Almost equivalence for Anosov flows

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Overview

Question (Fried 1982)

Does every transitive Anosov flow admit a finite collection of periodic orbits that form a genus-one fibered link?

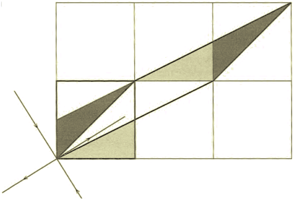
Question (Ghys 80's)

Given any two transitive Anosov flows, do they differ by a finite number of Dehn surgeries along periodic orbits?

1. Anosov flows
2. Birkhoff sections, Dehn surgeries, and almost equivalence
3. Equivalence of both questions (Minakawa, unpublished)
4. Positive answer for algebraic An. flows (D-Shannon, 2019)
5. Toward a positive answer for all Anosov flows (D, work in progress)

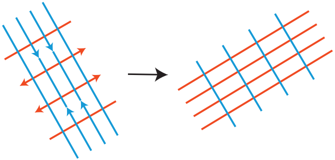
Anosov diffeomorphism

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ acting on } \mathbb{R}^2/\mathbb{Z}^2$$

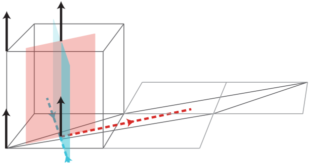


Definition

$f : M \to M$ is Anosov if $\exists(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ transverse f-invariant measured foliations and $\exists\lambda > 0$ such that $f^*\mu^s = \lambda^{-1}\mu^s$ and $f^*\mu^u = \lambda\mu^u$.



ANOSOV FLOW: EXAMPLE 1 - SUSPENSION

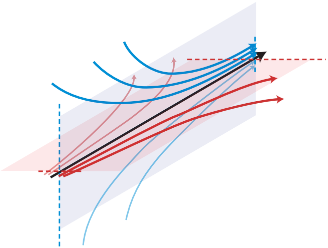


$$\begin{aligned} M &= \mathbb{T}^2 \times [0,1]/_{(p,1)\sim(A(p),0)} \\ &=: M_A \quad \text{mapping torus} \end{aligned}$$

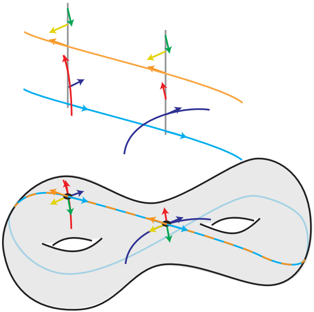
$X = \frac{\partial}{\partial z}$
$\varphi^t_{\text{sus}}((p,z)) = (p, z+t)$

Definition

φ^t is topologically Anosov if $\exists$ invariant 2-foliations $\mathcal{F}^s/\mathcal{F}^u$ tangent to φ^t that are exponentially contracted when $t \to +\infty/-\infty$.



Anosov flow: example 2 - geodesic flow

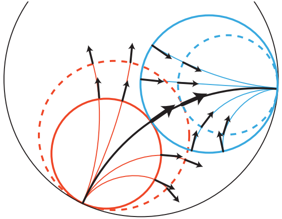


Σ Riemannian surface

$M = \mathrm{T}^1\Sigma = \{(p, v) \mid v \in \mathrm{T}\Sigma, ||v|| = 1\}$

$\varphi^t_{\text{geod}}((\gamma(0), \dot{\gamma}(0))) = (\gamma(t), \dot{\gamma}(t))$

Σ hyperbolic $\Longrightarrow$ φ^t_{geod} Anosov



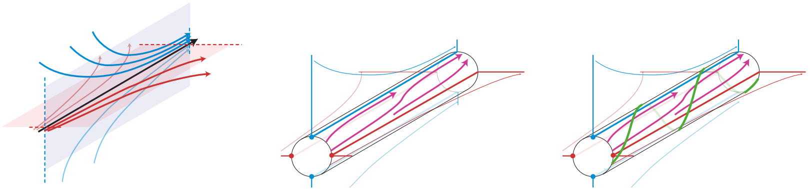
Other examples?

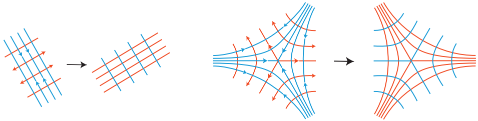
in dim ≥ 4, few other examples (see Barthelmé's lecture notes)
in dimension 3

- surgery constructions:
  - Handel-Thurston 1980
  - Dehn-Goodman-Fried 1983

    Dehn surgery on a periodic orbit
- Lego constructions: Béguin-Bonatti-Yu 2017

Dehn-Goodman-Fried surgery

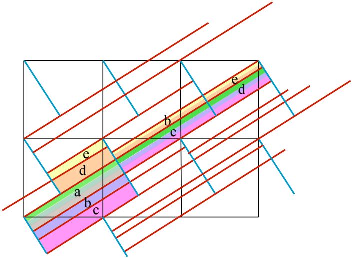




If the new meridian cuts the stable/unstable directions twice, the obtained 1-foliation is topologically Anosov.

