Braided surfaces and their characteristic maps

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Abstract

We show that branched coverings of surfaces of large enough genus arise as characteristic maps of braided surfaces, thus being 2-prems. In the reverse direction we show that any nonabelian surface group has infinitely many finite simple nonabelian groups quotients with characteristic kernels which do not contain any simple loops and hence the quotient maps do not factor through free groups. By a pullback construction, finite dimensional Hermitian representations of braid groups provide invariants for the braided surfaces. We show that the strong equivalence classes of braided surfaces are separated by such invariants if and only if they are profinitely separated.

Plan

- Let Σ denote a closed orientable surface. A braided surface over Σ is an embedding of a surface $j : S \to \Sigma \times \mathbb{R}^2$, such that the composition with the first factor projection

$$f : S \overset{j}{\hookrightarrow} \Sigma \times \mathbb{R}^2 \overset{p}{\to} \Sigma$$

is a branched covering. The composition $p \circ j$ is called the *characteristic map* of the braided surface S.
- Two braided surfaces $j_i : S \to \Sigma \times \mathbb{R}^2$, $i = 0, 1$ over Σ are *equivalent* if there exists some ambient isotopy $h_t : \Sigma \times \mathbb{R}^2 \to \Sigma \times \mathbb{R}^2$, $h_0 = id$ such that h_t is fiber-preserving (i.e. there exists a homeomorphism $\varphi : \Sigma \to \Sigma$ such that $p \circ h_t = \varphi \circ p$) and $h_1 \circ j_0 = j_1$. When φ can be taken to be isotopic to the identity rel the branch locus, we say that the braided surfaces are *strongly equivalent*.
- Viro, Rudolph '83, Kamada '94, Carter-Kamada, Nakamura '11
- Edmonds '99: f unramified covering, S contained in an orientable plane bundle, then its Euler class is torsion.

Geometric Lifting Problem:
When a ramified covering $f : S \to \Sigma$ lifts to a braided surface embedding $\varphi : S \to \Sigma \times \mathbb{R}^2$?

- We could instead ask φ be an immersion and the embedding might be smooth, PL topologically flat, topological, etc.
- One might take f be a generic smooth/PL map and ask if it lifts to an embedding. Melikhov '15.

- Two branched coverings $f_0, f_1 : S \to \Sigma$ are *equivalent* if there exist homeomorphisms $\Phi : S \to S$ and $\phi : \Sigma \to \Sigma$ such that
$$f_1 \circ \Phi = \phi \circ f_0$$
When ϕ is isotopic to the identity rel the branch locus, then the branched coverings are *strongly equivalent*.
- A degree n branched covering $f : S \to \Sigma$ of surfaces determines a holonomy homomorphism
$$f_* : \pi_1(\Sigma \setminus B) \to S_n$$
where B is the set of branch points.
- **Hurwitz branched coverings Classification**: Two branched coverings of surfaces are *strongly equivalent* if and only if their holonomy homomorphisms are *conjugate*. Moreover, they are *equivalent* if and only if the conjugacy classes of their holonomy homomorphisms are equivalent under the left action of the pure mapping class group $\Gamma(\Sigma \setminus B)$.

- A braided surfaces $\varphi : S \to \Sigma$ has degree n if its characteristic homomorphism $f : S \to \Sigma$ has degree n.
- A degree n braided surface $\varphi : S \to \Sigma \times \mathbb{R}^2$ of surfaces determines a holonomy homomorphism

$$\varphi_* : \pi_1(\Sigma \setminus B) \to B_n$$

where B is the set of branch points of its characteristic map and B_n is the braid group on n strands.
- **Braided surfaces Classification**: Two branched coverings of surfaces are *strongly equivalent* if and only if their holonomy homomorphisms are *conjugate*. Moreover, they are *equivalent* if and only if the conjugacy classes of their holonomy homomorphisms are equivalent under the left action of the pure mapping class group $\Gamma(\Sigma \setminus B)$.

- Our first result gives a positive answer to the geometric lifting problem, for large enough genus:

Theorem

There exists some $h_{n,m}$ such that every degree n branched covering $S \to \Sigma$ of a closed orientable surface Σ of genus $g \geq h_{n,m}$ with at most n branch points occurs as the characteristic map of some braided surface.

