The $(2,1)$-cable of the figure-eight knot is not slice

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joint with Sungkyung Kang, Abhishek Mallick, JungHwan Park, Matthew Stoffregen

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Motivation

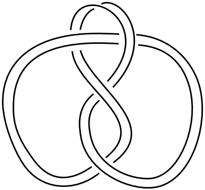

Figure: The $(2, 1)$-cable of the figure-eight knot.

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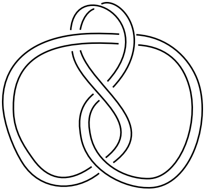

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Is $(4_1)_{2,1}$ *(smoothly) slice?*

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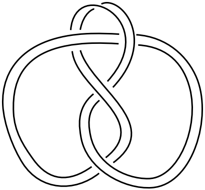

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Is $(4_1)_{2,1}$ (smoothly) slice?

Theorem (D.-Kang-Mallick-Park-Stoffregen 2022)

No.

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Theorem (Miyazaki 1994)

Let K be a fibered, negative amphichiral knot with irreducible Alexander polynomial. Then $K_{2n,1}$ is not ribbon for any $n \neq 0$.

- *For example,* $K = 4_1, 6_3, 8_{12}, 8_{17}, 10_{17}, \ldots$.

(Negative amphichiral = isotopic to reverse of mirror image.)

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Theorem (Kawauchi 1980, Cha 2007, Kim-Wu 2018)

Let K be a knot as above. Then $K_{2n,1}$ is (strongly) rationally slice.

Motivation

Definition

Recall K is rationally slice if K is slice into some $\mathbb{Q}HB^4$ W. We say K is *strongly* rationally slice if it admits a slice disk D in W such that

$$H_1(S^3 - \nu(K), \mathbb{Z}) \to H_1(W - \nu(D), \mathbb{Z})/\text{torsion}$$

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is an isomorphism.

- *Strong* rational sliceness implies algebraic sliceness (in particular, all of the Miyazaki cables $K_{2n,1}$ are algebraically slice).
- The figure-eight is rationally slice but *not* algebraically slice.
- $(4_1)_{2,1}$ is “closer” to being slice than 4_1.

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- Potential counterexamples to the slice-ribbon conjecture!
- What about Floer-theoretic or other modern invariants?
  - Usual knot Floer invariants (τ, Υ, and so on) are rational concordance invariants and thus vanish.
  - Hom-Kang-Park-Stoffregen (2022) used involutive knot Floer homology to establish linear independence of odd cables $(4_1)_{2n+1,1}$. However, they showed that for even cables $(4_1)_{2n,1}$, involutive knot Floer homology fails.
  - For $(4_1)_{2,1}$, the s-invariant from Khovanov homology fails.

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Definition

The $(\mathbb{Z}/2\mathbb{Z})$-homology cobordism group is

$$\{Y \text{ a } (\mathbb{Z}/2\mathbb{Z})HS^3\}/\sim,$$

where $Y_1 \sim Y_2$ if there is a $(\mathbb{Z}/2\mathbb{Z})$-homology cobordism W from Y_1 to Y_2.

Branched covers

- Studying the homology cobordism group is a very classical application of Floer homology/gauge theory. Has been studied using:
  - Rokhlin invariant.
  - Donaldson's theorem/Yang-Mills theory.
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- Unfortunately, in this case $\Sigma_2((4_1)_{2,1})$ *does* bounds a homology ball, as we will see in a moment.

Branched covers

Main Idea

Remember the branching involution gives a more refined obstruction by asking for the existence of an *equivariant* $\mathbb{Z}/2\mathbb{Z}$-homology ball.

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- For any knot K, can record the data of the branching involution $\tau\colon \Sigma_2(K) \to \Sigma_2(K)$.
- If K is slice with slice disk D, then the branched double cover $\Sigma_2(D)$ also inherits an involution $\tau_D\colon \Sigma_2(D) \to \Sigma(D)$ which restricts to τ on $\partial\Sigma_2(D) = \Sigma_2(K)$.

