K-Os 27/05/2021

Weinstein handlebody diagrams for complements of smoothed toric divisors

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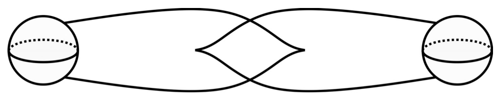
out of a WiSCON (Women in Symplectic and Contact Geometry) project.

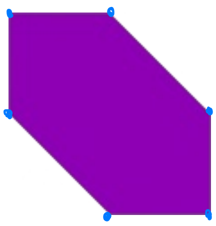
Goal

toric picture in symplectic geometry.

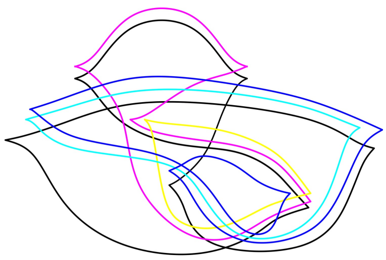
Weinstein handlebody diagram.

algorithm





algorithm



Context

Philosophy (M,ω) symplectic manifold

↳ closed non-deg. 2-form ω.

↳ decompose in easier to understand pieces

Look at $D \subset M$ a divisor ← a symplectic submanifold of codimension 2

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ \mathcal{N}(D) \arrow[d, "\text{fibre bundle}"] & & M \setminus \mathcal{N}(D) \\ D & & \end{tikzcd}$$

closed compact

hope: nicer.

structure of Weinstein domain

$\omega = d\lambda$

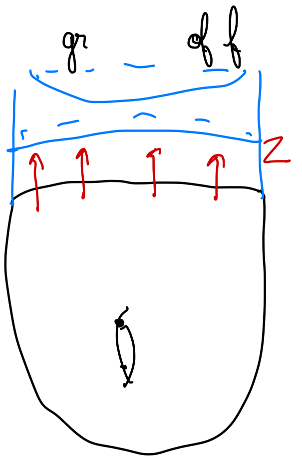
constructed by attaching handles.

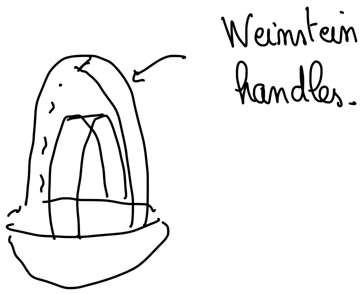
Same philosophy in mirror symmetry.

Weinstein manifolds

A Weinstein domain is a quadruple (W, ω, Z, ϕ) where

- (W,ω) is a symplectic manifold with boundary.
- Z is a Liouville vector field, ie $\mathcal{L}_Z\omega = \omega$ or $\omega = d\lambda$ where $\lambda = i_Z\omega = \omega(Z,\cdot)$.
  $Z \pitchfork \partial W$ and points outward of ∂W.
- $\phi : W \to \mathbb{R}$ is a Morse function on W such that ϕ is constant on ∂Z and Z is gradient-like for ϕ ie $d\phi(Z) > 0$ away from critical points and equal to the adient in the neighbourhood of critical points





A Weinstein manifold is obtained from a Weinstein domain by adding cylindrical ends (in order to make the Liouville vector field complete)

Symplectic toric manifolds

Definition A toric symplectic manifold is a symplectic manifold (M^{2n}, ω) endowed with an effective Hamiltonian action of a torus $\mathbb{T}^n$ and a choice of a moment map $\Phi : M \longrightarrow \mathbb{R}^n = \mathrm{Lie}(\mathbb{T}^n)$

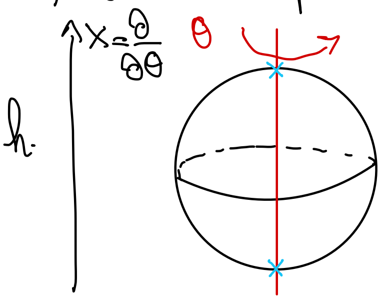
$\mathbb{Z}$-basis $\mathbb{Z}^n$

Examples

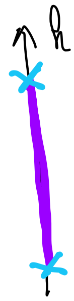
0) The most important one: $\mathbb{C}P^1 = S^2$ $\quad \omega = dh \wedge d\theta$

Hamiltonian act° w/ Ham. fct° h.

