Triviality of the J_4-equivalence relation among homology 3-spheres

Quentin Faes

Institut de Mathématiques de Bourgogne

7/04/2022

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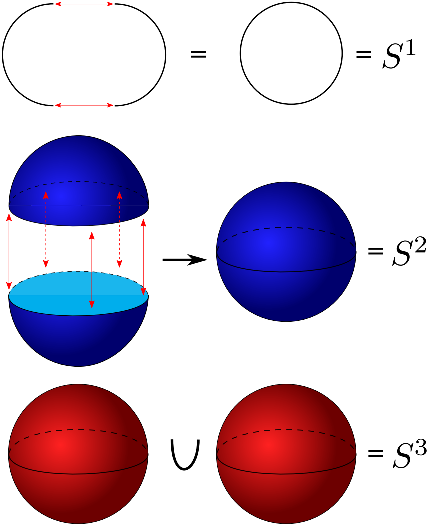
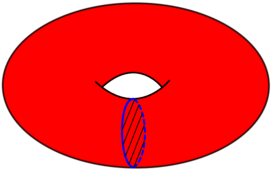
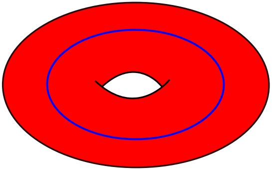
Gluing



Heegaard splittings

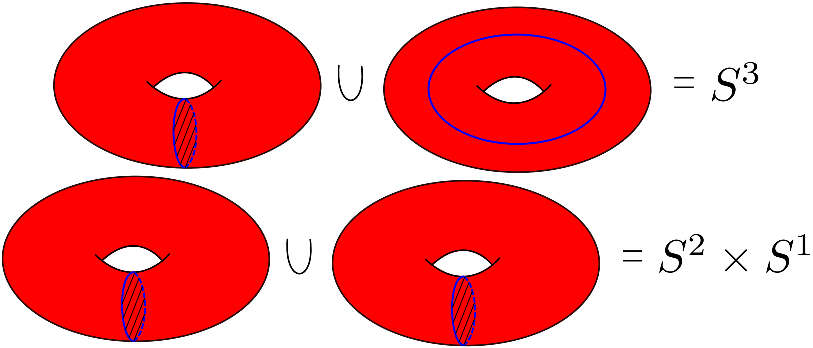


$\cup$



$= S^3$

Heegaard splittings



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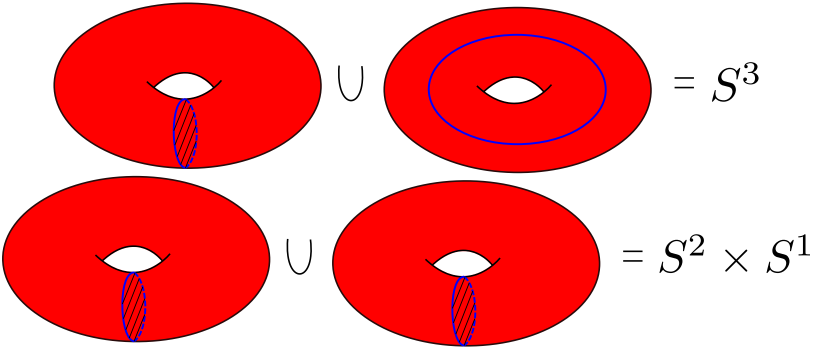
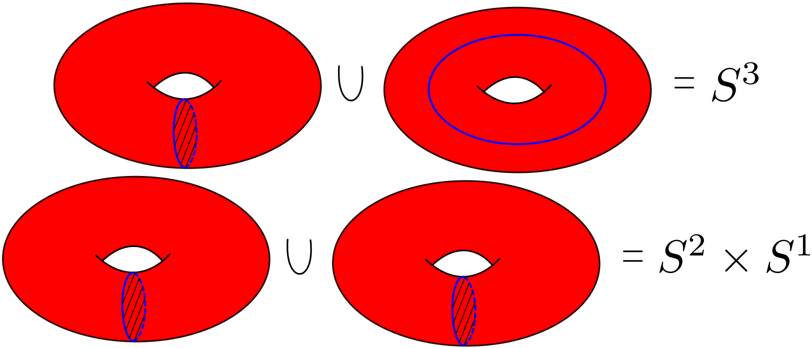


The way we glue matters, up to isotopy.

Heegaard splittings



The way we glue matters, up to isotopy.