Branched covers

- More generally, our previous homomorphism factors through some *equivariant* homology cobordism group

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Definition (D.-Hedden-Mallick 2020)

The *homology bordism group of involutions* $\Theta^{\tau}_{\mathbb{Z}/2\mathbb{Z}}$ is

$$\{Y \text{ a } (\mathbb{Z}/2\mathbb{Z})HS^3 \text{ with involution } \tau\colon Y \to Y\}/\sim,$$

where $(Y_1, \tau_1) \sim (Y_2, \tau_2)$ if there is a homology cobordism W from Y_1 to Y_2 equipped with an involution $\tau_W\colon W \to W$ restricting to τ_i on Y_i.

Topological interlude

Main Idea

Great fact: $\Sigma_2((K)_{2,1}) \cong S_{+1}(K\#K^r)$ (Akbulut-Kirby). Moreover, this homeomorphism identifies the branching action on the former with the involution on the latter induced by the strong inversion on $K\#K^r$ interchanging the two components.

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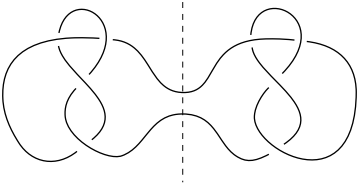

Figure: Strong inversion on $K\#K^r$.

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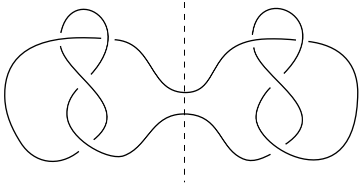

Figure: Strong inversion on $K\#K^r$.

- Since $4_1\#4_1^r$ is slice, $S^3_{+1}(4_1\#4_1^r)$ bounds a homology ball (in fact, a contractible manifold). However, $4_1\#4_1^r$ is not *equivariantly* slice.

Topological interlude

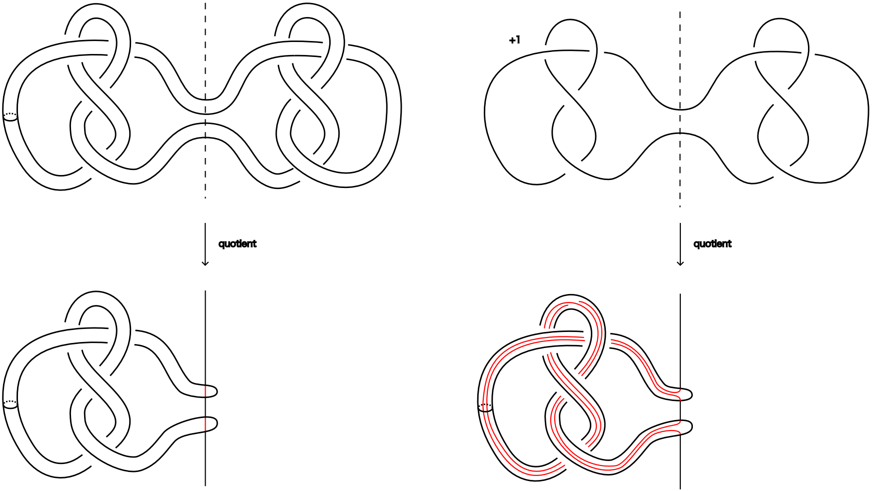

Figure: Sketch of $\Sigma_2((K)_{2,1}) \cong S_{+1}(K \# K^r)$.

Heegaard Floer homology

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Use (Heegaard) Floer homology to obstruct the bounding of an equivariant homology ball.

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Use (Heegaard) Floer homology to obstruct the bounding of an equivariant homology ball.

