[K-OS] Knot Online Seminar 14.1.21

Note Title 1/13/2021

Braids, quasimorphisms, and slice-Bennequin inequalities

(based on work in progress, featuring joint work with Hubbard (2017))

I) The writhe (wr) & the fractional Dehn twist coefficent: $B_n \to \mathbb{R}$

II) The closure of braids & slice-Bennequin inequalities

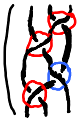
III) Homogenization of concordance homomorphisms for knots

IV) Questions

1-page summary

Writhe: $wr: B_n \longrightarrow \mathbb{Z} \subset \mathbb{R}$

$\beta \longmapsto \#\mathrm{poscr}(\beta) - \#\mathrm{negcr}(\beta)$

$wr\Big($  $\Big)$ $3 - 1 = 2$

- Grouphomomorphism ($wr(\alpha\beta) = wr(\alpha) + wr(\beta)$) unique up to scaling
- $wr(\beta) \leq 2g_4(\hat\beta) - 1 + n$ for $\beta \in B_n$ with $\hat\beta$ a knot.

fdtc: $\omega: B_n \longrightarrow \frac{1}{n}\mathbb{Z} \subset \mathbb{R}$

$\omega(\beta) =$ "amount of twisting of β"

- homogeneous quasimorphism (hqm)

Thm A: Characterization of ω among the many hqm $f: B_n \to \mathbb{R}$.

Thm B: $\omega(\beta) \leq 2g_4(\hat\beta) - 1 + n$

for $\beta \in B_n$ with $\hat\beta$ a knot.

I) Artin's braid groups:

B_1 trivial, $B_2 \cong \mathbb{Z}$

$n \geq 2$:

$$B_n = \left\langle a_1, \dots, a_{n-1} \;\middle|\; \begin{array}{ll} a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} & \\ a_i a_j = a_j a_i & |i-j| \geq 2 \end{array} \right\rangle$$

Facts
- $\mathrm{Center}(B_n) = \langle \Delta^2 \rangle$ for $n \geq 3$.
- $B_n \twoheadrightarrow Ab(B_n) \cong \mathbb{Z}$

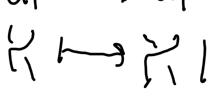
  $a_i \longmapsto 1$

Writhe: $B_n \longrightarrow \mathbb{Z}$

$a_i \longmapsto 1$

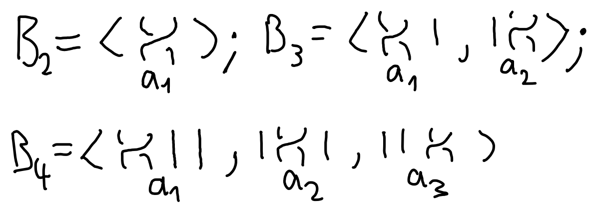
- $B_{n-1} \subset B_n$

  $a_i \longmapsto a_i$



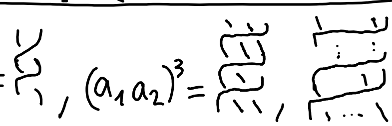
$B_2 = \langle a_1 \rangle$; $B_3 = \langle a_1, a_2 \rangle$;

$B_4 = \langle a_1, a_2, a_3 \rangle$



Rel: $a_1 a_2 a_1 = a_2 a_1 a_2$, $a_1 a_3 = a_3 a_1$

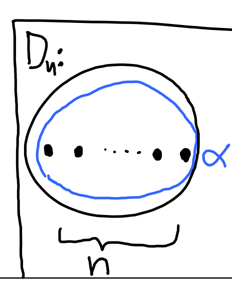
Δ^2: $a_1^2 =$, $(a_1 a_2)^3 =$,



$B_n = MCG(D_n) = \pi_0(\mathrm{Diffeo}^+(D_n))$

$\Delta^2 \longmapsto D_\alpha$

$D_\alpha =$ pos. Dehn twist along α



Fdtc: $w: B_n \longrightarrow \frac{1}{n}\mathbb{Z}$ is
hqm with Defect 1 s.t.