- Petersen '90 have proved that solvable unramified coverings can be lifted.

Algebraic Lifting Problem

Given a surjective group homomorphism $p : \widetilde{G} \to G$, when does a homomorphism $f : \pi_1(\Sigma) \to G$ lift to $\varphi : \pi_1(\Sigma) \to \widetilde{G}$?

- We will restrict to surjective homomorphisms f and Σ will be a closed orientable surface.

Definition

The Schur class $sc(f) \in H_2(G)$ is the image $f_*([\Sigma])$ of the fundamental class $[\Sigma]$ of Σ.

- The action of $\mathrm{Aut}(\pi_1(\Sigma)$ on $\mathrm{Hom}(\pi_1(\Sigma), G)$ preserves the Schur classes.
- Moreover, the G-conjugacy acts trivially on $H_2(G)$. Thus the Schur class descends to a function:

$$sc : \Gamma(\Sigma)\backslash\mathrm{Hom}(\pi_1(\Sigma), G)/G \to H_2(G)$$

- A homomorphism $\pi_1(\Sigma') \to \pi_1(\Sigma)$ is a *pinch* map if its is induced by the quotient (degree one) map $\Sigma' \to \Sigma$ which crushes several 1-handles to points.

Definition

A stabilization of $f : \pi_1(\Sigma) \to G$ is the composition with a pinch map.

- Two homomorphisms are stably equivalent if they have stabilization equivalent under the $\mathrm{Aut}^+(\pi_1(\Sigma)$ action. The stable equivalence descends also to G-conjugacy classes of homomorphisms.
- Observe that the image of a homomorphism is an invariant of its (stable) equivalence class. For this reason we shall restrict to surjective homomorphisms.
- Note that the Schur class of a homomorphism does not change under stabilization.

Theorem (Livingston '85, Zimmermann '87)

Surjective homomomorphisms are stably equivalent if and only if their Schur classes agree.

- If $\Omega_n(X)$ is the dimension n orientable bordism group of X, then Thom proved that the natural map $\Omega_n(X) \to H_n(X)$ is an isomorphism if $n \leq 3$ and an epimorphism, if $n \leq 6$. Two maps $f : \Sigma \to X$, $f' : \Sigma' \to X$ representing the same class in $H_2(X)$ are therefore bordant and thus there exists a 3-manifold M^3 whose boundary is $\Sigma \sqcup \Sigma'$ and a common extension $F : M^3 \to X$.
- Consider a Heegaard surface Σ'' in M^3, decomposing it into the union of two compression bodies $C \cup C'$, glued together along their common boundary Σ'' by means of a homeomorphism ψ. A compression body is obtained from $\Sigma'' \times [0, 1]$ by attaching 2-handles along disjoint nontrivial simple closed curves on $\Sigma'' \times \{1\}$.
- Then $F|_{\Sigma''}$ is a stabilization of f and f', up to equivalence, throughout the restrictions $F|_C$ and $F|_{C'}$.

The stable algebraic lifting problem has a solution:

Corollary

Given a surjective $p : \widetilde{G} \to G$*, then a surjective homomorphism* $f : \pi_1(\Sigma) \to G$ *lifts stably to* $\widetilde{G}$ *if and only if there exists some class* $a \in H_2(\widetilde{G})$ *such that* $p_*(a) = sc(f)$*.*

The Livingston-Zimmermann result was improved in the case when the target G is a *finite group*, as follows:

Theorem (Dunfield-Thurston '06)

If G *is finite, then there exists some* $g(G)$ *such that every two surjective homomorphisms* $\pi_1(\Sigma) \to G$ *with the same Schur class, for a closed orientable surface* Σ *of genus* $g \geq g(G)$*, are equivalent. In particular, every such surjective homomorphism* $f : \pi_1(\Sigma) \to G$ *lifts to* $\widetilde{G}$*, if there exists some* $a \in H_2(\widetilde{G})$ *with* $p_*(a) = sc(f)$*.*

A key ingredient is that for large enough genus every surjective f should be a stabilization.