EXISTENCE OF MARKOV PARTITIONS

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$



transition matrix

$$T = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

admissible words $\mathcal{L}_T \subseteq \{a, b, c, d, e\}^{\mathbb{Z}}$
semi-conjugacy $(\mathcal{L}_T, \sigma) \to (\mathbb{T}^2, (\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}))$
periodic admissible words $\to$ periodic orbits of $(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix})$

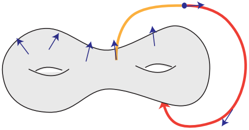
Theorem (Ratner 1968)

Anosov flows in dimension 3 admit Markov finite partitions.

Global sections

Definition

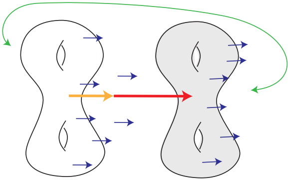
Global section for (M, φ) : compact surface S with no boundary such that



1. *S is embedded in M,*
2. *S is transverse to $X := \frac{d}{dt}(\varphi^t)|_{t=0}$*
   ($\iff S \pitchfork X \iff T_pM = T_pS \oplus \langle X(p) \rangle$),
3. *S cuts all orbits in bounded time*
   ($\iff \exists T > 0, \varphi^{[0,T]}(S) = M$).

$f : S \to S$ first-return map
up to reparametrization

$$\begin{aligned}(M, \varphi) &\simeq (S \times [0,1], \varphi_{\text{sus}})/_{(x,1)\sim(f(x),0)} \\ &= (M_f, \varphi_{\text{sus}})\end{aligned}$$

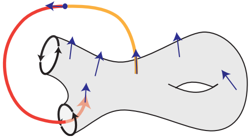


Birkhoff section

Definition

Birkhoff section for (M, φ) *:*

compact surface S *with boundary such that*



1. $\operatorname{int} S$ *is embedded in* M*,*
2. $\operatorname{int} S$ *is transverse to* $X := \frac{d}{dt}(\varphi^t)|_{t=0}$
   $(\iff \operatorname{int}(S) \pitchfork X$
   $\iff T_pM = T_pS \oplus \langle X(p) \rangle$ *si* $p \in \operatorname{int}(S))$*,*
3. ∂S *is a finite collection fo periodic orbits of* φ^t *(*$\iff \partial S /\!\!/ X$*),*
4. S *cuts all orbits in bounded time*
   ($\iff \exists T > 0, \varphi^{[0,T]}(S) = M$*).*

Up to reparametrization

$(M \setminus \partial S, \varphi) \simeq (\operatorname{int}(S) \times [0,1], \varphi_{\text{sus}})/_{(x,1)\sim(f(x),0)}$

$\to$ open book decomposition of M

$\to$ Dehn-Goodman-Fried surgery on ∂S (with appropriate coefficients) yields $(M_f, \varphi_{\text{sus}})$

Birkhoff sections and almost equivalence

S Bikhoff section for (M, X) with first-return map f
$\to$ Goodman-Fried surgery on ∂S yields $(M_f, \varphi_{\text{sus}})$

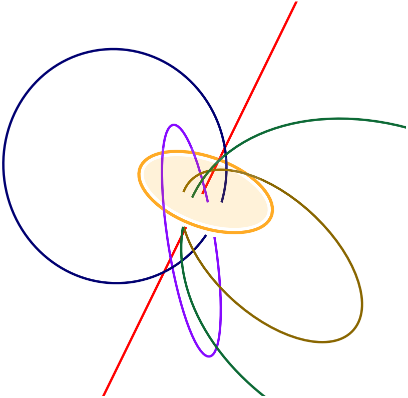
Definition

$(M_1, \varphi_1), (M_2, \varphi_2)$ are almost-equivalent if $\exists \gamma_1, \dots, \gamma_n$ periodic orbits of φ_1, $r_1, \dots, r_n \in \mathbb{Q}$ s. t. Dehn-Goodman-Fried surgery on $((\gamma_1, r_1), \dots, (\gamma_n, r_n))$ yields (M_2, φ_2).

S Birkhoff section for (M, φ) with first-return map $f : S \to S$
$\to$ (M, φ) almost-equivalent to $(M_f, \varphi_{\text{sus}})$.

