Link concordance, Reidemeister torsion

& much more

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X = n-dim'l mfd

$\partial X = R_- \cup R_+$ s.th.

$R_- \cap R_+ = \partial R_- = \partial R_+$

$$\tau(X, R_-) \underset{\tau}{=} \overline{\tau(X, R_+)}^{(-1)^{n+1}}$$

where α = unitary rep.

Ex: $K \subset S^3$ knot, $X_K = S^3 \setminus \nu K$

$\alpha: \pi_1(X_K) \to \langle t \rangle \to GL(1, \mathbb{Q}(t))$

$\Rightarrow \tau(X_K, \alpha) = \Delta_K(t)/(t-1)$

Reidemeister torsion

X compact manifold, $\alpha\colon \pi_1(X) \to \mathrm{GL}(n, \mathbb{K})$ representation

Definition of Reidemeister torsion:

(a) pick CW-structure ← need finite CW cplx

(b) $\mathbb{K}^n \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X})$ is a based $\mathbb{K}$-chain complex

(c) if acyclic, get Reidemeister torsion $\tau(X, \alpha) \in \mathbb{K}^* = \mathbb{K} \setminus \{0\}$, generalization of determinant

(d) well-defined up to multiplication by $\pm\det(\alpha(g))$ — $g \in \pi_1$

(e) two CW-structures that are simple homotopy equivalent give same torsion

(f) Chapman's theorem: all CW-structures are simple homotopy equivalent

can also define relative version $\tau(X, Y, \alpha)$

Key properties:

(a) $\tau(X, \alpha) = \tau(X, Y, \alpha) \cdot \tau(Y, \alpha)$

(b) $\tau(X, R_-, \alpha) = \overline{\tau(X, R_+, \alpha)}^{(-1)^{n+1}}$

Examples:

(a) $K \subset S^3$ knot, set $X_K = S^3 \setminus \nu K$ and $\pi_1(X) \to \langle t \rangle \to \mathrm{GL}(1, \mathbb{Q}(t))$ then $\tau(X_K, \alpha) = \Delta_K(t)/(t-1)$.

(a) $L \subset S^3$ an m-component link, $\rho\colon \pi_1(X_L) \to \mathrm{GL}(n, \mathbb{C})$, get $\pi_1(X_L) \to \mathrm{GL}(n, \mathbb{C}(t_1, \ldots, t_m))$ and $\tau(X_L, \rho) \in \mathbb{C}(t_1, \ldots, t_m)$, equals twisted Alexander polynomial, but smaller indeterminacy

Definition of Reidemeister torsion if not acyclic:

(a) need to pick basis of twisted homology

(b) cool observation by Turaev, given unitary
for X_L pick basis for H_1
equip H_2 with the dual basis
if the representation ρ is unitary
get invariant of X_L well-defined up to norm

$L = L_1 \ldots L_m \subset S^3$ link

$\alpha: \pi_1(X_L) \to GL(n, \mathbb{C})$, then get $\pi_1(X_L) \to GL(n, \mathbb{C}(t_1 \ldots t_m))$

$\Rightarrow \tau(X_L, \alpha)$ "=" "twisted Alex polynomial"

m-comp. links

Link concordance

Links $L, J \subset S^3$ are concordant if there exist disjoint *locally flat* annuli $A = A_1 \cup \cdots \cup A_m$ with $\partial A_i = L_i \sqcup -J_i$

Consider knot concordance and Reidemeister torsion

(a) Let A be concordance between knots L and J

(b) then $\tau(X_A) = \tau(X_A, X_L) \cdot \tau(X_L)$ and $\tau(X_A) = \tau(X_A, X_J) \cdot \tau(X_J)$

(c) and $\tau(X_A, X_L) = \overline{\tau(X_A, X_J)}^{-1}$

(d) thus get $\tau(X_L) \cdot \tau(X_j) = \tau(X_A) \cdot \overline{\tau(X_A)}$

X_A = top mfd, homeo to finite CW cplx

Example: Figure-8 knot is not concordant to unknot

Links and *twisted* Reidemeister torsion

(a) representations factoring through p-group extend to concordance

(b) picking basis and dual basis get $\tau(X_L, \alpha) \cdot \tau(X_j, \alpha) = \tau(X_A, \alpha) \cdot \overline{\tau(X_A, \alpha)}$