(1) $w(\Delta^2) = 1$
(2) $w(B_{n-1}) = \{0\}$

Ex: B_3: • $w(a_1) = 0 = w(a_2)$

"⤫ | ; | ⤫ does not twist"

• $w(a_1 a_2) = \frac{1}{3}$ $\quad (w((a_1 a_2)^3) = 1)$

$\Longrightarrow |w(a_1 a_2) - w(a_1) - w(a_2)| = \frac{1}{3}$

THM A(*): $\exists!$ hqm $f: B_n \to \mathbb{R}$ with defect ≤ 1 s.t. (1) & (2) hold.

Rem: $\forall n \ge 3 \ \forall \varepsilon > 0 \ \exists$ infinitely many linearly independent hqm $f: B_n \to \mathbb{R}$ with defect $\le 1+\varepsilon$ & (1) & (2) hold.
(Input in proof of Rem: Fact)

Homogeneous quasimorphism (hqm):

G group, $f: G \to \mathbb{R}$:

• homogeneous: $f(g^k) = k f(g) \ \forall g \in G, \forall k \in \mathbb{Z}$

• quasimorphism: $\underbrace{\sup_{g,h \in G} |f(gh) - f(g) - f(h)|}_{D_f = \text{Defect of } f} < \infty$

Fact (Brooks 81, Fujiwara 98, Bestvina-Fujiwara 02, ...)
$n \ge 3$: $\dim(\{f: B_n \to \mathbb{R} \mid f \text{ hqm}\}) = \infty$

fdtc:
where: Gabai-Oertel 89; Malyutin 04; Honda-Kazez-Matić 08; ...
why: • $w(\beta) > 0 \Longrightarrow \beta$ right veering
β right veering $\Longrightarrow w(\beta) \ge 0$
• $w(\beta) > 1 \Longrightarrow \beta$ cannot be destab.

II) Closure: $B_n \longrightarrow$ Link
$\beta \longmapsto \hat{\beta} \subset S^3$

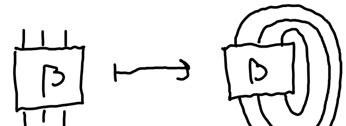
Q: How do braid invariants relate to properties of the closure?

Slice-Bennequin inequality for wr:
$wr(\beta) \le 2 g_4(\hat{\beta}) - 1 + n$ for $\beta \in B_n$ with $\hat{\beta}$ knot. (Rudolph 92)

Key input: loc. Thom conj. (proven by Kronheimer-Mrowka)
$g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$.

(Equivalently: $wr(\beta_{p,q}) = 2 g_4(\widehat{\beta_{p,q}}) - 1 + p$, where $\beta_{p,q} = (a_1 a_2 \cdots a_{p-1})^q \in B_p$.)

Closure of Braids:

$\beta =$  $= \hat{\beta}$

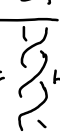
4-genus of knots:

$g_4(K) := \min\{\text{genus}(F) \mid F \subset B^4, \partial F = K\}$

$(2 g_4(K) - 1 = -\chi_4(K))$

• $g_4(K) \le g_3(K)$

Ex for $n=2$:

$\beta_{2,3} = \beta =$  $\longmapsto \hat{\beta} = \partial F \quad F =$ 

• $\chi(F) = 2 - 3 = -1$ • genus$(F) = 1$

What about $w(\beta)$ & $\hat\beta$?

- $|w(\beta)|>1 \Rightarrow \beta$ cannot be destab.

