A quantum obstruction to purely cosmetic surgeries

I Introduction: Dehn surgery and cosmetic surgeries

$K \subset S^3$ a knot

μ meridian

λ longitude zero winding number

Def. Dehn surgery $E_K(\frac{p}{q})$:

$$\forall \tfrac{p}{q} \in \mathbb{Q} \cup \{\infty\}, \quad E_K(\tfrac{p}{q}) = \underbrace{S^3 \setminus N(K)}_{E_K} \sqcup D^2 \times S^1$$

$S^1 \times \{1\} = p\mu + q\lambda$

Question: Rigidity of Dehn surgeries?

Thm: (Gordon-Luecke 89)

$\forall K$ nontrivial knot, $\forall s \neq \infty$

$$E_K(s) \neq S^3.$$

Def: Cosmetic surgeries

Let $s \neq s' \in \mathbb{Q} \cup \{\infty\}$, K a knot

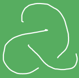
(i) If $E_K(s) \simeq E_K(s')$ then $\{s, s'\}$ is a purely cosmetic surgery pair.

positively homeo

(PCS)

(ii) If $E_K(s) \simeq \overline{E_K(s')}$ —— chirally cosmetic surgery pair

Ex: (1) $K \subset S^3$ achiral knot then $E_K(s) \simeq \overline{E_K(-s)}$

(2) $K =$  $\xrightarrow[\text{(Mathieu 92)}]{} E_K\left(\frac{18k+9}{3k+1}\right) \simeq \overline{E_K\left(\frac{18k+9}{3k+2}\right)}$ $k \geq 1$

Cosmetic surgery conjecture (Gordon 90)

No non-trivial knot admit a PCS.

II Some obstructions

Thm (Boyer-Lines 90)

If $\Delta_K''(1) \neq 0$ then K has no PCS.

(Tool: Surgery formula for Casson-Gordon invariants)

Thm: (Hanselman 2019)

If K has a PCS with slopes $s \neq s'$ then

(a) $s = -s'$ and $s \in \{\pm 2\} \cup \{\pm 1/k,\ k \geq 1\}$

(b) If $s \in \{\pm 2\}$ then $g(K) = 2$ (Seifert genus)

(c) If $s \in \{\pm \frac{1}{k}\}$ then

$$|k| \leq \frac{th(K) + 2g(K)}{2g(K)(g(K)-1)}$$

(d) The conjecture is true for knots with ≤ 16 crossings.

$1 + th(K) =$ number of non-zero "diagonals" of $\widehat{HF}(K)$

$$\widehat{HF}(K) = \bigoplus_{i,j} H_{i,j}$$

$th(K) = 0$ for alternating knots.

We will use $SO(3)$-Reshetikhin-Turaev invariants to get:

Thm (D 2021): If K has a PCS $\{\pm s\}$ then

(a) $s = \pm \frac{1}{5k}$, $k \geq 1$ unless $J_K(e^{\frac{2i\pi}{5}}) = 1$

(b) For $p \geq 5$ odd prime, $s = \pm \frac{1}{pk}$ unless if

$$Z_p(E_K) = \begin{pmatrix} 1 \\ -[2] J_{K,1}(\zeta_p) \\ \vdots \\ (-1)^{\frac{p-3}{2}} \left[\frac{p-3}{2}\right] J_{K,\frac{p-3}{2}}(\zeta_p) \end{pmatrix}$$ is orthogonal to

a vector in some explicit finite set $F_p \subset \mathbb{C}^{\frac{p-1}{2}}$ with $|F_p| \leq p^{\frac{p+1}{2}}$

ζ_p p-th root of 1

$$[n] = \frac{\zeta_p^{\,n} - \zeta_p^{\,-n}}{\zeta_p - \zeta_p^{\,-1}}$$

III SO(3)-WRT invariants

$p \geq 5$ odd prime. $\zeta_p = e^{\frac{2i\pi}{p}}$ $\quad A = -e^{\frac{i\pi(p+1)}{2p}}$

Def: Kauffman bracket $\langle L \rangle$

$L \subset S^3$ link

(1) $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

(2) $\langle L \sqcup \bigcirc \rangle = (-A^2 - A^{-2}) \langle L \rangle$

(3) $\langle \emptyset \rangle = 1$.

