Generalisations of Hecke Algebras from Loop Braid Groups

Joint work with P. Martin and E. Rowell

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Dahm 1962, Goldsmith 1982

A motion of N in M

is a continuous transformation of M (*ambient isotopy of M*) who brings N to its original position at the end of the transformation.

We can **compose** motions, and say when they are **equivalent**:

we have a **group** structure.

We call this group the:

Motion group of N in M

Encodes the topologically distinct ways of moving N in M, so that at the end of the motion N has returned in its starting position.

Formally: $\pi_1(Homeo^c(M), Homeo^c(M,N); id_M)$.

Motion groups

Unpacking

M oriented manifold; N oriented compact submanifold.

A **motion of N in M** is an ambient isotopy $f_t(x)$ of N in M such that:

$f_0(x) = \mathrm{id}_M$ and $f_1(N) = N$ as an oriented manifold.

A motion f is **stationary** for N if $f_t(N) = N$, for all $t \in [0,1]$.

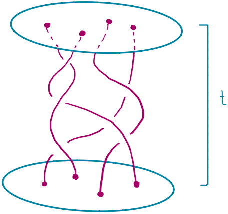
$f \simeq f'$ if $f^{-1} \circ f' \sim$ a stationary motion.

$$\boxed{\mathcal{M}(M,N) = \text{motions} / \simeq}$$

Motion groups

An example: braid groups B_n

When $M = B^2$ disc
$N = P$ set of n points in $\mathring{B}^2$



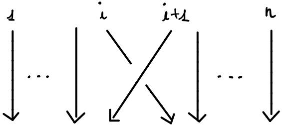
$M(B^2, P) = B_n$ the braid group on n strands

... as motions of points in a disc.

<u>Recall:</u>

$$B_n \cong \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j|>1 \end{array} \right\rangle$$ Artin's presentation

$\sigma_i \longmapsto$



Motion groups

Another example: loop braid groups LB_n

Goldsmith

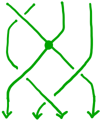
- **Group of motions** of B^3 **with an n-trivial link** C_n in its interior, with orientation preserved both on B^3 and C_n.

Savushkina

- **Group of automorphisms** β of the free group F_n of type $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$, with $\pi \in S_n$ and $a_i \in F_n$.

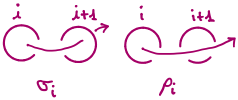
Fenn, Rimányi, Rourke

- $\left\langle \begin{matrix} \sigma_1, \ldots, \sigma_{n-1}, \\ \rho_1, \ldots, \rho_{n-1} \end{matrix} \middle| \begin{matrix} \textbf{braid grp} \text{ rels} \\ \textbf{symmetric grp} \text{ rels} \\ \textbf{mixed} \text{ rels} \end{matrix} \right\rangle$ + **welded diagrams**.



Brendle, Hatcher

- **Fundamental group** of the **configuration space of n circles** on parallel planes in $\mathbb{R}^3$.



- **Annuli** $S^1 \times I$ **braided in** $\mathbb{R}^4$.

Another example: loop braid groups LB_n

- **Group of motions** of B^3 **with an n-trivial link** C_n in its interior, with orientation preserved both on B^3 and C_n.
- **Group of automorphisms** β of the free group F_n of type $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$, with $\pi \in S_n$ and $a_i \in F_n$.
- $\left\langle \begin{array}{c} \sigma_1, \ldots, \sigma_{n-1}, \\ \rho_1, \ldots, \rho_{n-1} \end{array} \middle| \begin{array}{c} \textbf{braid grp} \text{ rels} \\ \textbf{symmetric grp} \text{ rels} \\ \textbf{mixed} \text{ rels} \end{array} \right\rangle$ + **welded diagrams**.
- **Fundamental group** of the **configuration space of n circles** on parallel planes in $\mathbb{R}^3$.
- **Annuli $S^1 \times I$ braided in $\mathbb{R}^4$.**

Theorem (D., 2016)

All these definitions are equivalent

The dream

R integral domain, $t \in R$ a unit

1- parameter Iwahori-Hecke algebra :

$\mathcal{B}_n$

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1- parameter Iwahori-Hecke algebra:

$$H_n^R(t) := R[\mathcal{B}_n] \Big/ \sigma_i^2 = (1-t)\sigma_i + t$$

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1- parameter Iwahori-Hecke algebra:

$$\begin{array}{ccc} \mathcal{B}_n & \xrightarrow[\text{homomorph.}]{\text{well-def'd}} & H_n^R(t) := R[\mathcal{B}_n] \big/ \bar{\sigma}_i^2 = (1-t)\bar{\sigma}_i + t \\ \sigma_i & \longmapsto & \bar{\sigma}_i \end{array}$$

- We can represent any element of $H_n^R(t)$ by a linear combination of braid diagrams: the quadratic relation can be seen as a skein relation.