$h \hookrightarrow \omega(\cdot, X) = dh.$



$\Phi = h$

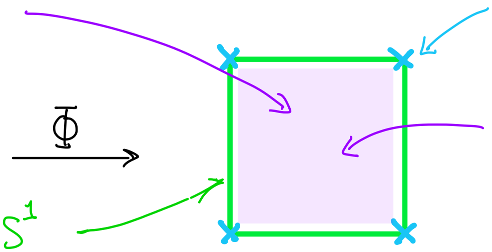


1) $S^2 \times S^2$

Lagrangian $\longrightarrow S^1 \times S^1$ $\qquad \omega|_{\text{Fiber}} = 0$ (isotropic)

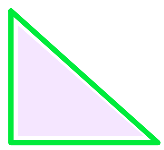
fixed pts

$S^2 \times S^2 \xrightarrow{\Phi}$



S^1

2) $\mathbb{C}P^2 \xrightarrow{\Phi}$



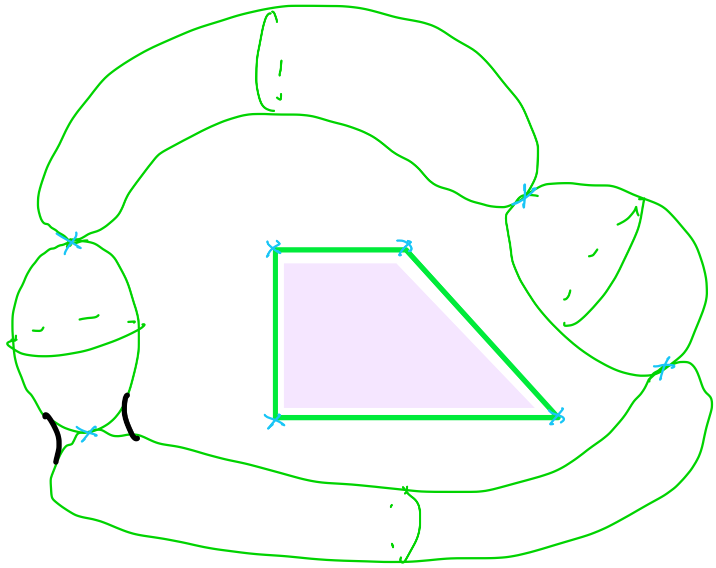
$$[z_0 : z_1 : z_2] \longmapsto \left(\frac{|z_1|^2}{\Sigma |z_i|^2}, \frac{|z_2|^2}{\Sigma |z_i|^2} \right)$$

Image: convex hull of the fixed pts (Guillemin-Sternberg)
Delzant polytope.

Divisors? toric divisors: preimage of the facets

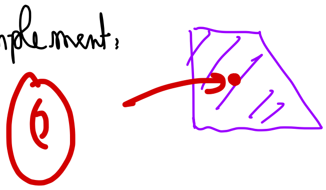
boundary of the polytope.

Today: dim $4 = 2 \times 2$. $\Rightarrow$ divisors of dim 2 (surfaces)



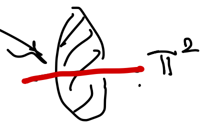
Toric divisor in dim 4: a necklace of S^2's meeting $\pitchfork$

Complement:



Arnold-Liouville thm

symplectomorphic $D(T^*\mathbb{T}^2)$



[Giroux] Choose $\underline{D}$ st $M \setminus D$ has a Weinstein structure

D PD $k[\omega]$

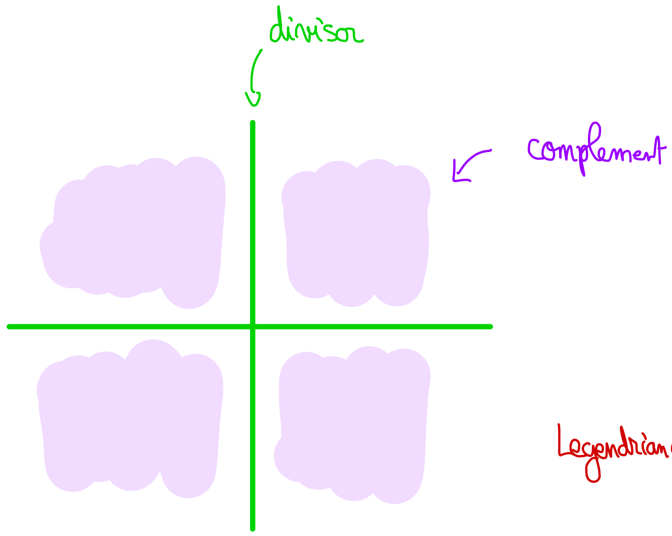
[Donaldson]

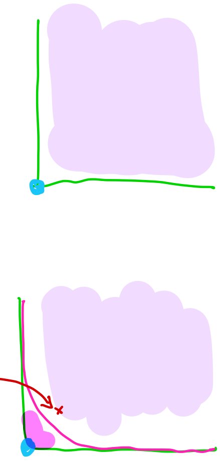
The pair we have (toric manifold, toric div $^{\text{smoothing}}$) : log Calabi-Yau

PD $(c_1(M))$

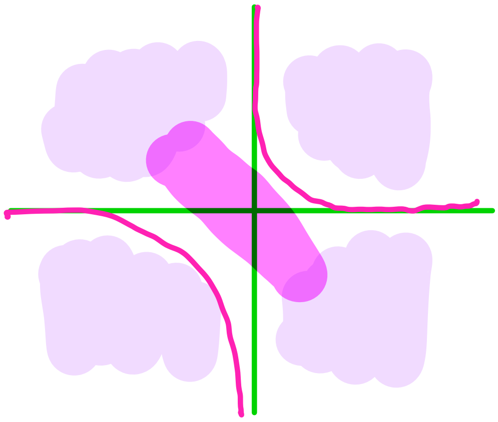
[Hacking - Keating]

Smoothing





Smoothing



$M \setminus D$

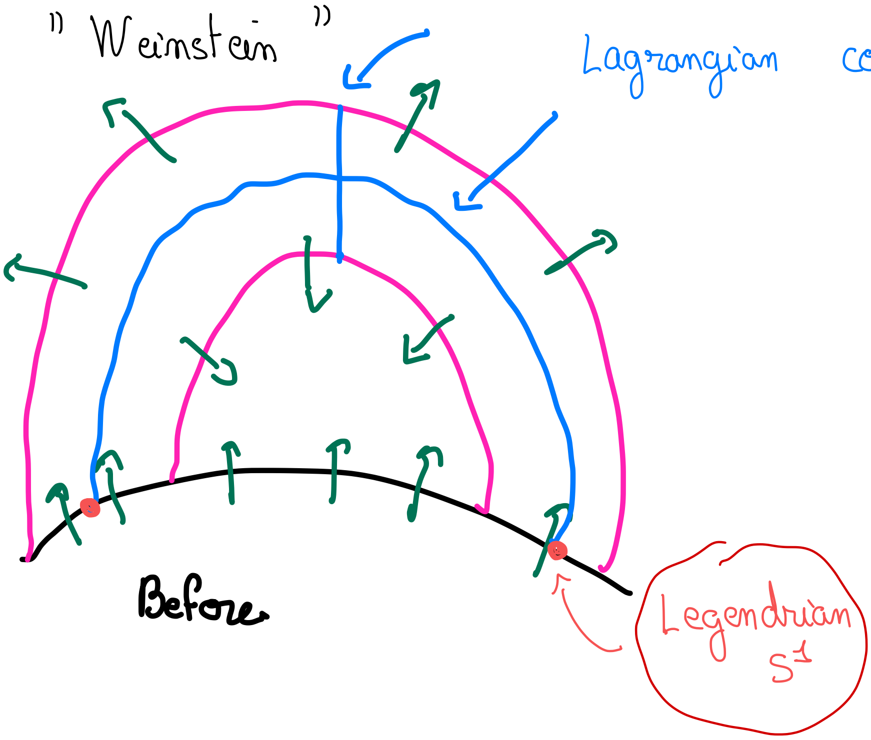
$\rightsquigarrow$

$M \setminus \tilde{D} = M \setminus D \cup$ 2-handle attached.