The last step, in the unramified case, is the following:

Lemma

If $G \subset S_n$ and $p : B_n \to S_n$ is the projection, then we have a surjective homomorphism in 2-homology:

$$H_2(p^{-1}(G)) \to H_2(G) \to 1$$

Proof: Let P_n denote the pure braid group on n strands. The five terms exact sequence in homology reads:

$$H_2(p^{-1}(G)) \to H_2(G) \to H_1(P_n)_G \to H_1(p^{-1}(G)) \to H_1(G)$$

observe that $H_1(P_n)_G \cong \mathbb{Z}S(n)_G \cong \mathbb{Z}[S(n)/G]$, where $S(n) = \{(i,j); 1 \leq i < j \leq n\}$. In particular, $H_1(P_n)_G$ is a torsion-free group, while $H_2(G)$ is torsion. Therefore every homomorphism $H_2(G) \to H_1(P_n)_G$ must be trivial.

- In the ramified case ($B \neq \emptyset$) we need to adapt the proof above to surjective homomorphisms $\pi_1(\Sigma \setminus B) \to G$, having prescribed value of the peripheral loops, i.e. encircling once every branch point.
- The characteristic homomorphism of a branched covering maps peripheral loops into *nontrivial* elements of S_n. What about braided surfaces?
- A link $L \subset S^1 \times D^2$ is *completely split* if there exist pairwise disjoint disks $D_i^2 \subset D^2$ such that each component L_i of the link L is contained in one solid torus $S^1 \times D_i^2$. A braid $b \in B_n$ is *completely splittable* if its closure L within the solid torus is completely split, while L is a trivial link in the sphere S^3.
- Kamada '96: Local monodromy of PL topologically flat embeddings φ around branch points completely splittable braids $A_n \subset B_n$.
- Schur classes for homomorphisms with fixed peripheral holonomy, Catanese-Lönne-Perroni '16 and extension of Livingston-Zimmermann and Dunfield-Thurston results.

Definition

We say that a homomorphism $f : \pi_1(\Sigma) \to G$ is *elementary*, if it factors through a *free* group.

By a well-known result of Stallings and Jaco '69, this is equivalent to the fact that f factors through the map $\pi_1(\Sigma) \to \pi_1(H)$, where H is the handlebody bounded by Σ.

Corollary

If G is finite, then there exists $g(G)$ such that any null-homologous surjective homomorphism $f : \pi_1(\Sigma) \to G$, where Σ is a surface of genus $g \geq g(G)$, is elementary. In particular, f lifts to any $\tilde{G}$.

- The *thickness* $t(f)$ of a nullhomologous f is the smallest g for which there exists some 3-manifold M^3 with boundary Σ and Heegaard genus g, and an extension $F : \pi_1(M^3) \to G$ Liechti-Marché '19 considered the torus case.

Proposition

$t(f)$ is the smallest genus of an elementary stabilization of f.

- Let $\pi_1(\Sigma) = \langle a_i, b_i; \prod_{i=1}^{g}[a_i, b_i]\rangle$ and $G = \mathbb{F}/R$ be a presentation of G. Set $ocl(f)$ to be the minimal n such that we can write

$$\prod_{i=1}^{g}[\widetilde{f(a_i)}, \widetilde{f(b_i)}] = \prod_{i=1}^{n}[r_j, f_j]$$

where $\widetilde{f(a_i)}, \widetilde{f(b_i)}$ denote lifts to $\mathbb{F}$ and $r_j \in R$, $f_j \in \mathbb{F}$.

Proposition (Hopf type formula)

$t(f) = ocl(f)$.

Conjecture (Wiegold)

For any finite simple nonabelian group G and $n \geq 3$ we have

$$|Out(\mathbb{F}_n)\backslash Epi(\mathbb{F}_n, G)/\mathrm{Aut}(G)| = 1$$

- McCullough-Wanderley '03: Epimorphisms become equivalent after $\mu(G)$ (= the minimal number of generators of G) stabilizations, for all finite G.
- McCullough-Wanderley '03: For large enough $n \geq 1 + |G| \log_2 |G|$ any two epimorphisms into G are equivalent.
- There exist nonequivalent epimorphisms onto infinite groups G.

Conjecture (virtual solvability – Lubotzky)

For any finite dimensional representation of $\mathrm{Aut}(\mathbb{F}_n)$*, the image of the inner automorphisms subgroup* $\mathbb{F}_n$ *is virtually solvable.*

Formanek-Procesi '92 proved that the image of the free subgroup of $\mathbb{F}_n$ on two standard generators is virtually solvable.