- Representations of $H_n^R(t)$ are equiv. to those of B_n for which the image of the generators satisfy the quadratic relation.

The dream

R integral domain, $t \in R$ a unit

<u>1- parameter Iwahori - Hecke algebra :</u>

$$H_n^R(t) := R[B_n] \Big/ \bar{\sigma}_i^2 = (1-t)\bar{\sigma}_i + t$$

knots can be represented
as closed braids (Alexander theorem)

$\downarrow$

Markov's theorem $\Rightarrow$ an R- valued link invariant
is equivalent to a function $B_\infty \longrightarrow R$
that is invariant under Markov moves.

$\rightsquigarrow$ Markov traces

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⇝ Markov traces

JONES POLYNOMIAL

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SUPERSYMMETRIC INVARIANTS

Deguchi, Akutzu, Martin, Kauffman-Saleur

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BIRMAN - WENZL - MURAKAMI ALGEBRAS

Reverse Jones process

Link invariants

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Deguchi, Akutzu, Martin, Kauffman - Saleur

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Well understood representation theory

BIRMAN - WENZL - MURAKAMI ALGEBRAS

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Loop Hecke-like algebra?

$R[LB_n] \big/ \bar{\sigma}_i^2 = (1-t)\bar{\sigma}_i + t$

How do we want it to be?

Iwahori-Hecke algebras

Some properties

1. $\mathcal{H}_n^R$ is finite dimensional

Iwahori-Hecke algebras

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Iwahori-Hecke algebras

Some properties

1. $\mathcal{H}_n^R$ is finite dimensional
2. $\mathcal{H}_n^R$ is free R-module of rank n!
3. its has a basis indexed on permutations
4. If $t = 1$ then $\mathcal{H}_n^R$ is the group algebra $R[S_n]$ of the symmetric group S_n. If R has characteristic zero and t is generic then $\mathcal{H}_n^R$ is isomorphic to $R[S_n]$
5. K a field, $\mathcal{H}_n^K$ is semisimple provided t is not a root of unity of order certain orders.
6. Very well understood representation theory over a field of characteristic 0.
7. connection with quantum groups arising from statistical mechanics, and with Yang-Baxter equations.

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$$R[LB_n] \big/ \bar{\sigma}_i^2 = (1-t)\bar{\sigma}_i + t$$

FINITE DIMENSION?

BRAID RELATIONS:

$\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i-j|>1$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $i=1,\dots,n-2$

SYMMETRIC GROUP RELATIONS:

$\rho_i \rho_j = \rho_j \rho_i$, $|i-j|>1$

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$$R[LB_n] \Big/ \bar{\sigma}_i^2 = (1-t)\bar{\sigma}_i + t$$

- $\sigma_i \rho_i$, $i \in 1, \ldots, n-1$
- $\sigma_i \rho_i \sigma_i$
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FINITE DIMENSION?

NO :(

We need to add some relations to kill

BRAID RELATIONS:

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Representations of B_n

- 1936 Burau $B_n \longrightarrow GL_n(\mathbb{Z}[t,t^{-1}])$

$$\sigma_i \longmapsto \left[\begin{array}{c|cc|c} I & & & \\ \hline & 1-t & t & \\ & 1 & 0 & \\ \hline & & & I \end{array}\right]$$

- 1990 - 2001 Bigelow, Krammer, Lawrence : faithful !

- 1980s Yang - Baxter equation - local representations

- 1980s → reps. of towers of quotients of $K[B_n]$:

  Iwahori - Hecke H_n, Temperley - Lieb TL_n,

  Birman - Murakami - Wenzl $BMW_n(r,q)$

- Unitary representations, representations from braided fusion categories, ...

Representations of LB_n

WHICH ONES EXTEND?

B_n representations	LB_n representations
Artin representation	✓
Burau	✓ Vershinin, Bardakov
Lawrence - Krammer - Bigelow	? Bardakov: not combinatorially
Local representations	Depends Rowell, Martin, ...

* Beautiful State of the art in 2020 Bellingeri - Soulié

The Loop Hecke algebra LH_n

$R = \mathbb{C}$, $t \in \mathbb{C}$ parameter ← dreaming of polynomial invariants

- We start from:

$$\mathbb{C}[\mathbf{LB_n}] \,/\, \boxed{\bar{\sigma}_i^2 = (1-t)\bar{\sigma}_i + t} \quad *$$

- Remark: (1) Relation $*$ is satisfied by the Burau matrices.

In fact: if we do "reverse Jones" starting from the Alexander poly, this is the quotient we get (remember link Burau - Alexander).

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- Remark: ① Relation $*$ is satisfied by the Burau matrices.

In fact: if we do "reverse Jones" starting from the Alexander poly, this is the quotient we get (remember link Burau - Alexander).