- Alfieri-Kang-Stipsicz (2019) studied Montesinos knots modulo torus knots.
- D.-Hedden-Mallick (2020) studied $\Theta^{\tau}_{\mathbb{Z}/2\mathbb{Z}}$ in the context of corks.
- Computational techniques and relation to equivariant knots in D.-Mallick-Stoffregen (2022) and Mallick (2022).
- Studying similar questions using Yang-Mills or Seiberg-Witten: Anvari, Baraglia, Hambleton, Hekmati, Konno, Montague, Miyazawa, Taniguchi.

What is Heegaard Floer homology?

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- Given a 3-manifold Y with spin^c-structure $\mathfrak{s}$, we get a chain complex $CF^-(Y, \mathfrak{s})$.
- Given a cobordism $(W, \mathfrak{s})$ from $(Y_1, \mathfrak{s}_1)$ to $(Y_2, \mathfrak{s}_2)$, we obtain a chain map $F_{W,\mathfrak{s}} : CF^-(Y_1, \mathfrak{s}_1) \to CF^-(Y_2, \mathfrak{s}_2)$.

Heegaard Floer chain groups

The chain complex $CF^-(Y, \mathfrak{s})$ satisfies certain properties.

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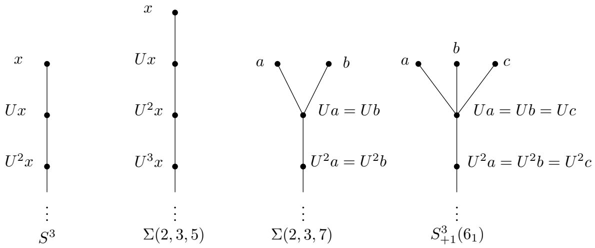

Figure: Some Heegaard Floer homologies.

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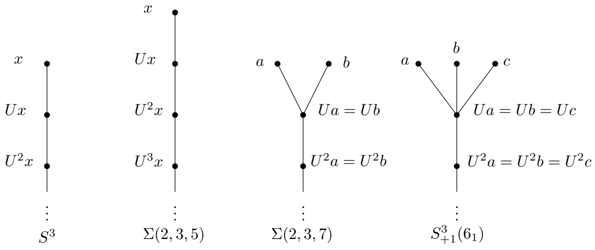

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- If Y is a $\mathbb{Q}HS^3$, $CF^-(Y, \mathfrak{s})$ satisfies a *localization condition*:

$$U^{-1}HF^-(Y, \mathfrak{s}) \cong \mathbb{F}[U^{-1}, U].$$

  This just means that sufficiently far down, $HF^-(Y, \mathfrak{s})$ is just a single U-nontorsion tower (as in the above examples).

Heegaard Floer chain maps

The chain maps also satisfy certain properties.

- $F_{W,\mathfrak{s}}$ is $\mathbb{F}[U]$-equivariant with grading shift

$$\mathrm{gr}(F_{W,\mathfrak{s}}) = \frac{1}{4}(c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W))$$

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- If W is negative-definite, it also satisfies a *localization condition*:

$$F_{W,\mathfrak{s}}\colon U^{-1}HF^-(Y_1, \mathfrak{s}_1) \xrightarrow{\cong} U^{-1}HF^-(Y_2, \mathfrak{s}_2).$$

  This just means that $F_{W,\mathfrak{s}}$ maps U-nontorsion elements to U-nontorsion elements.

Local equivalence

If W is a homology cobordism from Y_1 to Y_2, we thus have that:

- F_W has grading shift zero; and,
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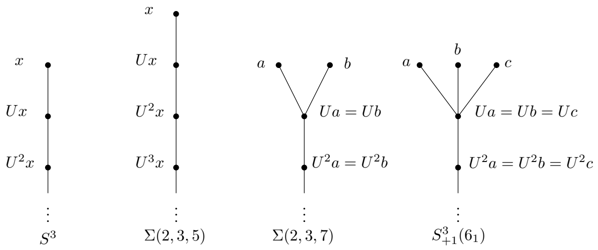
If two complexes C_1 and C_2 admit such maps between them, we say that C_1 and C_2 are *locally equivalent*.



