Knot Floer homology and surface diffeomorphisms

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Fibred knots

A knot K in a closed three-manifold Y is *fibred* if

- there is a locally trivial fibration $\pi\colon Y \setminus K \to S^1$, and
- in a tubular neighbourhood $N(K) \cong K \times D^2_{r,\theta}$,

$$\pi(x, r, \theta) = \theta.$$

There are:

- a surface with boundary S (the *fibre*) and
- a diffeomorphism $\varphi_0 \in \mathrm{Diff}^+(S, \partial S)$ (the *monodromy*)

such that

$$Y \setminus K = \mathrm{int}(S) \times [0,1]/(x,1) \sim (\varphi_0(x), 0).$$

The monodromy is well defined up to isotopy.

Knot Floer homology

Let $K \subset Y$ be a null-homologous knot. Knot Floer homology is a sequence of $\mathbb{Z}/2\mathbb{Z}$-vector spaces $\widehat{HFK}(Y, K, d)$, $d \in \mathbb{Z}$, defined by Ozsváth and Szabó and, independently, Rasmussen.

- It categorifies the Alexander polynomial (Ozsváth and Szabó)
- It detects the genus: $\widehat{HFK}(Y, K, d) = 0$ if $|d| > g$ and $\widehat{HFK}(Y, K, -g) \neq 0$ (O-Sz, Ni),
- It detect fibredness: K is fibred iff $\dim \widehat{HFK}(Y, K, -g) = 1$ (G., Ni),
- More generally, $\widehat{HFK}(Y, K, -g)$ is a measure of the complexity of the sutured manifold decompositions of the knot complement (Juhász).

The main result

If $\varphi\colon S \to S$ is a "good" diffeomorphism, $HF^\sharp(\varphi)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space which "counts" the fixed points of φ.

Theorem

If $K \subset Y$ is a fibred knot of genus g and φ is a "good" representative of the monodromy, then

$$\widehat{HFK}(-Y, -K, 1-g) \cong HF^\sharp(\varphi).$$

$-$ denotes reversing the orientation. In S^3 it corresponds to mirroring the knot.

A similar result has been obtained by Ni using the isomorphism between Heegaard Floer homology and periodic Floer homology (Lee-Taubes and Kutluhan-Lee-Taubes)

Some applications

From work of Thurston, Nielsen, Gautschi and Cotton-Clay we can derive the following corollaries.

Corollary (Baldwin and Vela-Vick)

If $K \subset Y$ is a fibred know to genus g, then $\widehat{HFK}(Y, K, 1-g) \neq 0$.

Corollary

if $K \subset Y$ is not the trivial knot in S^3 and φ_{th} is the Thurston representative of the monodromy, then

$$\mathrm{Fix}(\varphi_{th}) \leq \dim \widehat{HFK}(-Y, -K, 1-g) - 1$$

Corollary

If $K \subset S^3$ is an L-space knot, then the Thurston representative of the monodromy has no fixed points.

More applications

We denote by $\Sigma_n(Y, K)$ the n-fold cyclic cover of Y branched along K and by $K_n \subset \Sigma_n(Y, K)$ the ramification locus. If φ is the monodromy of K, then φ^n is the monodromy of K_n.

Corollary

If $K \subset Y$ is a fibred knot, the JSJ decomposition of $Y \setminus K$ contains a hyperbolic piece iff the dimension of $\widehat{HFK}(-\Sigma_n(Y, K), -K_n, 1-g)$ grows exponentially with n.

From results of A'Campo and McLean we obtain:

Corollary

If $K \subset S^3$ is the link of an irreducible isolated hypersurface singularity, then the multiplicity of the singularity is equal to

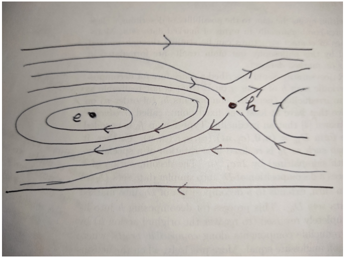
$$\min\{n > 0 | \dim \widehat{HFK}(-\Sigma_n(S^3, K), -K_n, 1-g) \neq 1\}.$$

Good representatives for the monodromy

We fix a diffeomorphism $\varphi\colon S \to S$ such that

- there exists a 1-form β on Σ such that $d\beta$ is an area form and $\varphi^*\beta - \beta$ is exact,
- in a collar A of ∂S, φ has the following form:
    - ∂S is rotated negatively by a small angle ε,
    - the other component of ∂A is rotated positively by a small angle ε,
    - in the interior of A there are only two fixed points: an elliptic one e and a hyperbolic one h,
- the fix points of φ are nondegenerate, and
- φ is isotopic to φ_0 by an isotopy which moves ∂S by less than ε (in either direction).

Focus on the collar



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The Floer homology group $HF^\sharp(\varphi)$

$CF^\sharp(\varphi)$ is the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the fixed points of φ *except for e*.