Study of 3-manifolds $\leftrightsquigarrow$ Study of mapping class groups

Strategy

- Questions about 3-manifolds $\rightsquigarrow$ Questions about the mapping class group (via splittings).

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- Compute.

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What happens if we restrict the set of gluing maps we are allowed to use ?

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Theorem (F.2021)

Any homology 3-sphere admits a Heegaard splitting such that the gluing map acts trivially on the 4-th nilpotent quotient of the fundamental group of the gluing surface.

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What happens if we restrict the set of gluing maps we are allowed to use ?

Theorem (F.2021)

Any homology 3-sphere admits a Heegaard splitting such that the gluing map acts trivially on the 4-th nilpotent quotient of the fundamental group of the gluing surface.

i.e. "J_4-equivalence is trivial among homology 3-spheres".

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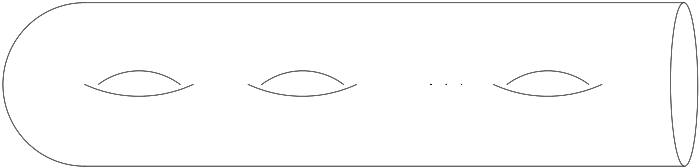
Framework

- $\Sigma_{g,1}$ is a surface of genus g with one boundary component. When one caps the boundary with a disk, one gets a closed surface Σ_g.



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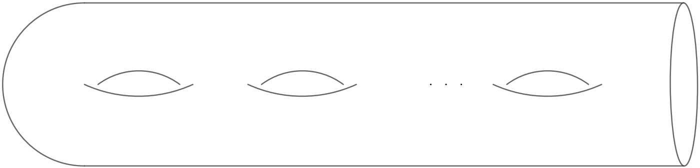

- $\mathcal{M}_{g,1}$ is the mapping class group of the surface:

$$\mathcal{M}_{g,1} := \mathrm{Homeo}^{+,\partial}(\Sigma_{g,1})/\textit{(isotopies fixing the boundary)}$$

Framework

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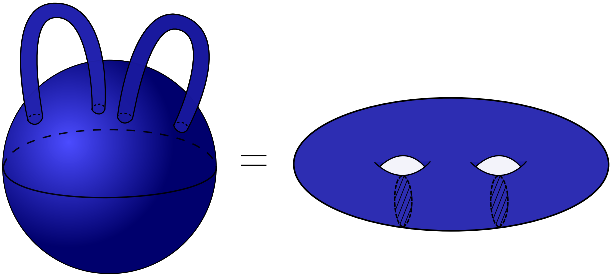

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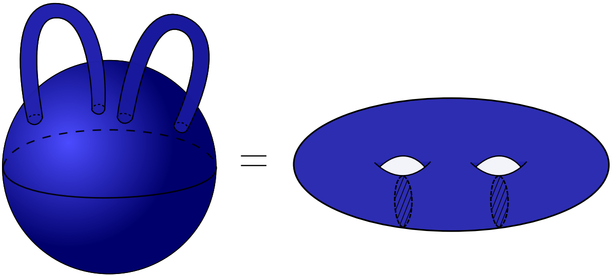

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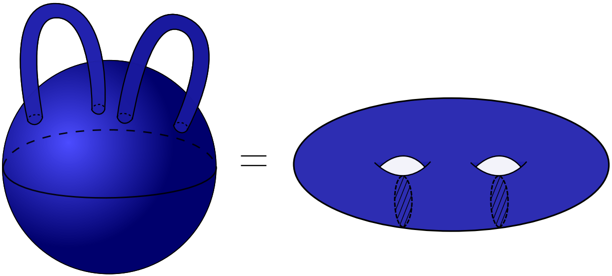

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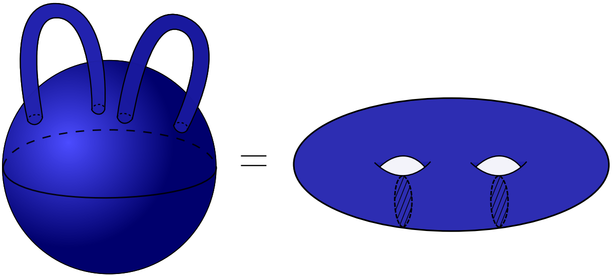