② We need to simplify words of the form $\sigma_i \rho_i \sigma_i \cdots$ and $\rho_i \sigma_i \rho_i \cdots$

IDEA: s_i, r_i extended Burau matrices. Do

$s_i r_i$ and $r_i s_i$

satisfy a nice quadratic relation?

BRAID RELATIONS:

$\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i-j| > 1$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $i = 1, \ldots, n-2$

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- We start from:

$$\mathbb{C}[LB_n] \Big/ \boxed{\bar{\sigma}_i^2 = (1-t)\bar{\sigma}_i + t} \quad *$$

③ Extended Burau matrices:

$$\sigma_i \longmapsto s_i = \left[\begin{array}{c|cc|c} I & & & \\ \hline & 1-t & t & \\ & 1 & 0 & \\ \hline & & & I \end{array}\right] \qquad \rho_i \longmapsto r_i = \left[\begin{array}{c|cc|c} I & & & \\ \hline & 0 & 1 & \\ & 1 & 0 & \\ \hline & & & I \end{array}\right]$$

LEMMA: s_i and r_i satisfy the relations:

$$r_i s_i = -t\, r_i + s_i + t$$

$$s_i r_i = -s_i + r_i + t$$

The Loop Hecke algebra LH_n

$R = \mathbb{C}$, $t \in \mathbb{C}$ parameter

dreaming of polynomial invariants

$$LH_n := \mathbb{C}[LB_n]$$

$$\bar{\sigma}_i^2 = (1-t)\bar{\sigma}_i + t$$

$$\bar{\rho}_i \bar{\sigma}_i = -t\bar{\rho}_i + \bar{s}_i + t$$

$$\bar{\sigma}_i \bar{\rho}_i = -\bar{\sigma}_i + \bar{\rho}_i + t$$

Proposition

LH_n is finite dimensional.

Ingredients of the proof.

- additional relations derived from defining relations:
  - $\rho_2\sigma_1\rho_2 = \rho_1\sigma_2\rho_1$,
  - $\sigma_2\rho_1\rho_2 = \rho_1\rho_2\sigma_1$,
  - $\sigma_2\rho_1\sigma_2 = \sigma_1\rho_2\rho_1 + \rho_1\sigma_2\sigma_1 - \rho_1\sigma_2\rho_1$
  - $\rho_2\sigma_1\sigma_2 = \sigma_1\rho_2\rho_1 + t\rho_1\sigma_2\rho_1 - t\rho_1\rho_2\rho_1 + \sigma_2\sigma_1 - \sigma_2\rho_1 - t\rho_2\sigma_1 + t\rho_2\rho_1$
- X_i = vector subspace of LH_n spanned by $\{1, \sigma_i, \rho_i\}$
  $LH_m^{\langle}$ = subalg. of LH_{n+1} gen'd by $\{\sigma_i, \rho_i \mid i = 1, \ldots, m-1\}$.

$$\text{Then } LH_{n+1} = LH_n^{\langle} X_n LH_n^{\langle}.$$

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□

<u>Remark:</u> From direct calculation, the dimension is bigger than the image of extended Burau.

Remark : playing with specialising the parameter, one finds the dimension can go up — contrast with Hecke.

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  Then $LH_{n+1} = LH_n^{\langle} X_n LH_n^{\langle}$.

□

Remark: From direct calculation, the dimension is bigger than the image of extended Burau.

Other properties of LH_n

- LH_n is **not** semisimple;

Some thoughts: bad news? Different from Hecke.

However: in the quest for (3+1)-TQFTs non semisimplicity seems desirable.

Other properties of LH_n

- LH_n is **not** semisimple;

- We can define a **local representation** related to extended Burau, and define the **Loop Burau-Rittenberg** LBR_n **algebra**.

The Loop Burau-Rittenberg LBR_n algebra

and what it tells us about LH_n

Define matrices:

$$R = \begin{bmatrix} 1 & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & -t \end{bmatrix}, \qquad S = R\big|_{t=1}$$

$$\sigma_i \longmapsto I_2^{\,i-1} \otimes R \otimes I_2^{\,n-i-1}$$

$$\rho_i \longmapsto I_2^{\,i-1} \otimes S \otimes I_2^{\,n-i-1}$$

$$\boxed{LBR_n \text{ algebra} := \langle R_i \,, S_i \rangle_{i=1}^{n-1}}$$

PROPERTIES:

- LBR_n / rad sum of simple matrix algebras of dimension given by Pascal diagram.
- The structure of LBR_n is independent of t, except $t=1$.
- For $t^2 \neq 1$ and $n < 8$, $LH_n \cong LBR_n$.

$\boxed{\text{CONJECTURE: For } t^2 \neq 1, \quad LH_n \cong LBR_n.}$

What next?

- Finite dimensional quotients from other motion groups?
- Topological invariants of surfaces in $\mathbb{R}^4$?
- Using non semisimplicity as a feature?