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Local equivalence

Main Idea

Heegaard Floer homology translates topology into algebra:

$$Y_1 \xrightarrow{\text{hom. cob. } W} Y_2 \Rightarrow HF^-(Y_1) \xrightarrow{\text{loc. map}} HF^-(Y_2).$$

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- In the most basic case, local equivalence governed by a single integer (d-invariant).
- First non-trivial application of the local equivalence formalism due to Stoffregen (2015) in the context of Pin(2)-equivariant Seiberg-Witten-Floer homology.

- Hendricks-Manolescu (2015) upgraded Heegaard Floer homology by introducing an additional symmetry

$$\iota\colon CF^-(Y,\mathfrak{s}) \to CF^-(Y,\mathfrak{s})$$

for any self-conjugate $\mathfrak{s}$.

Involutive Heegaard Floer homology

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$$\iota\colon CF^-(Y,\mathfrak{s}) \to CF^-(Y,\mathfrak{s})$$
for any self-conjugate $\mathfrak{s}$.
- Moreover, if $\mathfrak{s}$ is self-conjugate
$$F_{W,\mathfrak{s}} \circ \iota_1 \simeq \iota_2 \circ F_{W,\mathfrak{s}}.$$

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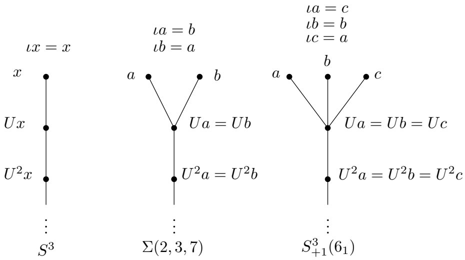

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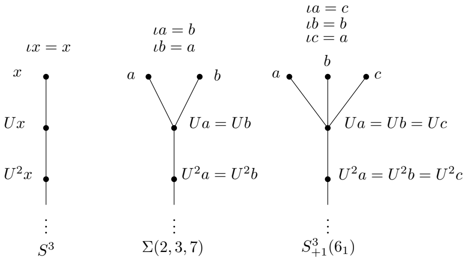

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- Used to successfully study $\Theta^3_{\mathbb{Z}}$:
  - First introduced by Hendricks-Manolescu-Zemke (2015)
  - $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$-summand (D.-Hom-Stoffregen-Truong; 2018)
  - $\Theta^3_{\mathbb{Z}}$ is not generated by Seifert fibered spaces (Hendricks-Hom-Stoffregen-Zemke; 2020)

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$$F_W \circ \tau_1 \simeq \tau_2 \circ F_W.$$
- To obstruct an *equivariant* homology cobordism, additionally require our local maps to be equivariant with respect to τ.

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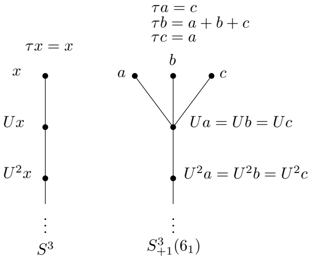

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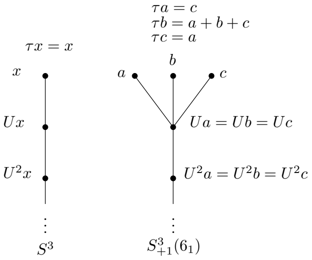

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- This was used to study equivariant homology cobordism by D.-Hedden-Mallick (2020).