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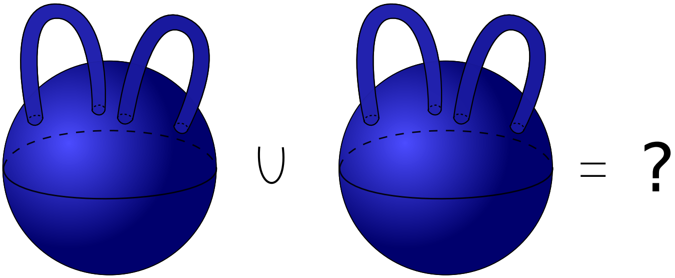
- $\mathcal{A}_{g,1}$ imbeds in $\mathcal{M}_{g,1}$ (the imbedding depends on the identification $\partial V_g \simeq \Sigma_g$).

Reidemeister-Singer Theorem

Theorem

Any closed connected compact oriented 3-manifold can be obtained by gluing two handlebodies together along their boundary.



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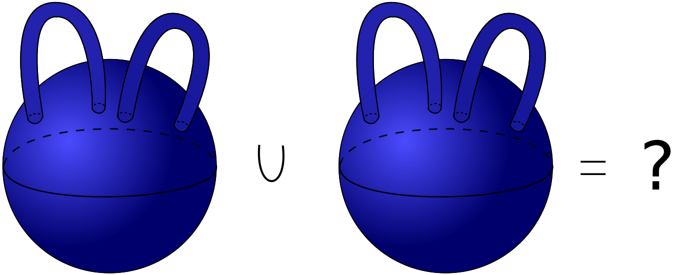

Any 3-manifold can be described by specifying a homeomorphism of some surface up to isotopy.

Reidemeister-Singer Theorem

Fix for all $g \geqslant 0$, compatible Heegaard splittings of the 3-sphere:

$$S^3 = V_g \bigcup_{\iota_g} (-V_g).$$

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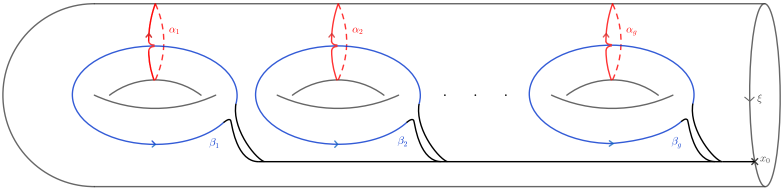
$\mathcal{A}_{g,1}$ = homeo extending to the "inner" handlebody V_g.
$\mathcal{B}_{g,1}$ = homeo extending to the "outer" handlebody $(-V_g)$.
There is a bijection:

$$\mathcal{R} : \lim_{g \to +\infty} \mathcal{A}_{g,1} \backslash \mathcal{M}_{g,1} / \mathcal{B}_{g,1} \longrightarrow \mathcal{V}(3)$$
$$\varphi \longmapsto S^3_\varphi.$$

Dehn-Nielsen Theorem

The mapping class group acts on the fundamental group $\pi := \pi_1(\Sigma_{g,1})$ of the surface.



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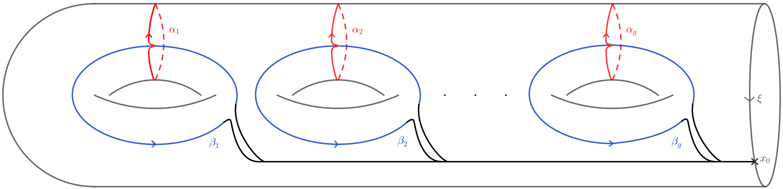

Theorem

There is a faithful representation

$$\mathcal{M}_{g,1} \xrightarrow{\rho} \mathrm{Aut}(\pi)$$

The Johnson filtration

- Let N_k be the k-th nilpotent quotient of π : $N_k := \pi/\Gamma_{k+1}\pi$. $\mathcal{M}$ acts both on π and all its nilpotent quotients:

$$\mathcal{M} \xrightarrow{\rho} \mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(N_k)$$

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- $\mathcal{I}_{g,1} := J_1$ is the *Torelli group*, $\mathcal{K}_{g,1} := J_2$ is the *Johnson Kernel*.
- Fundamental example: $\Gamma_k \mathcal{I}_{g,1} \subset J_k$

Problem

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Theorem (F., 2021)

Any homology 3-sphere is in the image of the restriction of the map $\mathcal{R}$ to $\lim\limits_{g \to +\infty} \mathcal{A}_{g,1} \backslash J_4(\Sigma_{g,1}) / \mathcal{B}_{g,1}$.