Example: Bing doubling

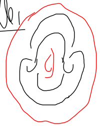
(a) Bing double $B(K)$ of a knot

(b) Conjecture $B(K)$ slice if and only if K is slice

(d) can show that Bing double of Figure-8 knot is not slice by using twisted Reidemeister torsion that factors through a 2-group

Where's the problem?

Bing double

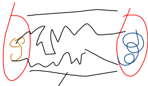


$\Rightarrow$ Wrap it around K, get $B(K)$

K, J conc $\Rightarrow B(K), B(J)$ conc

$\Leftarrow$ Conjecture

$L, J \subset S^3$ are concordant if there ex. locally flat collection of annuli $A_1 \ldots A_m \subset S^3 \times [0,1]$ s.th. $\partial A_i = L_i \cup -J_i$



S^3 $\quad S^3 \times [0,1] \quad$ S^3

Note (L, J knots)

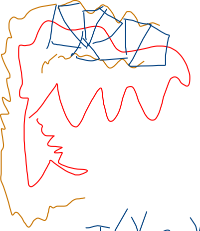
① $\tau(\underbrace{X_A}_{S^3 \times [0,1] \setminus \nu A}, X_L) = \tau(X_A)\, \tau(X_L)^{-1}$

② $\tau(X_A, X_J) = \tau(X_A)\, \tau(X_J)^{-1}$

③ $\tau(X_A, X_J) = \overline{\tau(X_A, X_L)}^{-1}$

$\Rightarrow \tau(X_L) \cdot \tau(X_J) = \tau(X_A)\, \overline{\tau(X_A)}$

$\Rightarrow L$ = Fig 8 knot, $J = O \Rightarrow$ not concordant



$\subset \mathbb{R}^n$

$\tau(X_A, X_L) = \overline{\tau(X_A, X_g}^{-1}$

$\shortparallel$

$\tau(X_A)\cdot\tau(X_L)^{-1}$

Issues dealt with in F-Nagel-Orson-Powell (no claim to originality)

(a) CW-complexes and topological manifolds
(b) simple homotopy type of topological manifolds
(c) connected sum of topological manifolds
(d) existence and uniqueness of tubular neighborhoods
(e) tubing submanifolds
(f) representing homology classes by submanifolds

Issues dealt with in lecture notes (no claim to originality)

(a) intersection of cycles computes cup product
(b) linking form symmetric (F-Herrmann)
(c) computation of linking form (Conway-F-Herrmann)
(d) ~~linking form and~~ intersection form (F-Herrmann)
(e) Novikov additivity
(f) can rotate knot so that projection is a diagram (F-Herrmann)
(g) cones are contractible

What still makes me nervous?

(a) twisted/local coefficients
(b) interplay between handles, CW-structures, triangulations etc.
(c) topological spaces that are not compact, Hausdorff
(d) PL-topology
(e) incoherent definitions

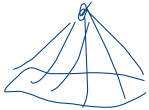
What would help?