Proof: $\beta = a_n\alpha \in B_{n+1}$, $\alpha\in B_n \subset B_{n+1}$

$$\Rightarrow |w(\beta) - \underbrace{w(a_n)}_{=\,w(a_1)\,\overset{(2)}{=}\,0} - \underbrace{w(\alpha)}_{\overset{(2)}{=}\,0}| \leq 1$$

$\square$

- Fi–Hubbard 17: If $|w(\beta)| > n-1$ for $\beta\in B_n$, then braid index$(\hat\beta)=n$

THM B (F): For all $\beta\in B_n$, $\hat\beta$ knot, $w(\beta) \leq 2g_4(\hat\beta) - 1 + n$.

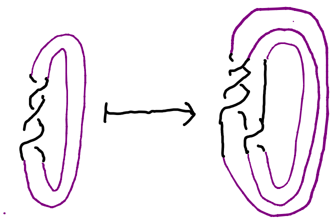
Q: Which hom $f: B_n \to \mathbb{R}$ satisfy a slice-Bennequin inequality?

Ex: $\frac{wr+w}{2}$, $\frac{1}{10}w$, ...

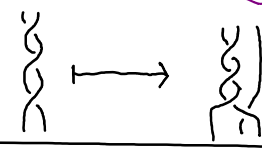
Stabilization:

$S_\pm : B_n \to B_{n+1}$, $\alpha \mapsto a_n^{\pm1}\alpha$

S_+:



S_-:



Markov's Thm: $\hat\alpha$ isotopic to $\hat\beta$ iff α & β are related by Stabilization, Destabilization, and conjugation

Braid index(L) $:= \min\{n \mid \exists\, \beta\in B_n \,\&\, \hat\beta = L\}$

Ex: Braid index(trefoil) $=2$

$\tilde{Q}$: Which hom $f: B_n \to \mathbb{R}$ with (1) & (2) satisfy

- $f(\beta) \leq 2g_4(\hat\beta) + B$?
- $f(\beta) \leq A\, g_4(\hat\beta) + B$?

Lemma. If $f(\beta) \leq A g_4(\hat\beta) + B$ $\forall \beta\in B_n$, $\hat\beta$ knot, then $D_f \leq A(n-1)$

Cor (Lemma & Thm A): If f satisfies (1), (2), & $f(\beta) \leq A\, g_4(\hat\beta) + B$ for $\beta\in B_n$, $\hat\beta$ knot, then $A > \frac{1}{n-1}$.

Conj: If $f(\beta) \leq 2 g_4(\hat\beta) + B$ $\forall \beta, \hat\beta$ knot, & (1) & (2), then $f = w$.

Ex: $\tilde\sigma_s : B_3 \to \mathbb{R}$ $s = e^{i\pi\frac{3}{4}}$

- $\tilde\sigma_s(a_1) = -\frac{3}{4}$ • $\tilde\sigma_\omega(\Delta^2) = -4$

Set $f := 2(\tilde\sigma_s + \frac{3}{4} wr)$

Then:
- (1) & (2) hold
- $f(\beta) \leq (4+3)\, g_4(\hat\beta) + \frac{4+3}{2}(-1+n)$ >2
- $f \neq w \Rightarrow D_f > 1$

(1) $f(\Delta^2) = 1$ (2) $f(B_{n-1}) = \{0\}$

Thm A: If $f: B_n \to \mathbb{R}$ is hom with defect ≤ 1 & (1) & (2), then $f = w$.

Tristram-Levine Signature of a link:

$\sigma_s(L) :=$ signature$((1-s)M + (1-\bar{s})M^T) \in \mathbb{Z}$, where $s\in S^1 \subset \mathbb{C}$ & M Seifert matrix of L.