Reformulation in terms of equivalence relation

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- The inclusions $\Gamma_{k+1}\mathcal{I}_{g,1} \subset \Gamma_k \mathcal{I}_{g,1} \subset J_k \supset J_{k+1}$ induce the following organization:

$$\begin{array}{ccccccccccc}
Y_1 & \Longleftarrow & Y_2 & \Longleftarrow & Y_3 & \Longleftarrow & \cdots & Y_k & \Longleftarrow & Y_{k+1} & \Longleftarrow & \cdots \\
\| & & \Downarrow & & \Downarrow & & & \Downarrow & & \Downarrow & & \\
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Results

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This is actually equivalent to

Theorem (F,2001)

There exists a homology 3-sphere which is J_4-equivalent to S^3 and has Casson invariant equal to 1.

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The Casson invariant

- The Casson invariant λ is an invariant of ZHS valued in $\mathbb{Z}$, defined by counting irreducible representations of the fundamental group of the homology 3-sphere into $\mathrm{SU}(2)$.

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- When performing a knot surgery, its variation can be computed from the Alexander polynomial of the knot.

Morita's formula

- Let $j : \Sigma_{g,1} \hookrightarrow S^3$ be an embedding of image the standard Heegaard surface, then we have a map $\varphi \mapsto S^3(j, \varphi)$ from $\mathcal{M}_{g,1}$ to the set of 3-manifolds.

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It is *not* a homomorphism.

Morita's formula

We will now restrict to $\mathcal{K} = J_2$. The restriction of λ_j to $\mathcal{K}$ is the sum of two homomorphisms.

$$\lambda_j \,|_{\mathcal{K}} = \frac{-d}{24} + q_j \circ \tau_2.$$

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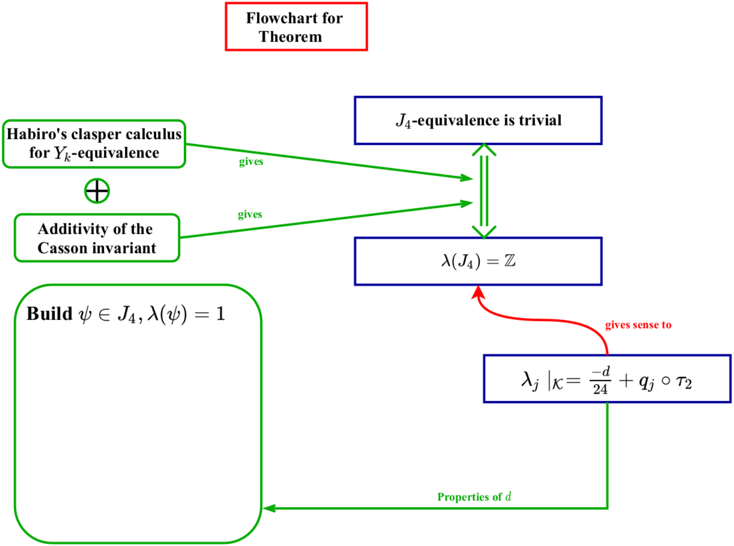
- The Casson invariant induces $\lambda : J_4 \to \mathbb{Z}$.


Flowchart for Theorem
Habiro's clasper calculus for Y_k-equivalence
Additivity of the Casson invariant
gives
gives
J_4-equivalence is trivial
$\lambda(J_4) = \mathbb{Z}$
gives sense to
$\lambda_j \mid_{\mathcal{K}} = \frac{-d}{24} + q_j \circ \tau_2$
Properties of d
Build $\psi \in J_4, \lambda(\psi) = 1$

Let us show:

$$\lambda(J_4) = \mathbb{Z}$$

$$\Updownarrow$$

$\exists M \in ZHS$ s.t. M is J_4-equivalent to S^3 and $\lambda(M) = 1$

$$\Downarrow$$

J_4-equivalence is trivial among ZHS^3

Proof.

Goussarov-Habiro clasper calculus imply that *two homology 3-spheres are Y_4-equivalent if and only if they have the same Casson invariant*.