Webpage with careful proofs, definitions, and references

$\Sigma = \mathbb{Q}HS^3$

$H_1(\Sigma)\times H_1(\Sigma)\to\mathbb{Q}/\mathbb{Z}$

$X \rightsquigarrow \mathrm{Cone}(X) \simeq \{*\}$



$$
\begin{tikzcd}
\mathrm{H}^2(M,Y;\mathbb{Z})\times\mathrm{H}^2(M,Y;\mathbb{Z}) \arrow[rr, "\cup"] \arrow[dr, "\varphi_*\times\mathrm{id}"] \arrow[ddd, "p^*\times p^*"] & \arrow[d, phantom, "\mathrm{(I)}"] & \mathrm{H}^4(M,Y;\mathbb{Z}) \arrow[r, "{\langle -,[M]\rangle}"] \arrow[d, "\varphi_*"] \arrow[dr, phantom, "\mathrm{(II)}"] & \mathbb{Z} \arrow[d, "\varphi"] \\
\arrow[ddr, phantom, "\mathrm{(III)}"] & \mathrm{H}^2(M,Y;\mathbb{Q})\times\mathrm{H}^2(M,Y;\mathbb{Z}) \arrow[r, "\cup"'] \arrow[d, "\mathrm{id}\times p^*"] \arrow[dr, phantom, "\mathrm{(IV)}"] & \mathrm{H}^4(M,Y;\mathbb{Q}) \arrow[r, "{\langle -,[M]\rangle}"'] \arrow[d, "="] \arrow[dr, phantom, "\mathrm{(V)}"] & \mathbb{Q} \arrow[d, "="] \\
 & \mathrm{H}^2(M,Y;\mathbb{Q})\times\mathrm{H}^2(M;\mathbb{Z}) \arrow[r, "\cup"'] \arrow[d, "p^*\times\mathrm{id}"] & \mathrm{H}^4(M,Y;\mathbb{Q}) \arrow[r, "{\langle -,[M]\rangle}"'] & \mathbb{Q} \arrow[dd, "="] \\
\boxed{\mathrm{H}^2(M;\mathbb{Z})\times\mathrm{H}^2(M;\mathbb{Z})} \arrow[r, "\varphi_*\times\mathrm{id}"] \arrow[ddd, "i^*\times i^*"] & \boxed{\mathrm{H}^2(M;\mathbb{Q})\times\mathrm{H}^2(M;\mathbb{Z})} \arrow[d, "(p^*)^{-1}\times\mathrm{id}"] & \mathrm{(VI)} & \\
\arrow[ddr, phantom, "\mathrm{(VII)}"] & \boxed{\mathrm{H}^2(M,Y;\mathbb{Q})\times\mathrm{H}^2(M;\mathbb{Z})} \arrow[r, "\cup"] \arrow[d, "\times\,\mathrm{id}"] \arrow[dr, phantom, "\mathrm{(VIII)}"] & \boxed{\mathrm{H}^4(M,Y;\mathbb{Q})} \arrow[r, "{\langle -,[M]\rangle}"] \arrow[d] \arrow[dr, phantom, "\mathrm{(IX)}"] & \boxed{\mathbb{Q}} \arrow[d] \\
 & \mathrm{H}^2(M,Y;\mathbb{Q}/\mathbb{Z})\times\mathrm{H}^2(M;\mathbb{Z}) \arrow[r, "\cup"] \arrow[d, shift left, "\times\, i^*"] \arrow[dr, phantom, "\mathrm{(X)}"] & \mathrm{H}^4(M,Y;\mathbb{Q}/\mathbb{Z}) \arrow[r, "{\langle -,[M]\rangle}"] \arrow[dr, phantom, "\mathrm{(XI)}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "="] \\
\mathrm{H}^2(Y;\mathbb{Z})\times\mathrm{H}^2(Y;\mathbb{Z}) \arrow[r, "\beta^{-1}\times\mathrm{id}"] & \mathrm{H}^1(Y;\mathbb{Q}/\mathbb{Z})\times\mathrm{H}^2(Y;\mathbb{Z}) \arrow[u, shift left, "\delta\times"] \arrow[r, "\cup"] \arrow[d, phantom, "\mathrm{(XII)}"] & \mathrm{H}^3(Y;\mathbb{Q}/\mathbb{Z}) \arrow[u, "\delta"] \arrow[r, "{\langle -,[Y]\rangle}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "="] \\
\mathrm{H}_1(Y;\mathbb{Z})\times\mathrm{H}_1(Y;\mathbb{Z}) \arrow[u, "\mathrm{PD}_Y\times\mathrm{PD}_Y"] \arrow[rrr, "\lambda_Y"'] & & & \mathbb{Q}/\mathbb{Z}.
\end{tikzcd}
$$