Conjecture (free factors - Lubotzky)

For finite simple nonabelian G *and surjective homomorphism* $f : \mathbb{F}_n \to G$, $n \geq 3$*, there exist a proper free factor* $H \subset \mathbb{F}_n$ *with* $f(H) = G$*, i.e. for any system of generators* $g_1, g_2, \ldots, g_n$ *of* $\mathbb{F}_n$*, we can drop one such that their images by* f *still generate* G*.*

Conjecture (characteristic quotients - Lubotzky)

There is no finite simple characteristic quotient of $\mathbb{F}_n$, $n \geq 3$*.*

- Gilman '77: There exists a large orbit of $\mathrm{Out}(\mathbb{F}_n)$ on $Epi(\mathbb{F}_n, G)/\mathrm{Aut}(G)$, whose size N goes to infinity with $|G|$ and on which the action is by the alternating group A_N or the symmetric group S_N.
- Therefore, if Wiegold's Conjecture holds, then there are no finite simple characteristic quotients of $\mathbb{F}_n$, $n \geq 3$.

Similar questions were asked by Lubotzky '11 about surface groups $\pi_1(\Sigma)$.

Theorem (F-Lochak '18)

For surface groups of genus $g \geq 2$ the virtual solvability conjecture and Wiegold-type conjecture do not hold. In particular, surface groups have infinitely many finite simple characteristic quotients.

It is unknown whether a single stabilization is enough to make equivalent nullhomologous epimorphisms of a surface group onto a finite simple nonabelian group.

Theorem

If $g \geq 2$, the there exist infinitely many epimorphisms $\pi_1(\Sigma) \to G$ onto finite simple nonabelian groups G, whose kernels do not contain any simple loop and hence are nonelementary.

- Livingston '00, Gabai: finite quotients without simple loops in the kernel
- Pikaart '01: finite caracteristic quotients without simple loops in the kernel

Proof sketch: If elementary, it has simple loops in the kernel, corresponding to (nonseparating) meridians of the handlebody. Since the kernel is characteristic, if it contains one nonseparating simple loop, it should contain all nonseparating simple loops and hence it would be trivial. The order of separating loops in the quotients are explicitly computed by using their TQFT description.

- Let $\gamma_0 G = G$, $\gamma_{k+1}G = [\gamma_k G, G]$ denote the lower central series of the group G. It is well-known that P_n is residually torsion-free nilpotent, namely $\bigcap_{k=0}^{\infty} \gamma_k P_n = 1$ and $A_k = \frac{\gamma_{k-1}P_n}{\gamma_k P_n}$ are finitely generated torsion-free abelian groups.
- We then have a series of *abelian extensions*

$$1 \to A_{k+1} \to \frac{B_n}{\gamma_{k+1}P_n} \to \frac{B_n}{\gamma_k P_n} \to 1$$

- Whether a homomorphism $f_k : \pi_1(\Sigma) \to \frac{B_n}{\gamma_k P_n}$ admits a lift to $f_{k+1} : \pi_1(\Sigma) \to \frac{B_n}{\gamma_{k+1} P_n}$ can be reformulated in purely cohomological terms.
- For every $k \geq 1$ there exist examples of homomorphisms f_k which admit no lift.

Proposition

Every homomorphism $f_0 : \pi_1(\Sigma) \to \frac{B_n}{\gamma_0 P_n} = S_n$ *admits a lift* $f_1 : \pi_1(\Sigma) \to \frac{B_n}{\gamma_1 P_n}$.

Every $f : \pi_1(\Sigma) \to S_n$ induces a $\pi_1(\Sigma)$-module on $A_1 = H_1(P_n)$. The key ingredient of the proof is the following

Lemma

Let $P : \pi_(\Sigma') \to \pi_1(\Sigma)$ *be a pinch map. Then*

$$P^* : H^2(\pi_1(\Sigma), A_1) \to H^2(\pi_1(\Sigma'), A_1)$$

is injective.

Then f_0 admits a lift f_1 if and only if the pull-back of the extension $\frac{B_n}{\gamma_1 P_n}$ of S_n by A_1 by f_0 is a split extension. Since every homomorphism lifts stably to B_n and hence to $\frac{B_n}{\gamma_1 P_n}$, the cohomological obstruction stably vanishes. Lemma above completes the proof.