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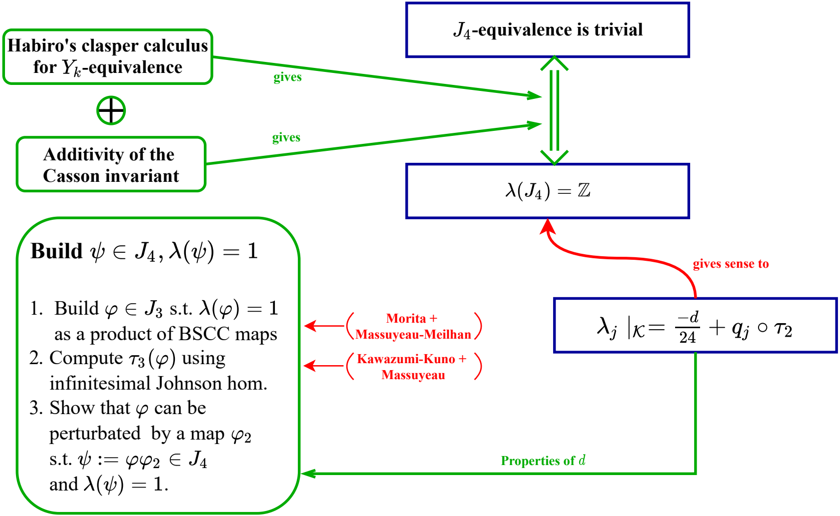
thus,

$$M \overset{Y_4}{\sim} S^3 \# P^k \overset{J_4}{\sim} S^3 \# {S^3}^k = S^3$$

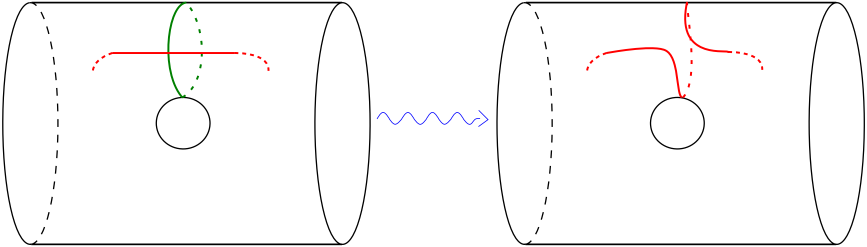
□

Flowchart for Theorem 2



Elements of the Johnson filtration

Dehn twists generate the mapping class group:



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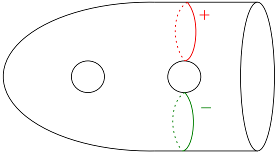
- $\mathcal{I}_{g,1} = J_1$ is generated by bounding pairs. ($g \geqslant 3$)



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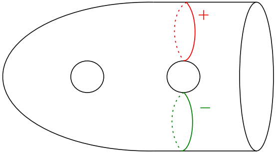

- $\mathcal{K}_{g,1} = J_2$ is generated by BSCC maps (Dehn twists along bounding simple closed curves). $(g \geqslant 2)$

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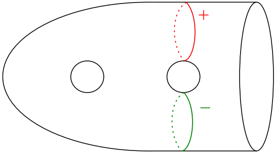

- $\mathcal{K}_{g,1} = J_2$ is generated by BSCC maps (Dehn twists along bounding simple closed curves). ($g \geqslant 2$)

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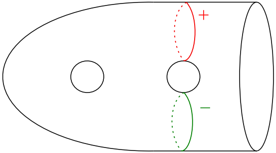

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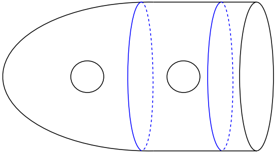

- No Dehn twist belong to J_3.

A slide with blackboxes

- To build an element deeper, one has to multiply Dehn twists, and check that the successive Johnson homomorphisms $\tau_k : J_k \to D_k(H)$ vanish. (because $\mathrm{Ker}(\tau_k) = J_{k+1}$).

A slide with blackboxes

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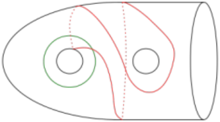
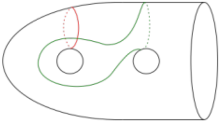
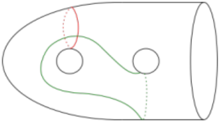
- $\rightsquigarrow$ A formula by Kawazumi and Kuno computes the action of a Dehn twist on the completion of the group algebra of the fundamental group.
- We do so by implementing it in a SageMath computer program.