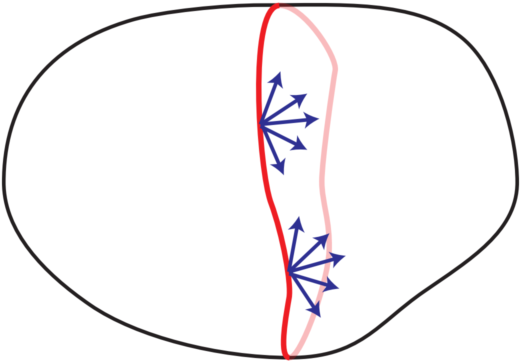
BIRKHOFF SECTION: EXAMPLE 1 - HOPF FLOW ON $\mathbb{S}^3$

$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$
$\varphi^t_{\mathrm{Hopf}}(z_1, z_2) = (e^{it}z_1, e^{it}z_2)$



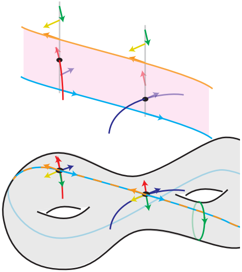
$(\mathbb{S}^3, \varphi_{\mathrm{Hopf}})$ almost-equivalent to $(\mathbb{D}^2 \times \mathbb{S}^1, e^{i2\pi t})$

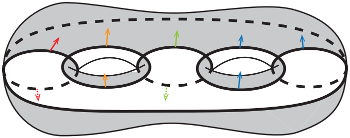
Birkhoff section: example 2 - geodesic flow on $T^1\mathbb{S}^2$ with positive curvature (Poincaré-Birkhoff)

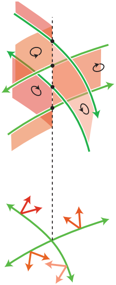


S is an annulus

Birkhoff section: example 3 - geodesic flow on a hyperbolic surface (Birkhoff-Fried)



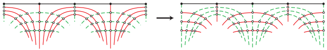




S is a torus

first-return is pseudo-Anosov

first-return: $\begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix}$



$\Longrightarrow \ (\mathrm{T}^1\Sigma_2, \varphi_{\mathrm{geod}})$

almost-equivalent to $(M_{\begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix}}, \varphi_{\mathrm{sus}})$

Fried's theorem

Theorem (Fried 1982)

Every transitive Anosov flow admits a Birkhoff section.

Corollary

Every transitive Anosov flow is almost-equivalent to the suspension flow of some pseudo-Anosov homeomorphism of some surface.

Question (Fried)

Does every transitive Anosov flow admit a genus-one Birkhoff section?

Question (Ghys)

Are all transitive Anosov flows pairwise almost-equivalent?

RESULTS

Theorem (Minakawa 2013?)

$\forall A \in \mathrm{SL}_2(\mathbb{Z}), \mathrm{tr}(A) \geq 3$, $(M_A, \varphi_{\mathrm{sus}}) \overset{a.e.}{\longleftrightarrow} (M_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)}, \varphi_{\mathrm{sus}})$.

Corollary

All transitive Anosov flows (with orientable foliations) admitting a genus-one Birkhoff section are almost-equivalent.

Theorem (Dehornoy-Shannon 2019)

For every hyperbolic 2-orbifold Σ, $(T^1\Sigma, \varphi_{\mathrm{geod}})$ *admits a genus-one Birkhoff section.*

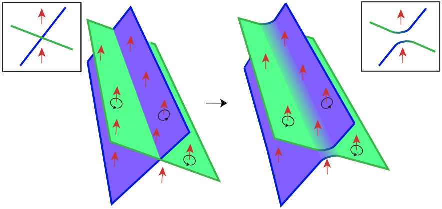
Theorem (Dehornoy 2020)

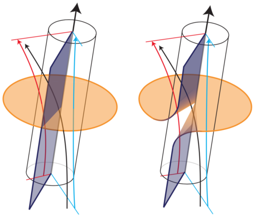
If a transitive Anosov flow (w. orient. foliations) admits a **genus-two** *Birkhoff section, then it admits a* **genus-one** *Birkhoff section.*

FRIED SUM OF TRANSVERSE SURFACES

S_1, S_2 transverse to X,

$S_1 \overset{F}{\cup} S_2 := S_1 \cup S_2$ desingularized transversally to the flow





$S_1^\circ := S_1 \setminus (\partial S_1 \cup \partial S_2), S_2^\circ := S_2 \setminus (\partial S_1 \cup \partial S_2)$