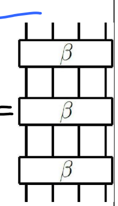
- $\sigma = \sigma_{-1}$ • $|\sigma_s(K)| \leq 2g_4(K)$

Homogenization of σ_s:

$$\tilde\sigma_s(\beta) := \lim_{k\to\infty} \frac{\sigma_s(\widehat{\beta^k})}{k} \in \mathbb{R}$$

- $|\tilde\sigma_s(\beta)| \leq 2g_4(\hat\beta) - 1 + n$

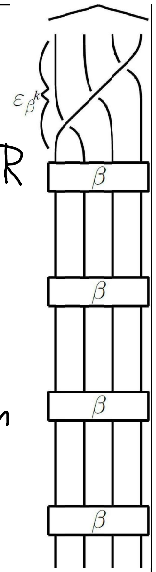
$\beta^3 =$



III) Let I be a concordance homomorphism.

$$\widetilde{I}(\beta) := \lim_{k\to\infty} \frac{I(\widehat{\beta^k \varepsilon_{\beta^k}})}{k} \in \mathbb{R}$$

$$\ell(\varepsilon_{\beta^k}) \leq n-1$$



Claim: $\widetilde{I}(\beta) \leq 2g_4(\hat{\beta})-1+n$

Ex: $2\widetilde{\tau} = -\widetilde{s} = wr$

Proof of THM B:

$$w(\beta) = \frac{\widetilde{I}(\beta)}{2} + \frac{wr(\beta)}{2} \overset{\text{Claim}}{\leq} \frac{2g_4(\hat{\beta})-1+n}{2} + \frac{2g_4(\hat{\beta})-1+n}{2}$$ □

Concordance homomorphisms:

$I: \text{Knots} \to \mathbb{R}$ s.t.

- $I(K\#J) = I(K) + I(J)$
- $|I(K)| \leq 2g_4(K)$

Rasmussen's s

Ex: • $I = \sigma$ • $I = 2\tau$ • $I = s$

Ozsváth-Szabó's tau

- $I = \frac{2\Upsilon(t)}{t}$, $t \in (0,1]$

Ozsváth-Stipsicz-Szabó's Upsilon

Thm B: $w(\beta) \leq 2g_4(\hat{\beta})-1+n$

F.-Hubbard 17: For $n \geq 2$, $\beta \in B_n$, $t = \frac{2}{n-1}$:

$$w(\beta) = \frac{\widetilde{\Upsilon}(t)(\beta)}{t} + \frac{wr(\beta)}{2}$$

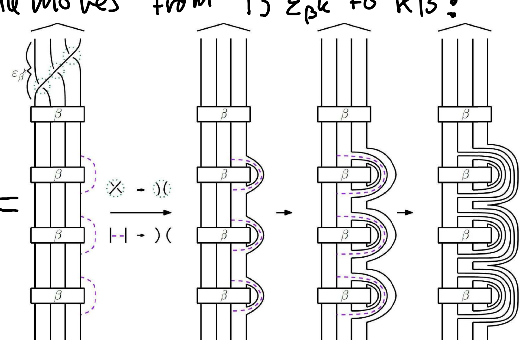
Proof of Claim: $\widetilde{I}(\beta) \leq 2g_4(\hat{\beta})-1+n$:

$$\widetilde{I}(\beta) = \lim_{k\to\infty} \frac{I(\widehat{\beta^k\varepsilon_{\beta^k}})}{k} \underset{\uparrow}{\leq} \lim_{k\to\infty} \frac{I(k\hat{\beta}) + (n-1)k}{k}$$ □

$\exists$ cobordism of genus at most $\frac{(n-1)(k-1)+n-1}{2}$ from $\widehat{\beta^k\varepsilon_{\beta^k}}$ to $k\hat{\beta} := \hat{\beta}\#\hat{\beta}\#\dots\#\hat{\beta}$

Proof of $\exists$ cobordism: $(n-1)(k-1)+n-1$ saddle moves from $\widehat{\beta^k\varepsilon_{\beta^k}}$ to $k\hat{\beta}$:

$\widehat{\beta^k\varepsilon_{\beta^k}} =$



IV)

Q: Let $f: B_n \to \mathbb{R}$ be a hom. Does there exist I & $r \in \mathbb{R}$ s.t. $f = r\widetilde{I}$?

Q: How does Thm A generalize to $f: MCG(F) \to \mathbb{R}$?