Shelukhin have an independent proof) that Calabi extends to a certain subgroup $Hameo \subset FHomeo$.

- Is the kernel of Calabi on Hameo simple?
- Does $Hameo = FHomeo$?
- What is the quotient $Homeo_0/FHomeo$?
- How do the PFH spectral invariants relate to the link spectral ones?