Knot Floer homology and surgery

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Can use $CFK(K)$ to compute $HF^-(S^3_n(K))$:

$$A_0^-(K) \simeq CF^-(S^3_n(K), \mathfrak{s}_0) \text{ (large } n\text{).}$$

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Theorem (Hendricks-Manolescu 2015)

Can use ι_K on $CFK(K)$ to compute ι on $HF^-(S^3_n(K))$:

$$(A_0^-(K), \iota_K) \simeq (CF^-(S^3_n(K), \mathfrak{s}_0), \iota) \text{ (large } n\text{).}$$

Knot Floer homology and surgery

Theorem (D.-Mallick-Stoffregen 2022)

A strong inversion τ_K on a knot K induces a (skew-graded, skew-$\mathbb{F}[U,V]$-equivariant) homotopy involution

$$\tau_K : CFK(K) \to CFK(K).$$

Theorem (Mallick 2022)

Can use τ_K on $CFK(K)$ to compute τ on $HF^-(S^3_n(K))$:

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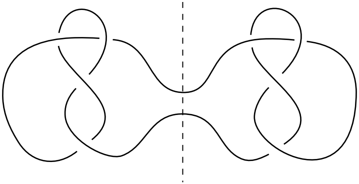

Figure: Need to understand $CFK(K\#K^r)$.

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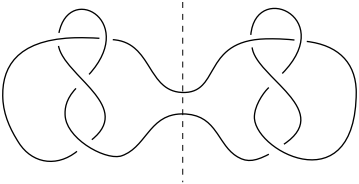

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Theorem (Ozsváth-Szabó 2004)

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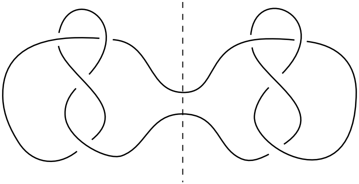

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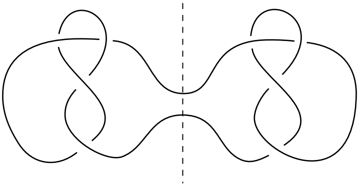
Theorem (Zemke 2017)

$$\iota_{K \# K^r} = (\mathrm{id} \otimes \mathrm{id} + \Psi \otimes \Phi) \circ (\iota_K \otimes \iota_{K^r}).$$

Knot Floer homology and surgery



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Theorem (D.-Mallick-Stoffregen 2022)

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Knot Floer homology and surgery

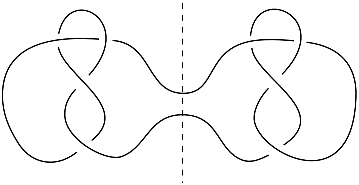

Figure: Need to understand $CFK(K \# K^r)$.

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- Here, $\tau_{K\#K^r}$ is the induced action of the summand-swapping involution.
- The map τ_{exch} is the "obvious" factor-swapping map on the tensor product $CFK(K) \otimes CFK(K^r)$.

Further results

Theorem (D.-Kang-Mallick-Park-Stoffregen 2022)

Let K be a Floer-thin knot with $\mathrm{Arf}(K) = 1$ and let $k \in \mathbb{N}$ be odd. Then $K_{2,k} \# - T_{2,k}$ has infinite order.

Further results

Theorem (D.-Kang-Mallick-Park-Stoffregen 2022)

Let K be a Floer-thin knot with $\mathrm{Arf}(K) = 1$ and let $k \in \mathbb{N}$ be odd. Then $K_{2,k} \# - T_{2,k}$ has infinite order.

- Generalize K: we can handle knots with convenient knot Floer homology.
  - For example, $K = 4_1, 6_3, 8_{12}, 8_{17}$. The smallest Miyazaki knot to which our theorem does not apply is 10_{17}.
- Consider other cabling parameters: if K is a knot as in Miyazaki's paper, then K is rationally slice (by work of Cha and Kim-Wu). Thus $K_{p,q} \# - T_{p,q}$ is rationally slice. These are likewise not ribbon (again by Miyazaki).
  - We are able to do $(2, k)$-cables since $\Sigma_2(K_{2,k}) \simeq S^3_k(K \# K^r)$.