The differential $\partial^\sharp\colon CF^\sharp(\varphi) \to CF^\sharp(\varphi)$ is defined by

$$\partial^\sharp(p_+) = \sum_{\substack{p_- \in \mathrm{Fix}(\varphi) \\ p_- \neq e}} \#\left(\mathcal{M}(p_+, p_-)/\mathbb{R}\right)_0 p_-$$

$(\mathcal{M}(p_+, p_-)/\mathbb{R})_0$ is the 0-dimensional part of the moduli space $\mathcal{M}(p_+, p_-)/\mathbb{R}$ (defined in the next slide).

$HF^\sharp(\varphi)$ is the homology of $(CF^\sharp(\varphi), \partial^\sharp)$.

Moduli spaces for $\partial^\sharp$

- T_φ mapping torus $[0,1] \times S/(1,x) \sim (0,\varphi(x))$,
- R transverse vector field,
- J almost complex structure on $\mathbb{R} \times T_\varphi$ such that:
  - $\mathbb{R} \times T_\varphi \to \mathbb{R} \times S^1$ is holomorphic,
  - $J(\partial_s) = R$, where ∂_s generate the translations in $\mathbb{R}$,
  - $\mathcal{L}_{\partial_s} J = 0$, and
  - J is compatible with $d\beta$ on each fibre.

Fixed points p of φ are in bijections with closed orbits γ_p of R which project to S^1 with degree one.

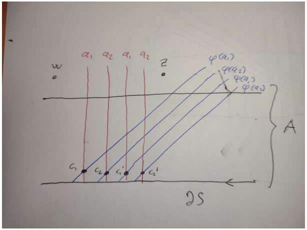
$\mathcal{M}(p_+, p_-)$ is the set of J-holomorphic sections $u \colon \mathbb{R} \times S^1 \to \mathbb{R} \times T_\varphi$ such that $\lim_{s \to \pm\infty} u(s,t) = \gamma_{p_\pm}(t)$.

$\mathbb{R}$ translations on $\mathbb{R} \times T_\varphi$ act on $\mathcal{M}(p_+, p_-)$ and $\mathcal{M}(p_+, p_-)/\mathbb{R}$ is the quotient.

Bases of arcs

let $\boldsymbol{a} = (a_1, \ldots, a_{2g})$ be a basis of arcs for S, i.e. a set of arcs which cut S into a disc, and let $\varphi(\boldsymbol{a}) = (\varphi(a_1), \ldots, \varphi(a_{2g}))$ be the image of $\boldsymbol{a}$ by the monodromy.

The intersection pattern between $\boldsymbol{a}$ and $\varphi(\boldsymbol{a})$ in the collar A looks like in the picture:



Knot Floer homology

$\widetilde{CFK}(S, \varphi(\boldsymbol{a}), \boldsymbol{a}, d)$ is the $\mathbb{Z}/2\mathbb{Z}$-vector space generated by $2g$-tuples of intersection points $\boldsymbol{x} = (x_1, \dots, x_{2g})$ such that

- each arc in $\boldsymbol{a}$ and each arc in $\varphi(\boldsymbol{a})$ contains exactly one intersection point, and
- exactly $g - d$ intersection points are in the collar A.

$\widehat{CFK}(S, \varphi(\boldsymbol{a}), \boldsymbol{a}, d)$ is the quotient of $\widetilde{CFK}(S, \varphi(\boldsymbol{a}), \boldsymbol{a}, d)$ by the relation $(\dots, c_i, \dots) \sim (\dots, c_i', \dots)$.

The differential $\widehat{CFK}(S, \varphi(\boldsymbol{a}), \boldsymbol{a}, d) \to \widehat{CFK}(S, \varphi(\boldsymbol{a}), \boldsymbol{a}, d)$ is defined by

$$\widehat{\partial}(\boldsymbol{x}_+) = \sum_{\boldsymbol{x}_-} \# \left(\mathcal{M}(\boldsymbol{x}_+, \boldsymbol{x}_-)/\mathbb{R} \right)_0 \boldsymbol{x}_-$$

and $\widehat{HFK}(-Y, -K, d)$ is the homology of $(\widehat{CFK}(S, \varphi(\boldsymbol{a}), \boldsymbol{a}, d), \widehat{\partial})$.

Moduli spaces for $\widehat{\partial}$

J almost complex structure on $\mathbb{R} \times [0,1] \times S$ such that:

- $\mathbb{R} \times [0,1] \times S \to \mathbb{R} \times [0,1]$ is holomorphic,
- $J(\partial_s) = \partial_t$, where s is the coordinate on $\mathbb{R}$ and t is the coordinate in $[0,1]$,
- $\mathcal{L}_{\partial_s} J = 0$, and
- J is compatible with $d\beta$ on each fibre.