Lemma

$$\chi((S_1 \overset{F}{\cup} S_2)^\circ) = \chi(S_1^\circ) + \chi(S_2^\circ)$$

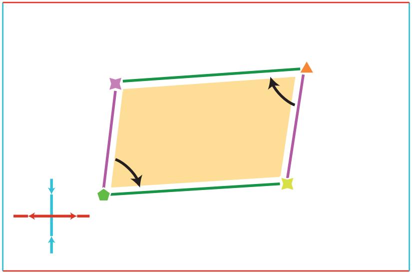
FRIED'S PANTS

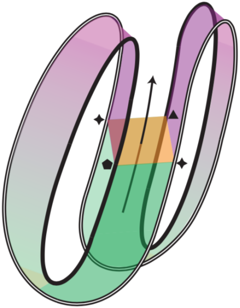
(M, φ) transitive Anosov flow

$R_1, \ldots, R_k$ Markov partition

periodic orbits $\leftrightarrow$ admissible periodic words $\subseteq \{1, \ldots, k\}^{\mathbb{Z}}$

$(aw_1)^{\mathbb{Z}}, (aw_2)^{\mathbb{Z}}$ periodic admissible words
$\implies (aw_1aw_2)^{\mathbb{Z}}$ periodic admissible





Fried's theorem

Theorem (Fried)

Every (M, φ) 3-dim transitive Anosov flow admits a Birkhoff section

Proof.

- Ratner: $\exists$ finite Markov partition $R_1, \dots, R_k$
- $\forall$ periodic point x in the interior of some R_i, $\exists \mathcal{P}_x$ Fried pair of pants containing x in the interior
  Actually one can prescribe the periodic orbits on the boundary of the rectangles
  $\implies \exists \mathcal{P}_x$ trough all points of M
- Compactness of $M \implies$ finite union $\cup_x \mathcal{P}_x$ intersects all orbits
- The Fried sum $\mathcal{P}_{x_1} \overset{F}{\cup} \dots \overset{F}{\cup} \mathcal{P}_{x_n}$ is a Birkhoff section.

$\square$

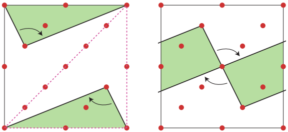
Minakawa's theorem

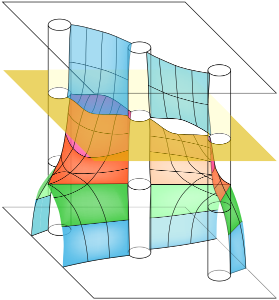
Theorem (Minakawa)

$\forall A \in \mathrm{SL}_2(\mathbb{Z}), \mathrm{tr}(A) \geq 3$, $(M_A, \varphi_{\mathrm{sus}}) \overset{a.e.}{\longleftrightarrow} (M_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)}, \varphi_{\mathrm{sus}})$.

Proof.

Find a
nice pair
of pants





Take the Fried sum $(\mathbb{T}^2 \times \{2/3\}) \overset{F}{\cup} \mathcal{P}$.

$$\begin{aligned} \chi((\mathbb{T}^2 \times \{2/3\})^\circ) &= -3 \\ \chi(\mathcal{P}^\circ) &= -1 \\ \chi((\mathbb{T}^2 \times \{2/3\}) \overset{F}{\cup} \mathcal{P})^\circ) &= -4 \end{aligned}$$

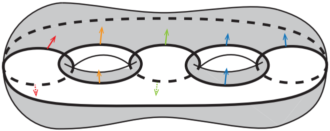
first-return map has less fixed points
(actually if $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B$ with B positive, first-return $= B$).

BIRKHOFF SECTIONS FOR GEODESIC FLOWS

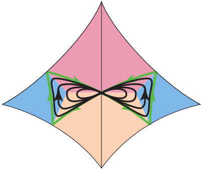
Theorem (Dehornoy-Shannon)

For every hyperbolic 2-orbifold Σ, $(T^1\Sigma, \varphi_{\text{geod}})$ admits a genus-one Birkhoff section.

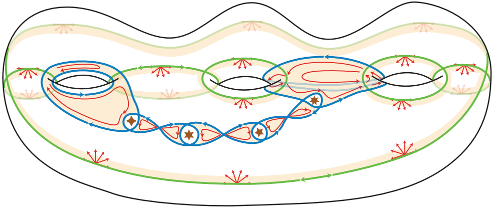
$\Sigma_{g;()}$ genus-g surface



$\Sigma_{0;(p,q,r)}$ genus-0 surface with 3 cone points



$\Sigma_{g;(p_1,\ldots,p_n)}$ general hyperbolic orbifold



Decreasing the genus of Birkhoff sections

Theorem (Dehornoy)

If (M, φ) is a transitive Anosov flow with orientable invariant foliations and with a genus-2 Birkhoff section 2, then it admits a genus-1 Birkhoff section.

Thanks to Minakawa, it is almost equivalent to $(M_{\left(\begin{smallmatrix}2 & 1\\ 1 & 1\end{smallmatrix}\right)}, \varphi_{\mathrm{sus}})$.