Def: Colored Kauffman bracket

$L = L_1 \sqcup \dots \sqcup L_n$ $\qquad p_i \in \mathbb{Z}[A^{\pm 1}][z]$

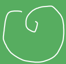
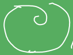
$\langle L, p_1, \dots, p_n \rangle$ is a framed link invariant

Def: SO(3) invariants

Assume $M = S^3(L)$

Define $Z_p(M) = \eta_p^{1+n(L)} \kappa_p^{\sigma(L)} \langle L, \omega_p, \dots, \omega_p \rangle$

- ω_p = Kirby color
- $n(L)$ = number of comp of L

- $\sigma(L)$ = signature
- $\eta_p = Z_p(S^3) = \dfrac{\zeta_p^2 - \zeta_p^{-2}}{\sqrt{-p}}$
- $\langle$  $, w_p \rangle = \eta_p^{-1} \kappa_p$ $\quad \langle$  $, w_p \rangle = \eta_p^{-1} \kappa_p^{-1}$

__Fact__: The invariants Z_p can be extended to TQFTs.

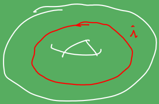
III Defn surgery formulas for Z_p

TQFT properties:

(1) Σ surface $\rightsquigarrow Z_p(\Sigma)$ $\mathbb{C}$-finite dim Hermitian vector space with $\langle \, , \rangle$

$\sum_i \lambda_i M_i = 0$ if $\sum_i \lambda_i Z_p(M_i \sqcup_\Sigma N) = 0$ $\forall N$ with $\partial N = \Sigma$

__ex__: If $\Sigma = T^2$ $\quad Z_p(T^2) = \mathrm{Span}\{e_0, \dots, e_{\frac{p-3}{2}}\}$

$e_i =$ 

(2) If $\partial M = \Sigma$ then $Z_p(M) \in Z_p(\Sigma)$

ex: $E_K = S^3 \setminus K$

$$Z_p(E_K) = \eta_p \begin{pmatrix} 1 \\ -[2]J_{K,1}(\zeta_p) \\ \vdots \\ (-1)^{\frac{p-3}{2}}\left[\frac{p-3}{2}\right]J_{K,\frac{p-3}{2}}(\zeta_p) \end{pmatrix}$$

(3) If $M = M_1 \cup_{\Sigma} M_2$ then $Z_p(M) = \kappa_p^{-\mu}\langle Z_p(M_1), Z_p(M_2)\rangle$

$\mu \in \mathbb{Z}$ Maslov index.

(4) We have quantum reps

$$\phi \in MCG(\Sigma) \longrightarrow \rho_p(\phi) \in PEnd(Z_p(\Sigma))$$

ex: $MCG(T^2) \simeq SL_2(\mathbb{Z})$ generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\rho_p(T)\, e_i = (-A)^{i^2-1}\, e_i$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_p(S)\, e_i = \sum_{0 \leq j \leq \frac{p-3}{2}} (-1)^{i+j}\left[(i+1)(j+1)\right] e_j$$

Rk: $\rho_p(T^p) = Id.$

Prop: We have

$$Z_p\left(E_K\left(\frac{1}{k}\right)\right) = \left\langle Z_p(E_K) , \rho_p(ST^{-k}S)\, e_0 \right\rangle$$

$$Z_p(E_K(2)) = K_p^{d_p} \left\langle Z_p(E_K), \rho_p(T^2S)\, e_0 \right\rangle$$

$$Z_p(E_K(-2)) = K_p^{d_p+2} \left\langle Z_p(E_K), \rho_p(T^{-2}S)\, e_0 \right)$$

Pf of main thm:

If $E_K\left(\frac{1}{k}\right) \simeq E_K\left(-\frac{1}{k}\right)$ then

$$Z_p(E_K) \perp \rho_p\underbrace{(ST^{-k}S)e_0 - \rho_p(ST^{k}}S)e_0$$

depends only on k mod p

$= 0$ si $k \equiv 0$ mod p

$\neq 0$ else

If $E_K(2) \simeq E_K(-2)$

$$Z_p(E_K) \perp \rho_p(T^2S)e_0 - K_p^2\, \rho_p(T^{-2}S)e_0 .$$

If $p=5$ we have $Z_p(E_K) = \eta_5 \begin{pmatrix} 1 \\ -[2]J_K(\zeta_5) \end{pmatrix}$

and

$$\big(\rho_5(ST^{-1}S) - \rho_5(STS)\big)e_0$$
$$\big(\rho_5(ST^{-2}S) - \rho_5(ST^{2}S)\big)e_0$$
$$\big(\rho_5(T^{2}S) - \rho_5(T^{-2}S)\big)e_0$$

are all colinear to $\begin{pmatrix} [2] \\ 1 \end{pmatrix}$

$$\Longrightarrow \quad J_K(\zeta_5) = 1 \quad \text{or} \quad p = \frac{1}{5k}$$

IV Numerical applications

Prop: If K has a PCS and ≤ 31 crossings then $J_K(\zeta_5) = 1$

Pf: Hanselman ineq $\Longrightarrow$ if you have $\pm\frac{1}{5k}$ PCS then

$$th(K) \geq 16$$

and $th(K) \leq \frac{c(K)}{2}$

Corollary: No knot with ≤ 17 crossings has a PCS.

- 97 prime knots with ≤ 17 crossing with $J_K(\zeta_5)=1$
- Only 2 have also have $\Delta_K''(1) = J_K''(1) = 0$

  and $J_K'''(1) = 0$

Knots $14_{nh00042}$ and $16_{ah000209}$ (Regina not)