" Weinstein "

Lagrangian core and co-core.

$\omega|_{\text{disks}} = 0$

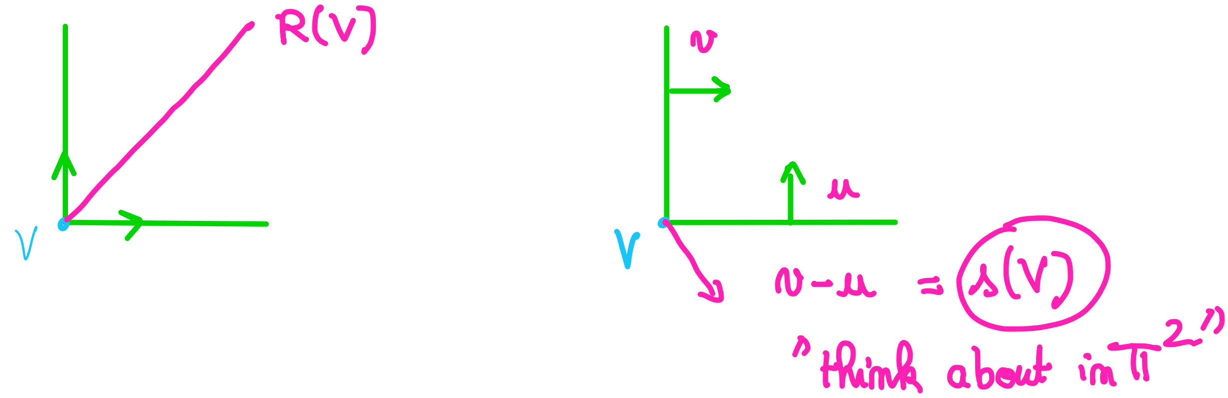


for $\lambda = i_Z \omega = \omega(Z, \cdot)$

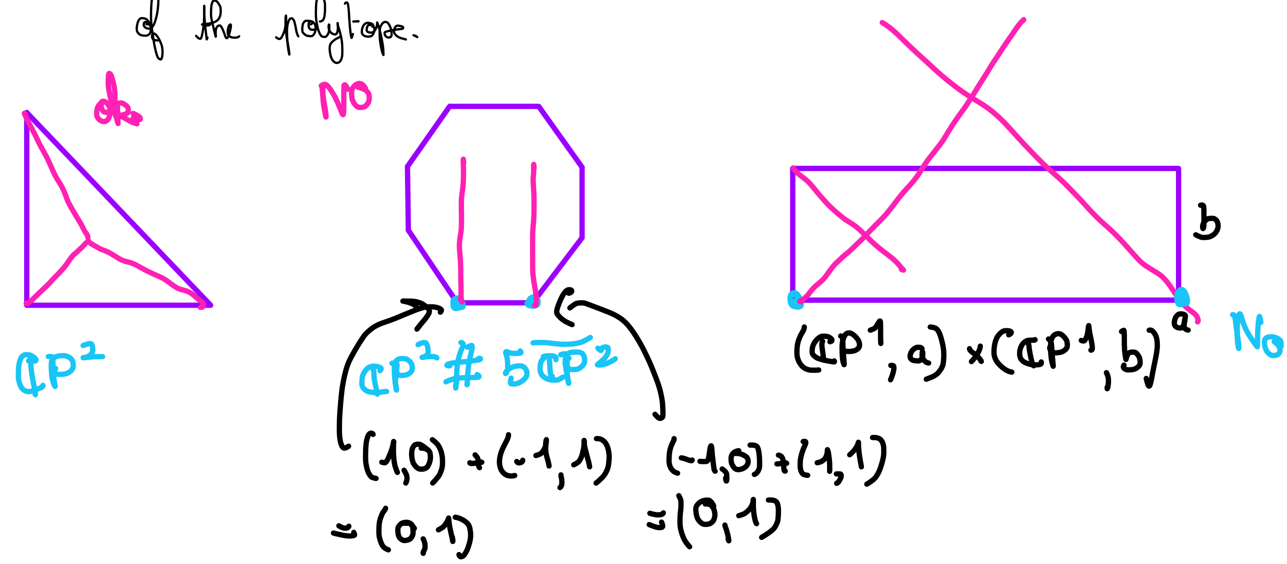
The attaching sphere when smoothing at a vertex