$\mathcal{M}(\boldsymbol{x}_+, \boldsymbol{x}_-)$ is the set of J-holomorphic embedded multi-sections

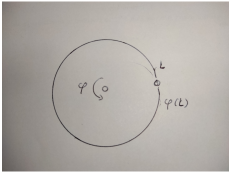
$$u\colon \mathbb{R} \times S^1 \to \mathbb{R} \times [0,1] \times S$$

with boundary on $\mathbb{R} \times \{0\} \times \varphi(\boldsymbol{a})$ and $\mathbb{R} \times \{1\} \times \boldsymbol{a}$ such that $\lim_{s \to \pm\infty} u(s,t) = \{t\} \times \boldsymbol{x}_\pm$.

$\mathbb{R}$ translations act on $\mathcal{M}(\boldsymbol{x}_+, \boldsymbol{x}_-)$ and $\mathcal{M}(\boldsymbol{x}_+, \boldsymbol{x}_-)/\mathbb{R}$ is the quotient; $(\mathcal{M}(\boldsymbol{x}_+, \boldsymbol{x}_-)/\mathbb{R})_0$ is the 0-dimensional part of $\mathcal{M}(\boldsymbol{x}_+, \boldsymbol{x}_-)/\mathbb{R}$.

The cobordism E

Denote $B = D^2 \setminus \{0, 1\}$ and let $\pi \colon E \to B$ be the flat bundle with fibre S and monodromy φ.



We fix an identification $\mathfrak{r} \colon (0, 1) \times S \to \pi^{-1}(\partial B) \subset E$ and define $L = \mathfrak{r}((0, 1) \times \boldsymbol{a})$.

We fix holomorphic identifications of:

- a punctured neighbourhood of $0 \in B$ with $(-\infty, 0) \times S^1$, and
- a punctured neighbourhood $1 \in B$ with $(0, +\infty) \times [0, 1]$.

Over these neighbourhoods we fix identifications of E with

$$(-\infty, 0) \times T_\varphi \quad \text{and} \quad (0, +\infty) \times [0, 1] \times S.$$

In the second neighbourhood L looks like

$$((0, +\infty) \times \{0\} \times \varphi(\boldsymbol{a})) \cup ((0, +\infty) \times \{1\} \times \boldsymbol{a}).$$

The chain map $\Phi^\sharp$

We define a chain map

$$\Phi^\sharp \colon \widehat{CFK}(S, \varphi(\boldsymbol{a}), \boldsymbol{a}, 1-g) \to CF^\sharp(\varphi)$$

as follows:

$$\Phi^\sharp(\boldsymbol{x}) = \sum_{\substack{p \in \mathrm{Fix}(\varphi) \\ p \neq e}} \#\mathcal{M}_0(\boldsymbol{x}, p)p$$

$\mathcal{M}_0(\boldsymbol{x}, p)$ is the 0-dimensional part of the moduli space of embedded J-holomorphic multi-sections $u \colon B \to E$ with boundary on L such that

- $\lim_{s \to +\infty} u(s,t) = \{t\} \times \{\boldsymbol{x}\}$ in a punctured neighbourhood of 1, and
- $\lim_{s \to -\infty} u(s,t) = (\gamma_p(t), \gamma_e(t), \dots, \gamma_e(t))$ in a punctured neighbourhood of 0.

Two exact sequences

Let $C \subset S$ be a nonseparating closed curve and τ_C the positive Dehn twist along C.
We can see C also as a framed knot in Y disjoint from K, with the framing induced by a fibre, and let $K_r \subset Y_r$ be the result of r-surgery on C.
Then we have exact triangles

$$\begin{array}{ccccc}
\widehat{HFK}(-Y_0, -K_0, 1-g) & \longrightarrow & \widehat{HFK}(-Y, -K, 1-g) & \longrightarrow & \widehat{HFK}(-Y_{+1}, -K_{+1}, 1-g) \\
\downarrow & & \downarrow \Phi^\sharp_* & & \downarrow \Phi^\sharp_* \\
HF(\varphi(C), C) & \longrightarrow & HF^\sharp(\varphi) & \longrightarrow & HF^\sharp(\varphi \circ \tau_C^{-1})
\end{array}$$

and the diagram commutes.

Proof of the theorem

Lemma

The map $\widehat{HFK}(-Y_0, -K_0, 1-g) \to HF(\varphi(C), C)$ *is an isomorphism.*

By the five-lemma, $\Phi^\sharp \colon \widehat{HFK}(-Y, -K, 1-g) \to HF^\sharp(\varphi)$ is an isomorphism iff $\widehat{HFK}(-Y_{+1}, -K_{+1}, 1-g) \to HF^\sharp(\varphi \circ \tau_C^{-1})$ is an isomorphism.

Every diffeomorphism of S can be decomposed as a product of Dehn twists $\varphi = \tau_{C_1}^{s_1} \circ \ldots \circ \tau_{C_n}^{s_n}, \quad s_i \in \{+1, -1\}$.

We argue by induction on the number of Dehn twists n:

- If $n = 0$ then $\varphi = \mathrm{id}$ and we check by hand that $\Phi^\sharp_*$ is an isomorphism.
- If the theorem true for $n-1$, then we have an isomorphism for $\varphi \circ \tau_{C_n}^{-s_n}$, and therefore an isomorphism for φ.