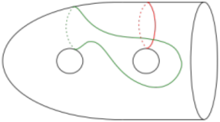
Building an element of J_4 with Casson invariant 1

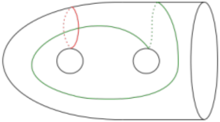
$$\psi := T_{\gamma_2}^{-3}\, T_1^{-1}\, T_2^{-1}\, T_3^2\, T_4^2\, T_5\, T_6^{-1}\, T_7^{-1}\, T_8\, T_9^{-1}\, T_{10}\, T_{11}^{-1}\, T_{12}^{-1}\, T_{13}\, T_{s1}^7\, T_{s2}^2$$

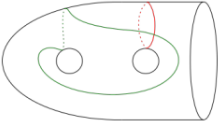

(a) Spine of T_1

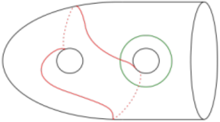

(b) Spine of T_2

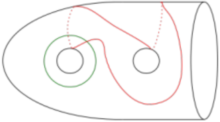

(c) Spine of T_3

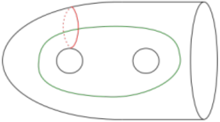

(d) Spine of T_4


(e) Spine of T_5


(f) Spine of T_6


(g) Spine of T_7


(h) Spine of T_8


(i) Spine of T_9

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- $\exists \psi_2 \in [\mathcal{K}, \mathcal{I}]$ s.t. $\varphi := \psi\psi_2 \in \mathrm{Ker}(\tau_3) = J_4$ and $\lambda(\varphi) = 1$.

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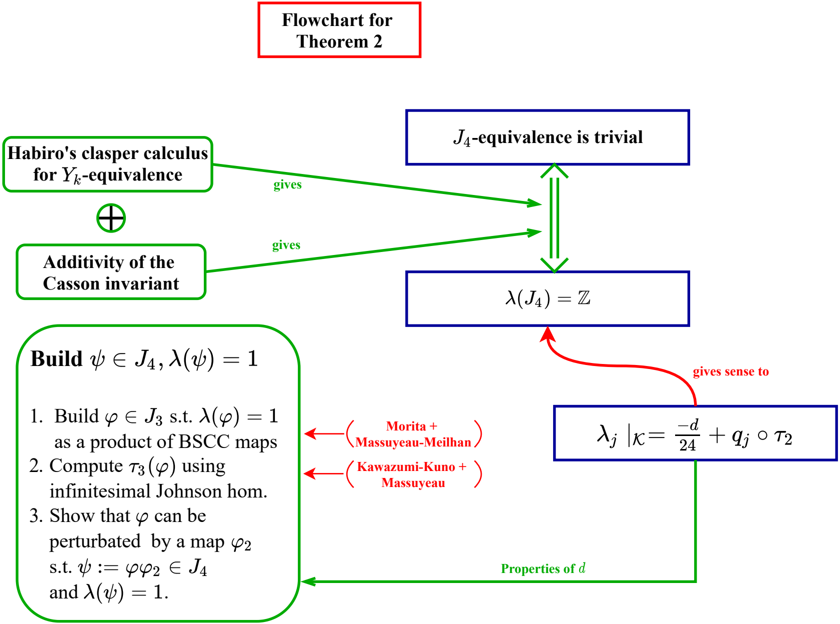
Flowchart for Theorem 2
J_4-equivalence is trivial
Habiro's clasper calculus for Y_k-equivalence
gives
$\bigoplus$
Additivity of the Casson invariant
gives
$\lambda(J_4) = \mathbb{Z}$
Build $\psi \in J_4, \lambda(\psi) = 1$
1. Build $\varphi \in J_3$ s.t. $\lambda(\varphi) = 1$ as a product of BSCC maps
2. Compute $\tau_3(\varphi)$ using infinitesimal Johnson hom.
3. Show that φ can be perturbated by a map φ_2 s.t. $\psi := \varphi\varphi_2 \in J_4$ and $\lambda(\psi) = 1$.
(Morita + Massuyeau-Meilhan)
(Kawazumi-Kuno + Massuyeau)
gives sense to
$\lambda_j \, |_{\mathcal{K}} = \frac{-d}{24} + q_j \circ \tau_2$
Properties of d

Perspectives and remarks

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- The element $\varphi \in J_4$ constructed here is not a commutator of elements of the lower terms of the filtration. The comparison $\Gamma_k \mathcal{I}_{g,1} \subset J_k$ is a central question.

Thank you for listening !

Selective bibliography

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