Definition Let (M,ω) be a symplectic toric manifold with moment polytope Δ. To each vertex v of Δ we associate

- the ray at V $R(V)$ of direction the sum of the direction of the edges meeting at V
- the slope of V $s(V)$: the difference of the unit inward normal vectors of the edges meeting at V



Definition The toric manifold (M,ω) is $\{V_1, \dots, V_n\}$-centered if the rays $R(V_1), \dots, R(V_n)$ all intersect at one point in the interior of the polytope.



always ok $\Leftrightarrow$ monotone ($\Leftrightarrow$ Fano).

Theorem [A C-S G M M S W]

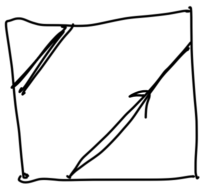
(1) Let (M,ω) be a $\{V_1,\dots,V_m\}$-centered toric 4-manifold and $\tilde{D}$ the divisor obtained by smoothing the toric divisor at the nodes $V_1,\dots,V_m$.

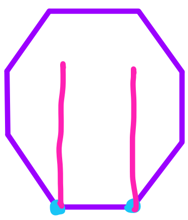
Then there exists arbitrarily small neighbourhoods $\mathcal{N}(D)$ of D such that $M \setminus \mathcal{N}(D)$ admits the structure of a Weinstein domain

(2) Moreover $M \setminus \mathcal{N}(D)$ is Weinstein homotopic to the Weinstein domain obtained by attaching Weinstein 2-handles to the unit disc cotangent bundle $\mathbb{D}^*\mathbb{T}^2$ along the Legendrian co-normal lifts of co-oriented curves of slope $s(V_1),\dots, s(V_m)$.

$$M \setminus \mathcal{N}(D) = \mathbb{D}^*\mathbb{T}^2 \cup h_{\Lambda_i}$$

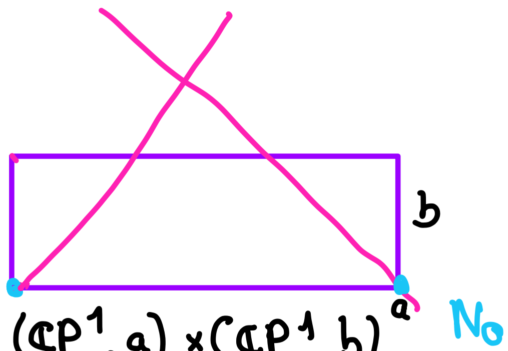
(a,b)





Not Weinstein

$\omega \neq d\lambda$



$(\mathbb{C}P^1, a) \times (\mathbb{C}P^1, b)$

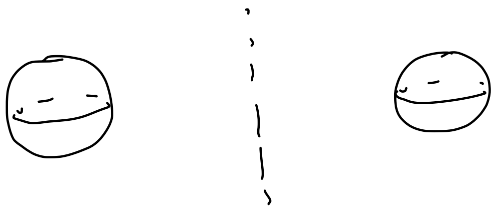
" lack of convexity at ∞ "

Kirby diagram

In dim 4.