- Equivalence classes of degree n braided surfaces with $B \neq \emptyset$ correspond to double cosets $B_n \backslash B_n^{m+1} / B_n$.
- Let $K \subseteq H$ be a pair of groups and $\rho : H \to U(V)$ be a finite dimensional representation of H preserving a Hermitian form $\langle , \rangle$. A *K-spherical function* on H is a matrix coefficient

$$\phi : H \to \mathbb{C}, \ \phi(x) = \langle \rho(x) v, w \rangle,$$

where v, w belong to the space of K-invariants vectors V^K. Then ϕ is bi-K-invariant, namely it descends to $K \backslash H / K$.

Proposition

For $K \subset H$ finite groups or compact connected Lie groups, the unitary K-spherical functions separate points of $K \backslash H / K$.

- Unitary representations $R : B_n \to U$ induce topological invariants of braided surfaces, pulling-back U-spherical functions under the map:

$$R_* : B_n \backslash B_n^{m+1} / B_n \to U \backslash U^{m+1} / U$$

- We can assemble all U-spherical functions in a single formal series Φ. Let i index the finite dimensional irreducible representations V_i of $SU(2)$. The space of invariant vectors $H^0(SU(2), V_{i_1} \otimes V_{i_2} \otimes \cdots V_{i_k})$ has a basis B_I indexed by the set of partitions $\alpha = (\alpha_{st})_{s,t=1,\ldots,k}$ with $\sum_t \alpha_{st} = i_s$.

$$\Phi = \sum_{I,(\alpha_{st})} \frac{1}{\alpha!\beta!} \prod_{s,t} \mathbf{x}^\alpha \mathbf{y}^\beta (\phi_{I,\alpha,\beta})$$

Here we set $\mathbf{x}^\alpha = \prod_{s,t} x_{st}^{\alpha_{st}}$, $\alpha! = \prod_{s,t} \alpha_{st}!$.
- Neretin '10 proved that:

$$\Phi(A) = \det(1 - AXA^{\perp}Y)^{-1/2}$$

for $A \in SU(2)^k$, where $X = (X_{ij})$, $Y = (Y_{ij})$ are matrices of blocks $X_{ij} = \begin{pmatrix} 0 & x_{ij} \\ -x_{ij} & 0 \end{pmatrix}$, $Y_{ij} = \begin{pmatrix} 0 & y_{ij} \\ -y_{ij} & 0 \end{pmatrix}$, $X_{ji} = -X_{ij}$, $Y_{ji} = -Y_{ij}$ for $i < j$, $X_{ii} = Y_{ii} = 0$ and x_{ij}, y_{ij} are variables.
- This provides a polynomial invariant $\Phi(A)^{-2}$ of degree 3 braided coverings via the unitary Burau representation.

- Consider the profinite completion of the group H:

$$\widehat{H} = \lim_{\leftarrow G \triangleleft H; |H/G| < \infty} H/G$$

This is a totally disconnected compact group. If $K \subset H$ is a subgroup, then $\overline{K}$ denotes the closure of K in the topological group $\widehat{H}$.

- Elements of $K \backslash H / K$ are *profinitely separated* if their images in $\overline{K} \backslash \widehat{H} / \overline{K}$ are distinct.

Theorem

If H is of finite type, then Hermitian K-spherical functions on $K \backslash H / K$ separate precisely those elements which are profinitely separated.

- Note that the conjugacy separability of B_n is unknown (for $n \geq 4$).

Problem

What can be said about braided surfaces (in particular their fundamental groups) which cannot be distinguished by unitary spherical functions?

- Unitary spherical functions are **not** invariants of strong equivalence classes, since mapping class group act ergodically on the moduli spaces of repersentations (see Goldman '99, Pickrell-Xia '03).
- However, we can construct strong equivalence invariants by using instead regular functions on moduli spaces of G-bundles

$$\Gamma(\Sigma \setminus B)\backslash \mathrm{Hom}^{\mathrm{stable}}(\pi_1(\Sigma \setminus B), G)/G$$

for suitable noncompact Lie groups, e.g. $SL(n, \mathbb{C})$.