0-handle $\qquad \mathbb{D}^4$

1-handle $\qquad \partial \mathbb{D}^4 = \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\} \qquad S^0 \times \{0\}$



2-handle $\qquad \mathbb{D}^2 \times \mathbb{D}^2$ attached along $\partial \mathbb{D}^2 \times \mathbb{D}^2 = \mathbb{S}^1 \times \mathbb{D}^2$ trivialisation

knot in $\mathbb{S}^3$ $\qquad$ normal bundle.

$\vdots$

contact structure.

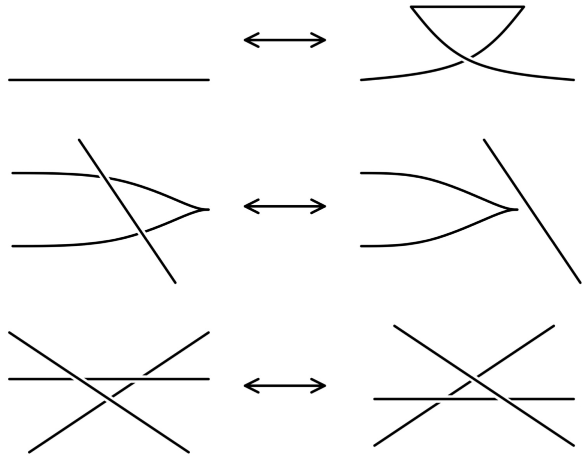
$e = tb - 1$

Weinstein [Gompf]

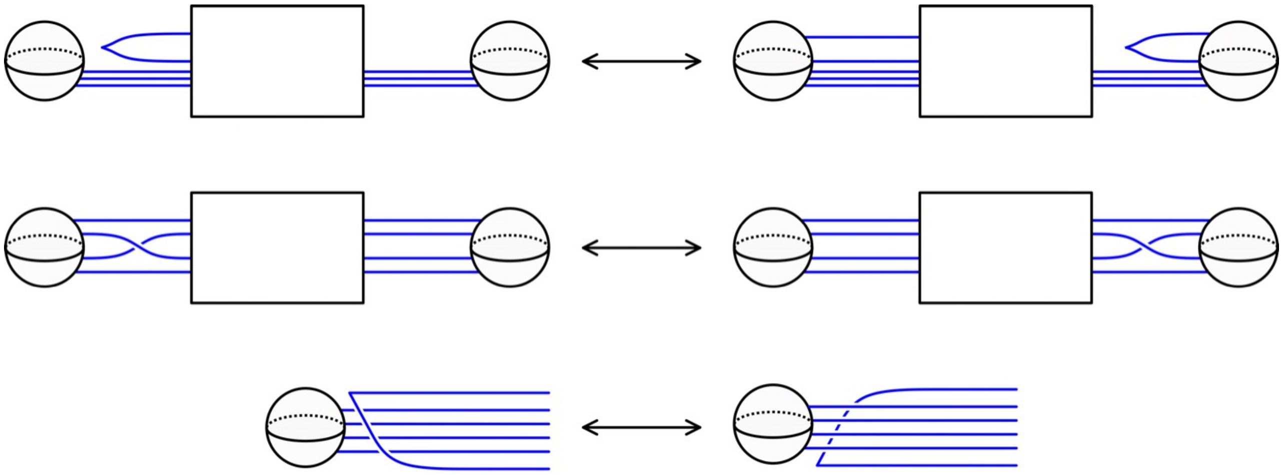
$\rightarrow$ index 0, 1, 2 handle.

$\rightarrow$ Legendrian knots / links

Legendrian Reidemeister moves

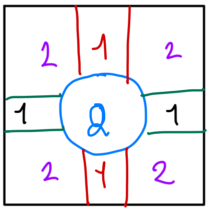


Gompf moves 4, 5, 6

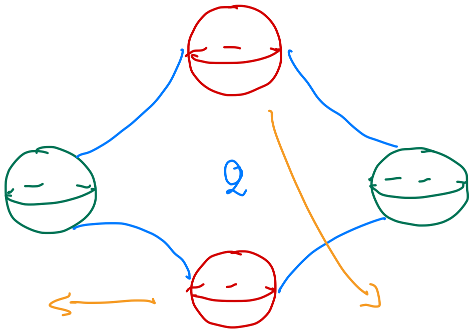


Example of D^*T^2

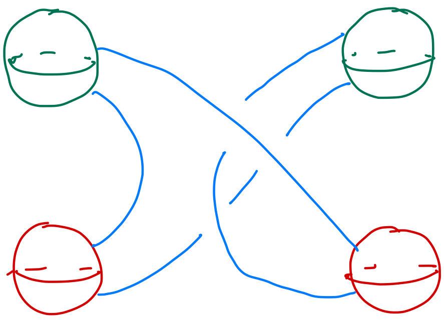
T^2



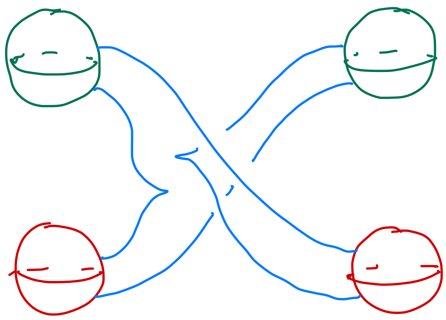
thickened



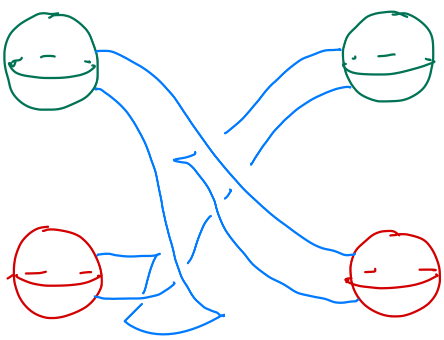
Kirby



Gompf



$e = -1$



$\leftarrow D(T^* T^2)$

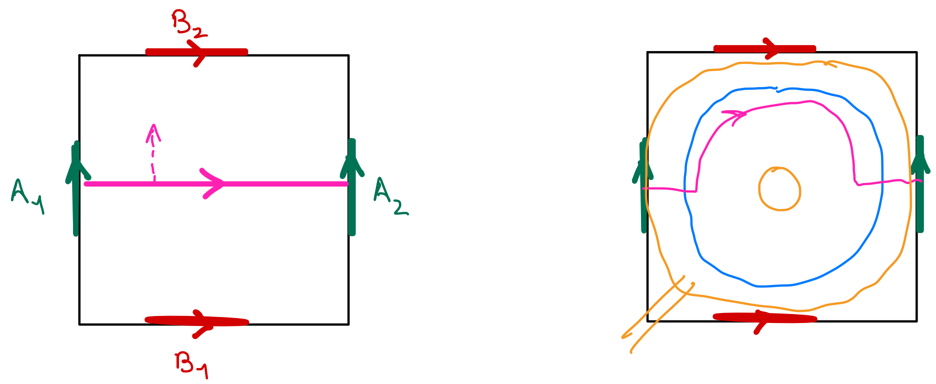
$e = 0$

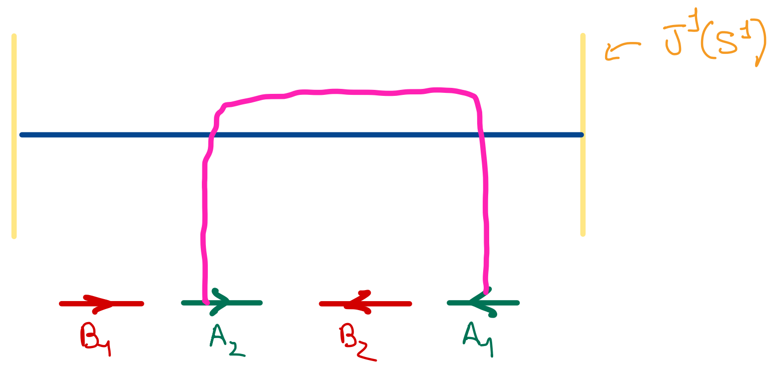
Theorem 2 [ACSGMMSW]

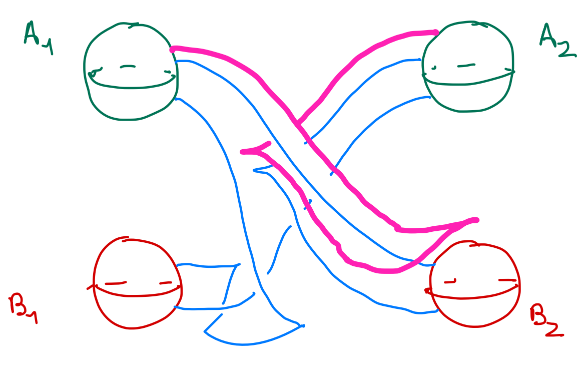
There exists an algorithm to produce the Weinstein handlebody diagram of $D^*T^2 \cup h_{\Lambda_i}$

Remark $\quad T^2 \leftarrow \left(\begin{array}{c}\text{closed}\\ \text{surface.}\end{array}\right)$

Example $\quad (1,0)$ -slope

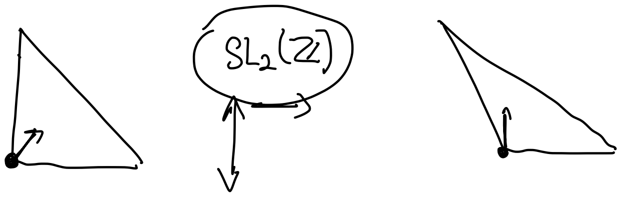






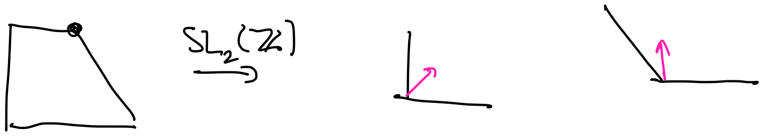
moves

(Casals-Murphy)

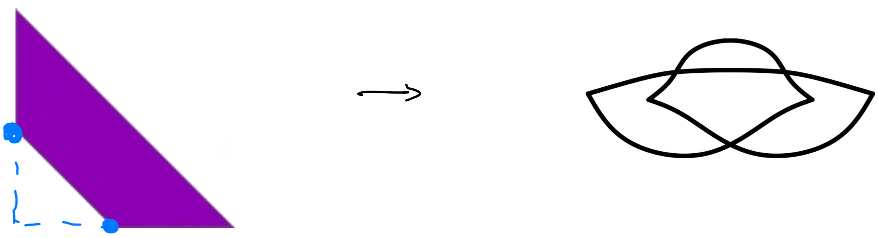


same toric mfld

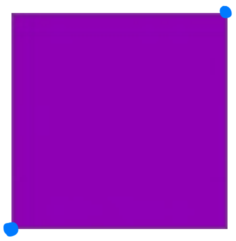
any (M,ω) smoothed once. has this handlebody diagram

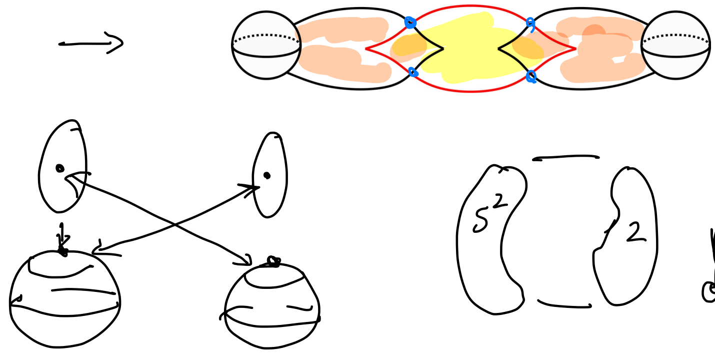


Other examples



$(\mathbb{CP}^1_a \times \mathbb{CP}^1, a)$





cyclic plumbing of 2 S^2

Ekholm–Ng $\rightsquigarrow$ CE dga Leg.

[Etgü $\hookrightarrow$ Lekili]

Extension to almost toric manifolds